From now on we will have an exercise sheet every 2 weeks

2.1 $\rho : \mathbb{R} \longrightarrow GL_2(\mathbb{R})$ not completely reducible

$x \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$V = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ subrep. $\Rightarrow$ ρ not irreducible.

If ρ is reducible, then $\mathbb{R}^2 = V_1 \oplus V_2$ with

$\dim V_1 = \dim V_2 = 1$ and both su[b]rep

However $V_i = \langle v_i \rangle$ means that v_i is eigenvector

of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $\forall x \in \mathbb{R} \Rightarrow v_i = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

But then $V_1 = V_2$, which is impossible.

2.2 $\rho : S^1 \longrightarrow SO_2(\mathbb{R})$

$$e^{i\theta} \longmapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

It is a group hom.

$$\rho(e^{i\theta_1})\rho(e^{i\theta_2}) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \dots \\ \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 & \dots \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} = \rho(e^{i(\theta_1+\theta_2)})$$

ρ injective: $\rho(e^{i\theta}) = 0 \Rightarrow \cos\theta = 1, \sin\theta = 0$

$$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta = \underline{1}$$

<u>ρ surjective</u>: We want to show that every element in $SO_2(\mathbb{R})$ is of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Let $A \in SO_2(\mathbb{R})$. We have $AA^T = Id$
$\det(A) = 1$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$AA^T = Id \Rightarrow \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $a^2+b^2 = 1$, $c^2+d^2=1$ we can find θ, φ with

$$a = \cos\theta, \ b = \sin\theta$$
$$c = \sin\varphi, \ d = \cos\varphi$$

and
$$\left.\begin{aligned} ac+bd = 0 &\Rightarrow \sin(\theta+\varphi) = 0 \\ ad-bc = 1 &\Rightarrow \cos(\theta+\varphi) = 0 \end{aligned}\right\} \Rightarrow \varphi = 2k\pi - \theta$$

$$\Rightarrow A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \square$$

2.3 $O_{p,q}(\mathbb{R})$ is a group.

$A, B \in O_{p,q}(\mathbb{R})$

$$(AB)^T I_{p,q} AB = B^T A^T I_{p,q} A B = I_{p,q}$$

$$\Rightarrow AB \in O_{p,q}(\mathbb{R})$$

$$(A^{-1})^T I_{p,q} A^{-1} = (A I_{p,q} A^T)^{-1} = I_{p,q}^{-1} = I_{p,q}$$

$$\Rightarrow A^{-1} \in O_{p,q}(\mathbb{R}).$$

If $A = (a_{ij})$

$$f_{ij} := \sum_{k=1}^{p} a_{ik} a_{kj} - \sum_{k=p+1} a_{ik} a_{kj}$$

f_{ij} are continuous function of $GL_n(\mathbb{R})$ and

$O_{p,q}(\mathbb{R}) = \{ A \mid f_{ij}(A) = 0 \ \forall i,j \}$ is closed.

$O_{p,q}(\mathbb{R})$ it is not compact.

Since $O_{p,q}(\mathbb{R}) \subset \mathbb{R}^{n^2}$ it is enough to show that it is not bounded.

Let's do it first for $p=q=1$. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A\begin{pmatrix}1&0\\0&-1\end{pmatrix}A^T=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \Rightarrow \begin{pmatrix} a^2-b^2 & ac-bd \\ ac-bd & c^2-d^2 \end{pmatrix}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$\forall t\in\mathbb{R}$ a solution is given by $b=c=t$, $a=d=\sqrt{t^2+1}$

The set $\left\{ \underbrace{\begin{pmatrix}\sqrt{t^2+1} & t \\ t & \sqrt{t^2+1}\end{pmatrix}}_{\overset{..}{=} A_t} \right\}$ is not bounded!

If $p,q\geq 1$ then

$$\left\{ \begin{pmatrix} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{p-1} & & \\ & \begin{matrix}\sqrt{t^2+1} & t \\ t & \sqrt{t^2+1}\end{matrix} & \\ & & \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{q-1} \end{pmatrix} \right\} \subset O_{p,q}(\mathbb{R})$$

and it is not bounded.

ACHTUNG! $O_n(\mathbb{R})=O_{n,0}(\mathbb{R})$ is compact!

It sends orthogonal bases to orthogonal bases

Its column vectors belong to $S^{n-1}=\{v\in\mathbb{R}^n \mid \sum v_i^2=1\}$

$\Rightarrow O_n(\mathbb{R})$ is a closed subspace of $(S^{n-1})^n$

$\Rightarrow$ It is compact.

$\frac{2.4}{1)}$ $e^{ABA^{-1}} = \sum_{m\geq 0} \frac{(ABA^{-1})^m}{m!} = \sum_{m\geq 0} A \frac{B^m}{m!} A^{-1}$

$$= A\left(\sum_{m\geq 0} \frac{B^m}{m!}\right) A^{-1} \Rightarrow A e^B A^{-1}.$$

2) $e^{Tr(B)} = \det(e^B)$

If $B = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$ then $Tr(B) = \sum \lambda_i$

and $e^B = \begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_m} \end{pmatrix}$, $\det(e^B) = e^{\sum \lambda_i}$

In general, $\exists\, A \in Gl_m(\mathbb{C})$ such that ABA^{-1} is triangular.

$\det(e^B) = \det(e^{A^{-1}TA}) = \det(A^{-1} e^T A) = \det(e^T)$

$e^{Tr(B)} = e^{Tr(A^{-1}TA)} = e^{Tr(T)}$ // !

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3) $sl_m(\mathbb{R}) = T_{Id}\, Sl_m(\mathbb{R})$.

"$\subseteq$" $A \in sl_m(\mathbb{R})$, i.e. $tr(A) = 0. \Rightarrow \overset{\forall t \in \mathbb{R}}{\det(e^{tA})} = e^{Tr(tA)} = e^0 = 1$

$\Rightarrow A \in T_{Id}\, Sl_m(\mathbb{R})$.

If $sl_m(\mathbb{R}) \subsetneq T_{Id}\, Sl_m(\mathbb{R})$, then $T_{Id}\, Sl_m(\mathbb{R}) = M_{m\times m}(\mathbb{R})$

so $Id \in T_{Id}\, Sl_m(\mathbb{R})$. But $\det(e^{Id}) = e^m \neq 1$ ↯