LECTURE 11 WEYL GROUPS AND ROOT SYSTEMS §5.3–5.5.

Lma S, T tori, Z connected top. space

We have a continuous of morphisms parametrized by Z,

$$\varphi_z : S \to T, \quad z \in Z$$

the φ_z does not depend on z.

Pf. $\varphi : Z \times S \to T$ is continuous

$$(z, s) \mapsto \varphi_z(s).$$

$$T[n] = \{ t \in T \mid t^n = e \}$$

$$\forall z \quad \varphi_z^{-1}(S[n]) = T[n].$$

We fix n, and $s \in S[n]$, $Z_{s,t} = \{ z \mid \varphi_z(s) = t \}$

closed sets

$$Z = \coprod_{t \in T[n]} Z_{s,t} \Rightarrow Z = Z_{s,t} \text{ for some } t \in T[n]$$

All the morphisms in Z are constant on s.

$\bigcup_{n \geq 0} S[n] \subset S$ dense subset

$\Rightarrow \varphi_z$ does not depend on z!

G Lie group $\supset S$ torus.

$$N_G(S) = \{ g \in G \mid gSg^{-1} = S \}$$

<u>Prop</u> $N_G(S)^\circ \subset Z_G(S)$

<u>Pf.</u> $Z = N_G(S)^\circ$

for $z \in Z$ $\quad \varphi_z : S \to S$

$$s \mapsto zsz^{-1}$$

$\Rightarrow \varphi_z$ does not depend on z

$\Rightarrow \varphi_z = \varphi_e \Rightarrow s = zsz^{-1} \ \forall z \in N_G(S)^\circ$

$\Rightarrow N_G(S)^\circ \subset Z_G(S)$.

<u>Def</u> K compact Lie group, T maximal torus

$$W(K,T) = N_K(T)/T$$

$W(K,T)$ is the Weyl group K.

<u>Rmk</u> $W(K,T)$ does not depend on the max. torus up to iso.

<u>Cor</u> $W(K,T)$ is a fin. group.

<u>Pf</u> $N_K(T)^\circ \subset Z_K(T)^\circ = T \Rightarrow W = N/T$ is a quotient

of $\underbrace{N_k(T)}_{\text{Compact}} / N_k(T)^0$ which is finite.

EXAMPLE

$$k = SO_3(\mathbb{R}) \supset T = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$N_k(T) = T \amalg Ts \quad , \text{where } s = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$N_k(T)/T \cong \mathbb{Z}/2\mathbb{Z} = \{1, \bar{s}\}$$

EXAMPLE

$$k = U(n) \supset T = \left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \middle| \theta_i \in \mathbb{R} \right\}$$

$g \in k$ s.t. $gTg^{-1} = T$?

$t \in T$ with pairwise distincts eigenvalue.

$gtg^{-1} \in T$ if and only $\exists\, w$ permutation matrix

$$gtg^{-1} = wtw^{-1}.$$

$\Rightarrow w^{-1}g \in Z_G(t) = T$ $\qquad W(k, T)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\bar{g} \in N/T \qquad \bar{g} = \bar{w}, \qquad N/T \cong S_n$

LATTICE REFLECTION GROUPS.

Let X free abelian group, f.g.

$$X \cong \mathbb{Z}^n \quad \text{for some } n \geq 0$$

We call such a group lattice.

We call (lattice) reflection $s : X \to X$ automorphism

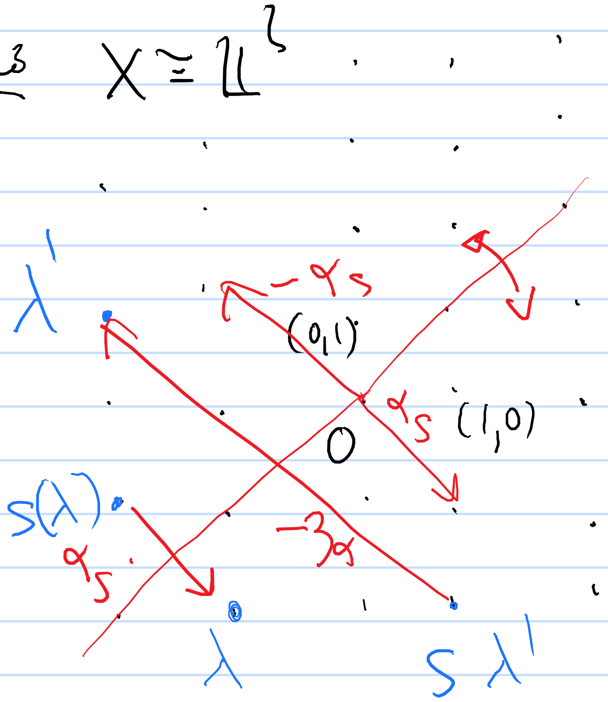
s.t. • $s^2 = \mathrm{Id}$

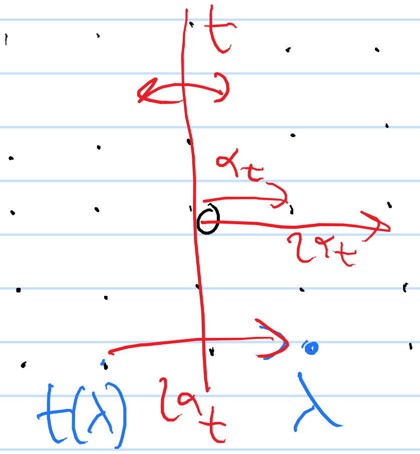
• $X^{-s} = \{ x \in X \mid s(x) = -x \} \cong \mathbb{Z}$

We call root of the reflection an element $\alpha \in X$

s.t. $\forall \lambda \in X \quad s(\lambda) - \lambda \in \mathbb{Z}\alpha \quad \forall \lambda \in X$

EXAMPLE $X \cong \mathbb{Z}^2$





X, s, α as above

$\alpha^\vee : X \to \mathbb{Z}$ is a group homomorphism.

which is defined by

$$s(\lambda) = \lambda - \underbrace{\langle \lambda, \alpha^\vee \rangle}_{= \alpha^\vee(\lambda)} \alpha \qquad \in \mathbb{Z}$$

We check that $\alpha^\vee$ is a group mor.

$$s(\lambda+\mu) = \lambda+\mu - \langle \lambda+\mu, \alpha^\vee \rangle \alpha$$

$$\shortparallel$$

$$s(\lambda) + s(\mu) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha + \mu - \langle \mu, \alpha^\vee \rangle \alpha$$

$$\Rightarrow \langle \lambda+\mu, \alpha^\vee \rangle = \langle \lambda, \alpha^\vee \rangle + \langle \mu, \alpha^\vee \rangle.$$

$\Rightarrow \alpha^\vee$ is a grp. morph.

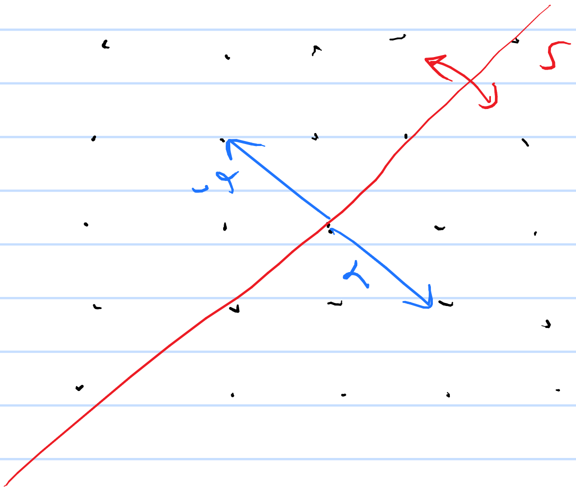
$\Rightarrow \alpha^\vee$ is called the coroot of α

Def: A finite lattice reflection group a finite subgroup of X generated by reflections.

. A root system R for the reflection group is a finite subset of $X \setminus \{0\}$ which is.

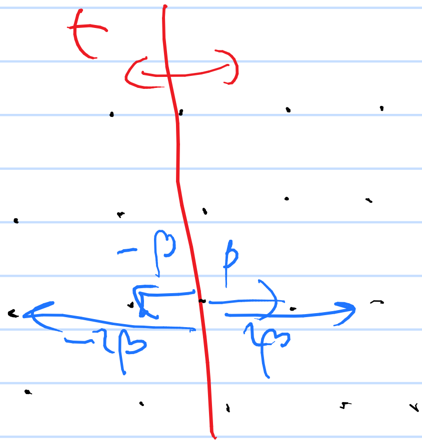
- stable under W.
- $\forall s \in W$, there are precisely two roots of s in R, one the negative of the other.
- all the elements in R are roots of some reflection in the reflection group.

EXAMPLE



$W = \langle s \rangle$ is a finite reflection group.
$\cong \mathbb{Z}/2\mathbb{Z}$

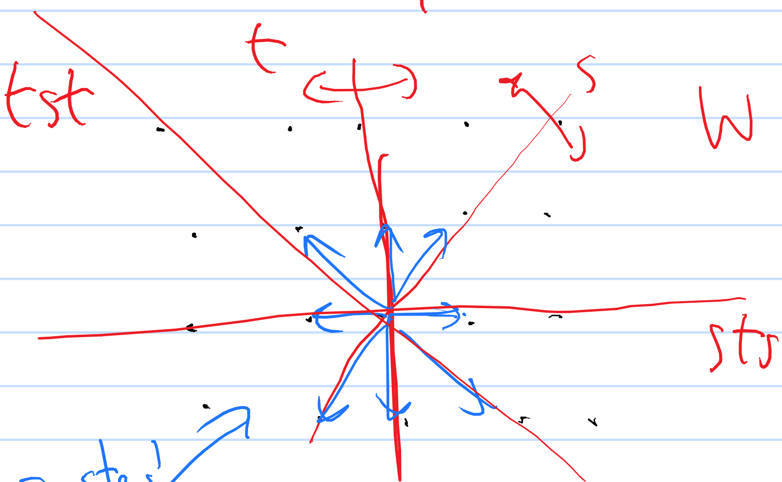
$R = \{\alpha, -\alpha\}$ is a root system.



$W = \mathbb{Z}/2\mathbb{Z} = \{Id, t\}$

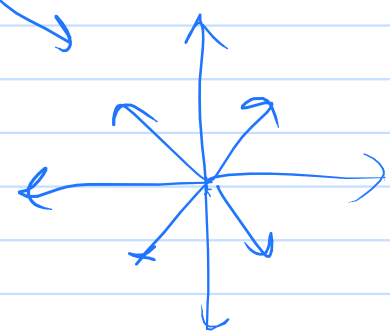
$\Phi = \{\beta, -\beta\}$

$\Phi' = \{2\beta, -2\beta\}$



$W = \langle s, t \rangle = D_4$

root systems for W



BACK TO COMPACT LIE GROUPS.

The character lattice

$\mathcal{X}(T) := \{ \varphi : T \to S^1 \text{ group hom.} \}$

$\mathcal{X}(T)$ is a free abelian group.

$T \cong (S^1)^n$, $\mathcal{X}(T) \cong \mathbb{Z}^n$

$R(K,T) \subset \mathcal{X}(T)$

$T \subset \mathrm{Lie}_{\mathbb{C}} K$.

Recall: Irred. Rep. of S^1 are $\rho_k : S^1 \to \mathbb{C}^*$, $k \in \mathbb{Z}$

$z \to z^k$

· Irred. rep. of $(S^1)^n$ are $\rho_{k_1,\dots,k_n} : (S^1)^n \to \mathbb{C}^*$

$z_1,\dots,z_n \mapsto z_1^{k_1} \cdots z_n^{k_n}$

· Irred. rep. of $\underset{\substack{\wr\wr \\ (S^1)^n}}{T}$, are parametrized by $\mathcal{X}(T) \cong \mathbb{Z}^n$

$T \circlearrowright \mathrm{Lie}_{\mathbb{C}} K$.

Decompose into subrepresentations of T

$$\mathrm{Lie}_{\mathbb{C}} K \cong \bigoplus_{\lambda \in X(T)} (\mathrm{Lie}_{\mathbb{C}} K)_\lambda$$

$$(\mathrm{Lie}_{\mathbb{C}} K)_\lambda = \{ X \in \mathrm{Lie}_{\mathbb{C}} K \mid tXt^{-1} = \lambda(t) X \}$$

$R(K,T) \subset X(T)$

is the subsets of $\lambda \in X(T) \setminus \{0\}$ s.t.

$(\mathrm{Lie}_{\mathbb{C}} K)_\lambda \neq 0$

<u>Thm</u> $\left\{ \begin{matrix} \text{compact connected} \\ \text{Lie groups} \end{matrix} \right\} / \cong \;\longleftrightarrow\; \left\{ \begin{matrix} \text{finite refl. groups} \\ \text{with root system} \end{matrix} \right\} / \cong$

$$K \longmapsto W(K,T) \circlearrowright X(T) \supset R(K,T)$$

<u>EXAMPLES</u> Let's see what the theorem tells us for groups of rk 1.

$K = S^1 = T$ $\quad W(K,T) = \{id\}$ $\quad X(T) \cong \mathbb{Z}$

$R(K,T) = \emptyset$

$$SU_2(\mathbb{C}) \supset T = \left\{ \underbrace{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}}_{=:\, t_\theta} \,\middle|\, \theta \in \mathbb{R} \right\}$$

$$W(G,T) = \mathbb{Z}/2\mathbb{Z} = \{1, s_\alpha\}$$

$$s_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$X(T) = \mathbb{Z}$$

$$\uparrow$$

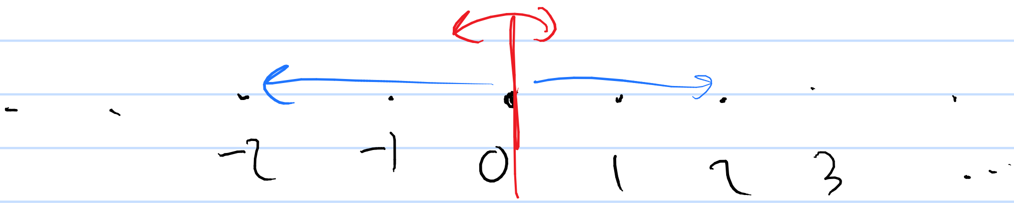
$$1 : e^{i\theta} \mapsto e^{i\theta}$$

$$\mathrm{Lie}_{\mathbb{C}}\, SU_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \ni X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$t_\theta X t_\theta^{-1} = \begin{pmatrix} 0 & e^{i2\theta} \\ 0 & 0 \end{pmatrix} = e^{i2\theta} X = (e^{i\theta})^2 X$$

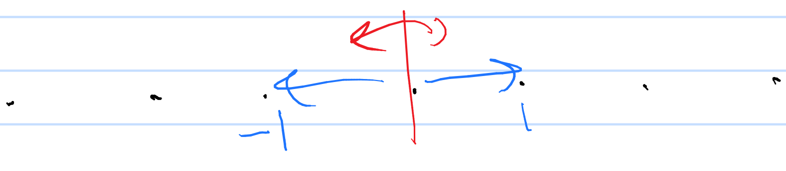
$X \in (\mathrm{Lie}_{\mathbb{C}}\, SU_2\,\mathbb{C})_2 \Rightarrow 2$ is a root

$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow -2$ is a roots



$SO_3(\mathbb{R})$ $\quad X(T) = \mathbb{Z}$

$$W(G,T) = \mathbb{Z}/2\mathbb{Z}$$



Prop G, H compact Lie groups. $\varphi : G \to H$ surjective morphism.

then the max. tori in H are the image of the max. tori in G.

Pf. $S \subset H$ max. torus.

$\Downarrow$

s top. generator, $S = \overline{\langle s \rangle}$

φ is surjective $\Rightarrow \varphi(G^0) = H^0$

$t \in \varphi^{-1}(s) \cap G^0$

$\exists$ max. torus $T \subset G$ which contains t.

$\Rightarrow \varphi(T) \supset S \Rightarrow \varphi(T) = S$ since S is maximal.

Pf. $\varphi : G \to H$ surjective morphism of cpt groups.

$\ker \varphi \subset Z(G)$.

$S \subset H$ max. torus $\Rightarrow \varphi^{-1}(S)$ max. torus in G

Pf G is connected

$Z(G) \subset T$, for every T max. torus.

T max. torus. $\varphi^{-1}(\varphi(T)) = \langle T, \ker \varphi \rangle = T$

because $\ker \varphi \subset T$.

S is a max. torus in H

$\Rightarrow S = q(T)$ for some max. torus in K

$\Rightarrow q^{-1}(S) = q^{-1}(q(T)) = T.$

<u>Cor</u> $q^{-1}(N_H(S)) = N_K(q^{-1}(S))$

$W(H,S) \quad W(K,T)$

$$W(H,S) = N_H(S)/S \cong q^{-1}(N_H(S))/q^{-1}(S) = N_K(T)/T = W(K,T)$$

T max. torus $\Rightarrow T = Z_K(T)^0$

<u>Prop</u> K compact connected Lie group $\supset T$ max. torus

$$T = Z_K(T) = Z_K(T)^0$$

<u>Pf.</u> S torus $\subset K$

$z \in Z_K(S) \qquad B = \overline{\langle z, S\rangle}$ abelian compact group.

B/B^0 is top. generated by $\bar{z}$

B/B^0 is top. cyclic $\Rightarrow B$ is top. cyclic.

but B compact $\Rightarrow B/B^0$ is finite

$\Rightarrow$ B/B^0 is cyclic.

So we can take $k>0$ s.t. $\bar{z}^k = \mathrm{id} \in B/B^0$

$\Rightarrow z^k \in B^0$ ← compact abelian connected $=$ torus

$\Rightarrow a \in B^0$ s.t. $a^k = z^k$

$z' = a^{-1} z$, $c \in B^0$ s.t. c^k is a top. generator of B^0.

$\rightsquigarrow$ cz' top. generates B.

$\overline{\langle cz' \rangle} \ni c^k \Rightarrow z = c^{-1} a z', z \in \overline{\langle cz' \rangle}$

$\Rightarrow z, B^0 \in \overline{\langle cz' \rangle} \Rightarrow B = \overline{\langle cz' \rangle}$

$B = \overline{\langle z, S \rangle} = \overline{\langle b \rangle} \subset T$ for some max. torus T.

$\forall z \in Z_h(S)$ $\exists$ max. torus $T \supset S$.

$Z_h(S) \subseteq \bigcup_{\substack{T \text{ max. torus} \\ T \supset S}} T$, $\supset$ because T is abelian

$\Rightarrow Z_h(S) = \bigcup_{T \supset S} T \Rightarrow Z_h(S)$ connected $\square$

Prop K compact connected Lie group.

$T \subset K$ max. torus.

$R = R(K,T) \subset X(T)$, $W := W(K,T)$

$\forall \alpha \in R$

1) $(\mathrm{Lie}_{\mathbb{C}} K)_\alpha$ is of dim 1, and if $\alpha, \beta \in R$ s.t $\alpha = n\beta$ with $n \in \mathbb{N}$ then $n = 1$

2) $\forall$ root $\alpha \in R(K,T)$ $\exists! \, s_\alpha \in W$ s.t. $s(\alpha) = -\alpha$

3). $\exists! \, \alpha^\vee : X(T) \to \mathbb{Z}$ s.t. $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$