REPRESENTATION THEORY OF $SU_2(\mathbb{C})$ AND $SO_3(\mathbb{R})$. §2.3

$SU_2(\mathbb{C}) \cong S^3$ as a diff. mfld.

$SO_3(\mathbb{C}) \cong SU_2(\mathbb{C}) / \{\pm \mathrm{Id}\}$

Today classify irr. complex rep.

Thm
$$\left\{ \begin{array}{c} \text{irr. cx. rep} \\ \text{of } SU_2(\mathbb{C}) \end{array} \right\} \Big/ \cong \;\xrightarrow{\sim}\; \mathbb{Z}_{\geq 0}$$
$$V \longmapsto \dim V$$
$$V(n) \longleftarrow n+1$$

EXAMPLE $\mathbb{C}$ is the trivial rep. $V(0)$

$V(1) := \mathbb{C}^2 \circlearrowleft SU_2$ natural rep.

$\mathrm{Ad} : SU_2(\mathbb{C}) \circlearrowleft \mathfrak{su}_2(\mathbb{C}) \cong V(2)$.

$$\left\{ \begin{array}{c} \text{irr. cx. rep.} \\ \text{of } SU_2(\mathbb{C}) \end{array} \right\} \Big/ \cong \;\hookrightarrow\; \left\{ \begin{array}{c} \text{irr. cx. rep.} \\ \text{of } \mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \end{array} \right\} \Big/ \cong$$

$\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$

What is $\mathfrak{su}_2(\mathbb{C}) \otimes \mathbb{R}$?

$$\mathfrak{su}_2(\mathbb{C}) = \left\{ A \in M_2(\mathbb{C}) \;\middle|\; \begin{array}{l} \operatorname{tr} A = 0 \\ A + \bar{A}^t = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

Claim

$$\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$$

$$\mathfrak{sl}_2(\mathbb{C}) = \{ A \in M_2(\mathbb{C}) \mid \operatorname{tr} A = 0 \}$$

Both have dim $3/\mathbb{C}$

Enough to find inj. morphism

$$\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{sl}_2(\mathbb{C}).$$

We have an inclusion $\mathfrak{su}_2(\mathbb{C}) \hookrightarrow \mathfrak{sl}_2(\mathbb{C})$

$$\Phi : \mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathfrak{sl}_2(\mathbb{C})$$

$$A \otimes z \longmapsto zA$$

Why Φ inj?

$\mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}i$ as a vector space.

So $\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{su}_2(\mathbb{C}) \oplus i\,\mathfrak{su}_2(\mathbb{C})$.

If Φ not inj., then $\exists A, A' \in \mathfrak{su}_2(\mathbb{C})$

$\Phi(A + iA') = 0$

$A + iA' = 0$ but $iA' \in \mathfrak{su}_2(\mathbb{C}) \Rightarrow$

if $iA' = \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \in \mathfrak{su}_2(\mathbb{C}) \Rightarrow$ $a, b, c = 0 \Rightarrow A' = 0$

$\Rightarrow \Phi$ inj $\Rightarrow \Phi$ isomorphism.

$\mathfrak{su}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$

REPR. THEORY OF $\mathfrak{sl}_2(\mathbb{C})$

Thm $\left\{ \begin{matrix} \text{cx. irr. rep.} \\ \text{of } \mathfrak{sl}_2(\mathbb{C}) \end{matrix} \right\} \Big/ \cong \quad \xrightarrow{\sim} \quad \mathbb{Z}_{\geq 0}$

$$V \longmapsto \dim V$$

$$L(n) \longleftarrow\!\shortmid \; n+1.$$

$L(0)$ trivial.

$L(1)$ natural.

$L(2)$ adjoint $\mathfrak{sl}_2(\mathbb{C}) \circlearrowleft \mathfrak{sl}_2(\mathbb{C})$.

<u>Pf</u> $\mathfrak{sl}_2(\mathbb{C})$ has a basis.

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$$

We construct a rep. of $\mathfrak{sl}_2(\mathbb{C})$ of dim $m+1$.

Consider the nat. rep. of $SL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2 = \langle X, Y \rangle$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY$$

$$V(m) =: \mathbb{C}[X, Y]^m$$ v.sp. of homogeneous polynomials of degree m.

$\dim V(m) = m+1$,

Basis is $\{ \underbrace{X^{m-k} Y^k}_{=: w_k} \}_{0 \le k \le m}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^{m-k} Y^k = \underbrace{(aX + cY)^{m-k} (bX + dY)^k}_{\in V(m)}.$$

We can derive and obtain a rep. of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

How does e act?

$$\gamma(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{C}), \quad \dot\gamma(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e$$

$$\frac{d}{dt}\gamma(t).(\underbrace{X^{m-k}Y^k}_{=\,\omega_k})\Big|_{t=0} = \frac{d}{dt} X^{m-k}(Y+tX)^k\Big|_{t=0} =$$

$$= X^{m-k}\cdot X\, \underbrace{kY^{k-1}}_{=\,k\,\omega_{k-1}} = X\frac{\partial}{\partial y}(X^{m-k}Y^k)$$

f acts as $Y\dfrac{\partial}{\partial x}$

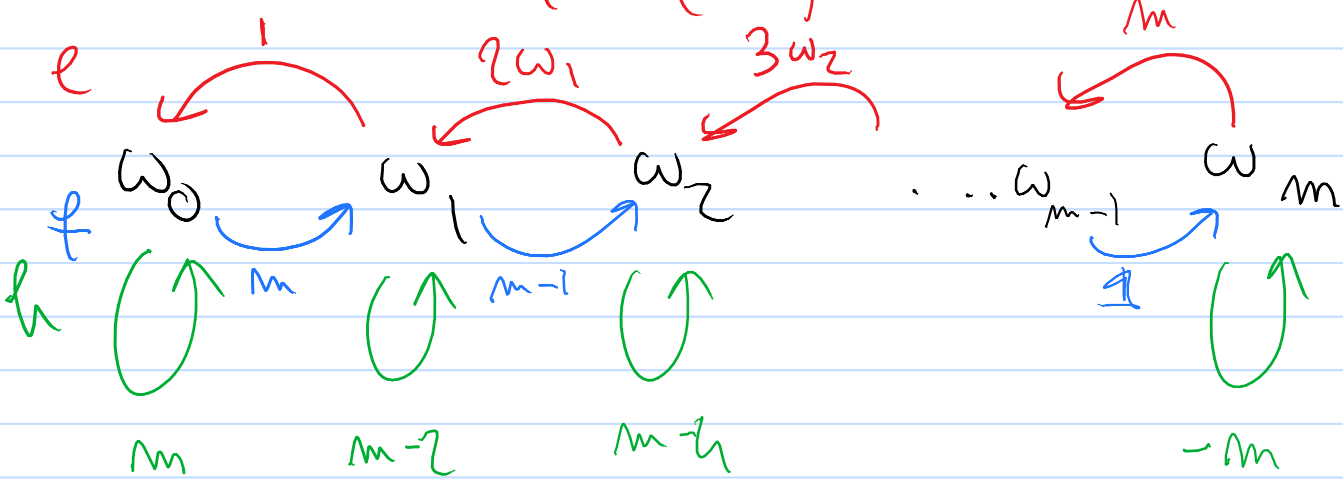
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

$$\frac{d}{dt}\gamma(t)(X^{m-k}Y^k)\Big|_{t=1} = \frac{d}{dt}(tX)^{m-k}(t^{-1}Y)^k\Big|_{t=1}$$

$$= X^{m-k}Y^k \frac{d}{dt}(t^{m-2k})\Big|_{t=0} =$$

$$(m-2k)X^{m-k}Y^k.$$

A Picture of $V(m)$



Claim $V(m)$ is an irr. $\mathfrak{sl}_2(\mathbb{C})$ rep.

$U \subset V(m)$ stable under $\mathfrak{sl}_2(\mathbb{C})$.

U stable under h, $\exists\, v \in U$ eigenvector.

$\Rightarrow \exists\, k$ s.t. $\omega_k \in U$.

$$\Rightarrow \underset{\substack{\parallel \\ \omega_{k-1}}}{e\omega_k},\ \underset{\substack{\parallel \\ \omega_{k-2}}}{e^2\omega_k},\ \underset{\substack{\parallel \\ \omega_{k-3}}}{e^3\omega_k}, \dots \in U$$

$$\underset{\substack{\parallel \\ \omega_{k+1}}}{f\omega_k},\ \underset{\substack{\parallel \\ \omega_{k+2}}}{f^2\omega_k}, \dots \in U$$

$\Rightarrow \omega_i \in U \quad \forall i \Rightarrow U = V(m)$ $\square$

It remains to show that every two rep. of dim m of $\mathfrak{sl}_2(\mathbb{C})$ are isomorphic.

Assume we have irr. cx. rep. V of $\mathfrak{sl}_2(\mathbb{C})$.
For $\lambda \in \mathbb{C}$

$$V_\lambda = \lambda\text{-eigenspace of } h = \{v \in V \mid h \cdot v = \lambda v\}$$

We have $e(V_\lambda) \subset V_{\lambda+2}$.

<u>Pf</u> $[h, e] = 2e$

$v \in V_\lambda$. $hv = \lambda v$

$[h, e]v = hev - ehv \iff$

$2ev = h(ev) - \lambda ev \iff$

$\iff h(ev) = (\lambda+2)ev \iff ev \in V_{\lambda+2}$.

$f(V_\lambda) = V_{\lambda-2}$.

V contains an eigenvector for h.

$\exists\, V_\lambda \neq 0$ $\quad v \in V_\lambda$, $ev = 0$ or ev eigenvector.

We can find λ such that $V_\lambda \neq 0$, but $V_{\lambda+2} = 0$.

$v \in V_\lambda \Rightarrow ev = 0$.

$U = \text{span}\langle f^i v \mid i \geq 0 \rangle$

<u>Claim</u> U is a subrep. of V.

We need to show that U is stable under e, h, f.

$$f \cdot f^k v = f^{k+1} v \in U \quad \checkmark$$

$$h \cdot f^k v = (\lambda - 2k) f^k v \in U \quad \checkmark$$

$$e \cdot f^k v = \begin{cases} 0 & \text{if } k=0 \\ e f f^{k-1} v = \underbrace{\underbrace{[e,f]}_{= h} f^{k-1} v}_{\in U} + f \underbrace{e f^{k-1} v}_{\in U} \quad \checkmark \end{cases}$$

$\square$

You can show by induction $\boxed{\text{Cor } U = V}$

$$e f^k v = k(\lambda - k + 1) f^{k-1} v.$$

Since V is fin. dim. $\exists$ minimal d such that

$$f^d v = 0, \quad f^{d-1} v \neq 0.$$

so a basis of V is

$$\{v, fv, f^2 v, \dots, f^{d-1} v\} \text{ basis.}$$

$$f^d v = 0 \Rightarrow e f f^{d-1} v = 0$$

$$\Rightarrow d(\lambda - d + 1) f^{d-1} v = 0$$

$$\Rightarrow \lambda = d - 1$$

So all the eigensp. of a rep. of $\mathfrak{sl}_2(\mathbb{C})$ have int. eigenvalue.

V is the unique rep. of dim d (up to isomorphism).

$\square$

$$\left\{ \begin{array}{l} \text{irr. cx. rep.} \\ \text{of } SU_2(\mathbb{C}) \end{array} \right\} \Big/ \cong \;\xrightarrow{\sim}\; \left\{ \begin{array}{l} \text{irr. cx. rep.} \\ \text{of } \mathfrak{su}_2(\mathbb{R}) \otimes \mathbb{C} \\ \qquad \cong \\ \qquad \mathfrak{sl}_2(\mathbb{C}) \end{array} \right\} \Big/ \cong$$

Want to show it is a bijection.

We need to construct representations of $SU_2(\mathbb{C})$ then, when derived, give us the representations $L(m)$ of $\mathfrak{sl}_2(\mathbb{C})$.

<u>Obs</u> $SU_2(\mathbb{C}) \hookrightarrow SL_2(\mathbb{C})$

So $V(m) := \mathbb{C}[X,Y]^m$ hom. pol. of degree m

is a rep. of $SU_2(\mathbb{C})$.

Why $V(m)$ irr. as $SU_2(\mathbb{C})$ rep?

Enough to check what is as a $su_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ rep!

$$su_2(\mathbb{C}) = \left\langle \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{= \varphi}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle$$

How does φ act on $V(m)$?

$$\gamma(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad \dot{\gamma}(0) = \varphi.$$

$$\frac{d}{dt} \gamma(t) \left(X^{m-k} Y^{k} \right) \Big|_{t=0} = \frac{d}{dt} \left(\cos t X - \sin t Y \right)^{m-k} \left(\sin t X + \cos t Y \right)^{k} \Big|_{t=0}$$

$$(m-k) X^{m-k-1} (-Y) Y^{k} + Y^{k-1} X^{m-k} k X =$$

$$= (e-f) X^{m-k} Y^{k}.$$

$\rightsquigarrow V(m) \cong L(m)$ of $sl_2(\mathbb{C})$

$\Rightarrow V(m)$ is irreducible. $\square$

Rep. of $SO_3(\mathbb{R})$.

$$SU_2(\mathbb{C}) \xrightarrow{s} SO_3(\mathbb{R})$$

If $\rho : SO_3(\mathbb{R}) \to GL(V)$ irr. rep.

$\rho \circ s$ is an irr. rep. of $SU_2(\mathbb{C})$.

If $\rho' : SU_2(\mathbb{C}) \to GL(V)$ irrep. of $SU_2(\mathbb{C})$

s.t. $-\mathrm{Id} \in \ker \rho$

$\Rightarrow$ ρ' descends to a rep. of $SO_3(\mathbb{R})$.

$$\left\{ \begin{array}{c} \text{irrep. of} \\ SO_3(\mathbb{R})/\mathbb{C} \end{array} \right\}_{/\simeq} \longleftrightarrow \left\{ \begin{array}{c} \text{irrep. } \rho \text{ of} \\ SU_2(\mathbb{C})/\mathbb{C} \text{ s.t.} \\ -\mathrm{Id} \in \ker \rho \end{array} \right\}_{/\simeq}$$

For which m does $-\mathrm{Id}$ lies in the kernel?

How does $-\mathrm{Id}$ act on $V(m)$?

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} X^{m-k} Y^{k} = (-X)^{m-k} (-Y)^{k} =$$

$$= (-1)^{m-2k} X^{m-k} Y^{k}$$

the action of (-1) is trivial $\Leftrightarrow$ m even

$$\left\{ \begin{matrix} \text{irrep of} \\ SO_3(\mathbb{R}) / \mathbb{C} \end{matrix} \right\} / \cong \longrightarrow \{1, 3, 5 \dots\}$$

$$V \longmapsto \dim V$$