irreducible = no subrepresentations

completely reducible = direct sum of irreducible repre.

$S^1 \circlearrowright \mathbb{R}^2$ $\quad$ $\{0\}$ and $\mathbb{R}^2$ are the only $S^1$-stable subspaces

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

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1.1 $(\mathbb{R}^*, \cdot)$ it is clear a group.

We want to show that mult and inv are continuous maps

$inv(x) = x^{-1} \rightsquigarrow$ this is a continuous map.

$mult(x, y) = xy.$

Want to show that preimage of a open is open

$U \subset \mathbb{R}^\times$ open

$\forall\ (x,y) \in mult^{-1}(U)$. We need to find an open nbhd of $(x,y)$ in $mult^{-1}(U)$.

Can assume $x, y > 0$

$$\mathrm{mult}\Big(B(x,\delta) \times B(y,\delta')\Big) \subset B(xy, \delta' x + \delta y + \delta\delta')$$

$U$ is open $\Rightarrow \exists\, \varepsilon > 0 \quad B(xy, \varepsilon) \subset U$

Take $\varepsilon < 1$

Can find $\delta, \delta'$ small enough so that

$$B(xy, \delta' x + \delta y + \delta\delta') \subset B(xy, \varepsilon)$$

$$\delta' = \min\left\{\frac{1}{4x}\varepsilon, \frac{1}{2}\varepsilon\right\}$$

$$\delta = \min\left\{\frac{1}{4y}\varepsilon, \frac{1}{2}\varepsilon\right\}$$

$$\delta' x + \delta y + \delta\delta' \leq \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon^2 < \varepsilon.$$

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**1.2** $G$ top. group. $H \subset G$ subgrp.

1) Want to show

$H$ open $\Leftrightarrow \exists\, U$ open nbhd in $G$ of $\mathrm{Id} \in H$ contained in $H$.

"$\Rightarrow$" $\mathrm{id} \in H \Rightarrow H$ is open nbhd of $\mathrm{Id}$

"$\Leftarrow$" By mult by $h \in H$ also $hU = \{hx \mid x \in U\}$ is open and contained in $H \Rightarrow hU$ open nbhd of $h$

$$H = \bigcup_{h \in H} hU \leftarrow \text{open} \quad \Rightarrow H \text{ open}$$

2) $H$ open $\Rightarrow$ $H$ closed

$$G = \bigcup_{g \in G} gH \quad , \quad gH \cap H = \emptyset \text{ iff } g \notin H$$

open $\forall g$

$$G \setminus H = \bigcup_{g \in G\setminus H} gH \quad \Rightarrow G\setminus H \text{ open}$$

$$\Rightarrow H \text{ closed.}$$

---

3) $G$ connected $\Rightarrow$ $G = H$

$$G = \underbrace{(G\setminus H)}_{\text{both open}} \amalg \underbrace{H}_{\ni id} \Rightarrow G\setminus H = \emptyset$$

$$\Rightarrow G = H$$

---

$\boxed{1.3}$ $(\mathbb{Z},+)$ with metric

$$d(m,n) = \frac{1}{5^k} \quad \text{when } k = \max\{5^k \mid m-n\}.$$

$$B(m,\varepsilon) = \left\{ n \in \mathbb{Z} \,\middle|\, \begin{array}{l} \exists k \text{ with } 5^k \mid m-n \\ \text{and } \dfrac{1}{5^k} < \varepsilon \end{array} \right\}$$

$$\Updownarrow$$

$$5^k > \frac{1}{\varepsilon} \iff k \geq \left\lceil \log_5\left(\frac{1}{\varepsilon}\right) \right\rceil$$

So all the balls are of the form

$$B'(m,k) = \{ n \in \mathbb{Z} \mid 5^k \mid m-n \} = \{ m + a5^k \mid a \in \mathbb{Z} \}$$

$B'(m,h)$ are a basis of the topology.

inv is continuous

$\mathrm{inv}(n) = -n$

$\mathrm{inv}^{-1}(B'(n,k)) = B'(-n, h)$

$\mathrm{mult}(m, n) = m+n$

$U$ open in $(\mathbb{Z}, +)$, Want to show that

$\mathrm{mult}^{-1}(U)$ is open.

$\overset{\Downarrow}{(m,n)}$. Since $U$ open $\exists h$ s.t.

$$B'(m+n, k) \subset U$$

$$\mathrm{mult}(\underbrace{B'(m,h) \times B'(n,h)}_{\{m+a5^k\} \times \{n+b5^k\}}) = \underbrace{\{m+n+a5^h+b5^h \mid a,b \in \mathbb{Z}\}}_{B(m+n, 5^k)}$$

$\Rightarrow$ $\mathrm{mult}^{-1}(U)$ is open because contains an open nbhd of every point.

## 1.4 Rep. of $S^1 \times S^1$.

$\forall m, n \in \mathbb{Z}$ $\quad \rho_{m,n} : S^1 \times S^1 \to GL_1(\mathbb{C})$

$$\rho_{m,n}(z, z') = z^m (z')^n$$

Need to show that it is a grp. hom.

$\forall z, z', \tilde{z}, \tilde{z}' \in S^1$

$$\rho_{m,n}(z,z')\rho_{m,n}(\tilde{z},\tilde{z}') = \rho_{m,n}(z\tilde{z}, z'\tilde{z}')$$

$$z^m (z')^n \cdot (\tilde{z})^m (\tilde{z}')^n = (z\tilde{z})^m (z'\tilde{z}')^n$$

2) $\rho_{m,n} \cong \rho_{m',n'} \Rightarrow \exists f : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$

$f(v) = \lambda v \quad \forall \lambda \in \mathbb{C}^*$

s.t. $\forall z, z' \in S^1$ $\forall v \in \mathbb{C}$ we have

$$\rho_{m',n'}(z,z') f(v) = f(\rho_{m,n}(z,z')v)$$

$$\Leftrightarrow \cancel{\lambda} z^{m'} z'^{n'} \cancel{v} = \cancel{\lambda} z^m z'^n \cancel{v}$$

Take $v \neq 0$ $\Rightarrow \forall z, z' \in S^1$ $\quad z^{m-m'} = (z')^{n'-n}$

Take $z = 1 \Rightarrow n' = n$

$z' = 1 \Rightarrow m' = m$

$\rho_{m',n'} \cong \rho_{m,n} \Rightarrow m = m'$, $n' = n$

3) I should have said continuous representation

$$\begin{tikzcd} S^1 \arrow[r] & S^1 \times S^1 \arrow[r, "\rho"] & GL_1(\mathbb{C}) \\ z \arrow[r, mapsto] \arrow[rr, bend right] & (z,1) \arrow[r] & \rho(z,1) \end{tikzcd}$$

If $\rho$ rep of $S^1\times S^1 \Rightarrow$ ↑ is a cont. rep of $S^1$

$\Rightarrow \exists\, m \in \mathbb{Z}$ s.t. $\rho(z,1) = z^m$

$$S^1 \to S^1\times S^1 \to GL_1(\mathbb{C})$$

$$z \mapsto (1,z)$$

$\Rightarrow \exists\, n \in \mathbb{Z}$ s.t. $\rho(1,z) = z^n$.

$$\rho(z,z') = \rho(z,1)\,\rho(1,z') = z^m (z')^n$$

$\rho \cong \rho_{m,n}$ for some $m,n \in \mathbb{Z}$

With the same proof we can also classify representations of $(S^1)^k$ over $\mathbb{C}$, $\forall h \in \mathbb{C}$

this is called torus.

$$\rho_{\vec{m}}(\vec{z}) = z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k}$$

for some $\vec{m} \in (\mathbb{Z})^k$

This will turn out to be important! We can understand representations of Lie groups by looking what happens on a maximal torus.

For example, when studying rep of $SL_n(\mathbb{C})$

we can restrict to what happens

on the maximal torus $T \cong (S^1)^{n-1}$ of

diagonal matrices w/ entries in $S^1$.