COMPACT LIE ALGEBRAS & KILLING FORMS

Def Let $\mathfrak{g}$ be fin. dim Lie algebra over a field $/k$.

The Killing form on $\mathfrak{g}$ is a bilinear form defined via
$$h(X,Y) = tr\left(ad(X)\,ad(Y)\right) \quad \forall X,Y \in \mathfrak{g}$$

$$ad(X) : \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$Y \longmapsto [X,Y]$$

Since $tr(AB) = tr(BA) \Rightarrow h$ is symmetric.

<u>EXAMPLE</u> $\mathfrak{sl}_2(\mathbb{R})$ $\quad e, h, f$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$ad(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{in basis } \{e, h, f\}$$

$$h(e,e) = tr\left(ad(e)^2\right) = 0$$ so it is not definite

Def A Lie algebra $/\mathbb{R}$ is <u>compact</u> if the Killing form h is negative definite.

Def The center of a Lie algebra $\mathfrak{g}$

$$z(\mathfrak{g}) = \{ X \in \mathfrak{g} \mid [X,Y] = 0 \}$$

$\mathfrak{g} = \mathrm{Lie}\, G$ then

$$z(\mathfrak{g}) = \mathrm{Lie}(\ker \mathrm{Ad}\, G)$$

$\ker \mathrm{Ad}\, G \subset Z(G)$

$=$ if G is connected.

$$\mathrm{Ad} : G \to GL(\mathfrak{g})$$

$$g \longmapsto (X \mapsto g X g^{-1})$$

Prop Let G be a compact Lie group w/ finite $Z(G)$, then $\mathfrak{g} = \mathrm{Lie}\, G$ is a compact Lie algebra

Pf G acts on $\mathfrak{g}$ via Ad

G compact $\Rightarrow \exists$ G-invariant scalar product on $\mathfrak{g}$

$$\mathrm{Ad}\, G \subset O(\mathfrak{g})$$

$X \in \mathrm{Lie}\, G \Rightarrow \forall t \in \mathbb{R}$ $e^{Xt} \in G \Rightarrow$

$$\Rightarrow \mathrm{Ad}(e^{Xt}) \in O(\mathfrak{g}) \Rightarrow \mathrm{Ad}(e^{Xt})\, \mathrm{Ad}(e^{Xt})^{T} = \mathrm{Id}$$

We choose an orthonormal basis of $\mathfrak{g}$ wrt

$$\frac{d}{dt}\left(\mathrm{Ad}(e^{Xt})\right)\Big|_{t=0} = \mathrm{ad}(X) \in \mathrm{Lie}\, O(g)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\{A \in gl(g) \mid A + A^T = 0\}$$

$\Rightarrow \forall X \in g$, $\mathrm{ad}(X)$ is antisymmetric,

$\mathrm{ad}(X)$ is diagonalizable

$\Rightarrow$ all eigenvalues of $\mathrm{ad}(X)$ are purely imaginary

$$\left(Av = \lambda v, \quad \underset{\lambda \langle v,v\rangle}{\langle Av, v\rangle} = \underset{-\bar{\lambda}\langle v,v\rangle}{-\langle v, Av\rangle} \Rightarrow \lambda \in \mathbb{R}i \right.$$

$$\Rightarrow h(X,X) = \mathrm{tr}\left(\mathrm{ad}(X)^2\right) \leq 0$$

and $h(X,X) = 0$ only if $\mathrm{ad}(X) = 0$

$Z(G)$ finite $\Rightarrow \ker(\mathrm{Ad}\, G)$ is finite.

$$\Rightarrow \mathrm{Lie}(\ker \mathrm{Ad}(G)) = 0$$

$$\| $$

$$z(g)$$

So $\mathrm{ad}(X) = 0 \Rightarrow X = 0$

So the Lie algebra of a compact Lie group w/ trivial center is compact. $\square$

Thm There is a bijection

$$\left\{ \begin{array}{c} \text{connected compact Lie groups} \\ \text{w/ trivial center} \end{array} \right\} \Big/ \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{compact Lie} \\ \text{algebras} / \mathbb{R} \end{array} \right\} \Big/ \cong$$

connected component of the identity

$$K \longrightarrow \operatorname{Lie} K$$

$$(\operatorname{Aut} \mathfrak{k})^0 \longleftarrow \mathfrak{k}$$

Pf Let $\mathfrak{k}$ compact Lie algebra,

$$\operatorname{Aut} \mathfrak{k} = \left\{ A \in GL(\mathfrak{k}) \;\middle|\; \begin{array}{c} A([X,Y]) = [AX, AY] \\ \forall X, Y \in \mathfrak{k} \end{array} \right\}$$

is a closed subgroup of $GL(\mathfrak{k})$, so it is a Lie group.

$$A := (\operatorname{Aut} \mathfrak{k})^0$$

wrt Killing form is compact!

We want to show that $A \subset O(\mathfrak{k})$

$f \in A$. $X, Y \in \mathfrak{k}$

$$\kappa(f(X), f(Y)) = \operatorname{tr}\left(\operatorname{ad}(f(X)) \operatorname{ad}(f(Y))\right)$$

$Z \in \mathfrak{k}$

$$\operatorname{ad}(f(X)) \operatorname{ad}(f(Y)) f(Z) =$$

$$[f(X), [f(Y), f(Z)]] = [f(X), f([Y,Z])] =$$

$$= f([X,[Y,Z]]) = f(\operatorname{ad} X \operatorname{ad}(Y) Z)$$

$$\Rightarrow \operatorname{ad}(f(X)) \operatorname{ad} f(Y) = f(\operatorname{ad} X \operatorname{ad} Y) f^{-1}$$

$$\Rightarrow h(\operatorname{ad} f(X) \operatorname{ad} f(Y)) = h(\operatorname{ad} X \operatorname{ad} Y)$$

$$\Rightarrow h(f(X), f(Y)) = h(X, Y)$$

$$\Rightarrow f \in O(h) \Rightarrow A \text{ is compact.}$$

What is $Z(A)$?

A connected $\Rightarrow$ $Z(A) = \ker(\mathrm{Ad})$

ASSUMPTION (we show it later)

- $\boxed{\text{Lie } A \cong K \text{ as Lie algebra and as representations of } A}$

$\ker(\mathrm{Ad}) = \ker(A \to GL(K)) = \{\mathrm{Id}\} \Rightarrow Z(A) = \{e\}$

We need to show that the two maps are inverse to each other.

k compact Lie algebra. $\quad \mathrm{Lie}(\underset{\overset{\text{\rotatebox{90}{$\cong$}}}{A}}{\mathrm{Aut}\,k)^0}) = K$. $\leftarrow$ follows by assumption.

K compact Lie group, connected w/ trivial center

Need to show:

$K \cong \mathrm{Aut}(\mathrm{Lie}\,K)^0$

$\mathrm{Ad}: K \longrightarrow GL(\mathrm{Lie}\,K)$

K connected $\Rightarrow$ $\ker(\mathrm{Ad}) = Z(A) = e$

$\Rightarrow$ Ad injective.

$g \in K$, $X, Y \in \mathrm{Lie}\,K$, $[gXg^{-1}, gYg^{-1}] = g[X,Y]g^{-1}$.

So we get an injective morphism

$$\begin{tikzcd} \mathrm{Lie}\,K \arrow[r, "\sim"] \arrow[d, "\exp"'] & \mathrm{Lie}\,\mathrm{Aut}(\mathrm{Lie}\,K)^0 \arrow[d, "\exp"] \\ K \arrow[r] & \mathrm{Aut}(\mathrm{Lie}\,K)^0 \end{tikzcd}$$

So this is also surjective $\Rightarrow$ a bijective morphism of Lie groups is an isomorphism. $\square$

It remains to show

<u>Prop</u> $\mathrm{Lie}\,A \cong K$ as Lie algebras.

<u>Pf</u> We introduce a third Lie algebra

$$\mathrm{Der}(K) = \{ d \in \mathrm{End}(K) \mid d([a,b]) = [da,b] + [a,db] \}$$

<u>EXERCISE</u> $\mathrm{Der}(K) \cong A$ (so in particular $\mathrm{Der}(K)$ is a Lie algebra)

So we need to show $K \xrightarrow{\sim} \mathrm{Der}\,K$

We have a map $\mathrm{ad}: K \to \mathrm{Der}\,K =: D$
$$X \to \mathrm{ad}(X)$$

ad(X) is a derivation

$\forall Y, Z \in K$

$$[X,[Y,Z]] = -[Y,[Z,X]] - [Z,[X,Y]]$$

$$\begin{aligned} &\overset{\shortparallel}{} \\ &[Y, ad(X)Z] - [Z, ad(X)Y] = \\ &= [ad(X)Y, Z] + [Y, ad(X)Z]. \end{aligned}$$

K compact $\Rightarrow$ ad is injective

$ad\, X = 0 \Rightarrow k(X,X) = 0 \Rightarrow X = 0.$

$D := \mathrm{Der}\, K$, let k_D be the killing form on D.

We want to show $\underset{\underset{K}{\cong}}{ad(K)} = D$.

ad(K) is compact Lie algebra
and it is an ideal of D.

$Y, X \in K$, $d \in \mathrm{Der}\, K$

$$\begin{aligned} [d, ad\, X](Y) &= d([X,Y]) - [X, dY] \\ &= [dX, Y] + \cancel{[X, dY]} - \cancel{[X, dY]} = ad(dX)(Y) \end{aligned}$$

$[d, \mathrm{ad}(X)] = \mathrm{ad}(dX)$.

<u>EXERCISE</u> Let I an ideal of a Lie algebra g then the restriction of k_g to I is k_I.

$$k_{D}|_{\mathrm{ad}(K)} = k_{\mathrm{ad}(K)} \text{ is negative definite.}$$

$$\mathrm{ad}(K)^{\perp} =: I \quad \text{orthogonal of } \mathrm{ad}(K) \text{ wrt } k_D.$$

<u>I is an ideal.</u> Follows from

$$k(X, [Y,Z]) = k([X,Y],Z)$$

if $X \in I$, $d \in D$, $\forall a \in \mathrm{ad}(K)$

$$k(X,[d,a]) = k([X,d],a) = 0$$

$$\Rightarrow [X,d] = -[d,X] \in I.$$

$$\Rightarrow D = \mathrm{ad}(K) \oplus I$$

$$[\mathrm{ad}K, I] \subset I \Rightarrow [\mathrm{ad}K, I] = \{0\}$$

$$\mathrm{ad}K \cap$$

But if we take $\delta \in I$, $\forall x \in K$

$$0 = [\delta, \mathrm{ad}x] = \mathrm{ad}\,\delta x \Rightarrow \delta x = 0 \Rightarrow \delta = 0 \Rightarrow I = 0$$

$\Rightarrow D \cong \mathrm{ad}(K) \cong K$ $\square$

Thm $K \xrightarrow{\sim} \mathrm{Der} K = \mathrm{Lie} A$ is isomorphism of A-representations.

$x \longrightarrow \mathrm{ad}\, x$

$\forall a \in A, x \in K, y \in K$

$\mathrm{ad}(ax)(y) = [ax, y]$

$a(\mathrm{ad}\, x)a^{-1}(y) = a[x, a^{-1}y] = [ax, y]$ $\square$