Ex. 3.1 $T_I Sp_{2n}(\mathbb{R}) = \{M \in M_{2n}(\mathbb{R}) \mid M^T = SMS\}$

"$\supseteq$" Let $M \in M_{2n}(\mathbb{R})$ with $M^T = SMS$.

We want to show that for every $t \in \mathbb{R}$ $e^{Mt} \in Sp_{2n}(\mathbb{R})$.

$$M^T = SMS. \Rightarrow e^{M^T t} = e^{S(Mt)S} = e^{S(-Mt)S^{-1}} = Se^{-Mt}S^{-1}$$

$$\Leftrightarrow e^{M^t t} S e^{Mt} = S \Rightarrow e^{Mt} \in Sp_{2n}(\mathbb{R}) \ \forall t$$

"$\subseteq$" $\Rightarrow M \in T_I Sp_{2n}(\mathbb{R})$

Let $M \in T_I Sp_{2n}(\mathbb{R})$. Then $e^{Mt} \in Sp_{2n}(\mathbb{R}) \ \forall t$

$$\Rightarrow e^{M^T t} S e^{Mt} = S$$

$$\Rightarrow \frac{d}{dt}\left(e^{M^T t} S e^{Mt}\right)\Big|_{t=0} = \frac{d}{dt}(S)\Big|_{t=0} = 0$$

$$\Rightarrow M^T S + SM = 0 \Rightarrow M^T = -SMS^{-1} = SMS.$$

3.2 $X \in \text{lie}\, G,\ Y \in \text{lie}\, N \Rightarrow [X, Y] \in \text{lie}\, N.$

$e^{Xt} \in G \ \forall t \in \mathbb{R}, \ e^{Ys} \in N \ \forall s \in \mathbb{R}.$

N normal $\Rightarrow e^{Xt} e^{Ys} e^{-Xt} \in N \ \forall s, t \in N.$

$$\Rightarrow \frac{d}{dt}\left(e^{Xt} e^{Ys} e^{-Xt}\right)\Big|_{t=0} \in \text{lie}\, N \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left(e^{Xt}e^{Ys}e^{-Xt}\right)\Big|_{t=0} \in \text{lie}\,N \Rightarrow$$

$$\Rightarrow X e^{Ys} - e^{Ys} X \in \text{lie}\,N \Rightarrow$$

$$\Rightarrow \frac{d}{ds}\left(Xe^{Ys} - e^{Ys}X\right)\Big|_{s=0} \in \text{lie}\,N \Rightarrow$$

$$\Rightarrow XY - YX \in \text{lie}\,N \Rightarrow [X,Y] \in \text{lie}\,N,$$

3.3 G abelian $\Rightarrow$ lie G abelian.

$$X, Y \in \text{lie}\,G \Rightarrow e^{Xt}, e^{Ys} \in G \quad \forall\, t,s \in \mathbb{R}$$

$$e^{Xt}e^{Ys} - e^{Ys}e^{Xt} = 0 \quad \forall\, s,t \in \mathbb{R}$$

$$\Rightarrow \frac{d}{ds}\left(\frac{d}{dt}\left(e^{Xt}e^{Ys} - e^{Ys}e^{Xt}\right)\Big|_{t=0}\right)\Big|_{s=0} = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{ds}\left(Xe^{Ys} - e^{Ys}X\right)\Big|_{s=0} = XY - YX = 0 \Rightarrow [X,Y] = 0$$

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exp: lie $G \to G$ is a group homomorphism.

$$X, Y \in \text{lie}\,G. \Rightarrow e^X e^Y = e^{X+Y}$$

In fact, if $XY = YX$ we have

$$e^{X+Y} = \sum_{n\geq 0} \frac{(X+Y)^n}{n!} = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{X^k Y^{n-k}}{n!}\right) =$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{X^k}{k!}\cdot\frac{Y^{n-k}}{(n-k)!}\right) = \left(\sum_{k\geq 0}\frac{X^k}{k!}\right)\left(\sum_{h\geq 0}\frac{Y^h}{h!}\right) = e^X e^Y$$

So exp is a group hom. If G connected G is generated by $\exp(\mathrm{Lie}\, G)$, but since $\exp(\mathrm{Lie}\, G)$ is a group, the group generated by $\exp(\mathrm{Lie}\, G)$ is again $\exp(\mathrm{Lie}\, G)$. Hence $G = \exp(\mathrm{Lie}\, G)$

G connected $\Rightarrow$ $\exp: \mathrm{Lie}\, G \to G$ surjective group hom.

$\Rightarrow G \cong \mathrm{Lie}\, G / \ker(\exp)$.

Now $\mathrm{Lie}\, G \cong \mathbb{R}^n$ as a group.

$\Gamma := \ker(\exp)$ is a discrete subgroup of $\mathbb{R}^n$.

In *fact*, exp is a diffeomorphism on a neighborhood U of $0 \in \mathbb{R}^n$, so $\Gamma \cap U = \{0\}$ meaning that Γ is discrete.