4.1 G Lie group. $g \in G$.

$$\mathrm{Lie}\, Z(G) = \{ X \in \mathrm{Lie}\, G \mid \mathrm{Ad}(g) X = X \}.$$

"$\supseteq$". $X \in \mathrm{Lie}\, G$ s.t. $\mathrm{Ad}(g) X = X \iff g X g^{-1} = X$

$$e^{tX} \in G \Rightarrow g e^{tX} g^{-1} = e^{t g X g^{-1}} = e^{tX}$$

$$\Rightarrow e^{tX} \in Z(g) \ \forall t \in \mathbb{R}.$$

"$\subseteq$" For $v \in \mathrm{Lie}\, Z(g)$ we can find

$\gamma(t)$ curve in $Z(g)$ s.t. $\dot\gamma(0) = v$ and $\gamma(0) = \mathrm{Id}$

$$g \gamma(t) g^{-1} = \gamma(t) \Rightarrow \frac{d}{dt}\left(g \gamma(t) g^{-1}\right) = \dot\gamma(0)$$

$$\Rightarrow \mathrm{Ad}(g)\, \dot\gamma(0) = \dot\gamma(0)$$

4.2 $(V_1, \rho_1), (V_2, \rho_2)$ rep. of Lie group G

- $\rho_1 \otimes \rho_2$ is rep. on $V_1 \otimes V_2$.

$\forall g, h \in G \quad \forall v_1 \in V_1, \forall v_2 \in V_2$

$$(\rho_1 \otimes \rho_2)(g) \cdot (\rho_1 \otimes \rho_2)(h)\, (v_1 \otimes v_2) =$$

$$= \rho_1(g) \rho_1(h) v_1 \otimes \rho_2(g) \rho_2(h) v_2$$

$$= \rho_1(gh) v_1 \otimes \rho_2(gh) v_2$$

$$= (\rho_1 \otimes \rho_2)(gh)(v_1 \otimes v_2).$$

(the fact that $\rho_1 \otimes \rho_2$ is diff. follows from the fact that the embedding

$$\mathrm{End}(V_1) \times \mathrm{End}(V_2) \longrightarrow \mathrm{End}(V_1 \otimes V_2)$$
$$(f, g) \longmapsto f \otimes g$$
is bilinear, hence diff.)

$d(\rho_1 \otimes \rho_2) : \text{Lie}\, G \longrightarrow gl(V_1 \otimes V_2).$

$X \in \text{Lie}\, G$. $\gamma(t)$ curve in G with $\dot\gamma(0) = X$ and $\gamma(0) = \text{Id}$.

$$d(\rho_1 \otimes \rho_2)X(v_1 \otimes v_2) = \frac{d}{dt}(\rho_1 \otimes \rho_2)(\gamma(t))(v_1 \otimes v_2)\Big|_{t=0} =$$

$$= \frac{d}{dt}\rho_1(\gamma(t))v_1 \otimes \rho_2(\gamma(t))v_2\Big|_{t=0} =$$

$$= d\rho(\dot\gamma(0))v_1 \otimes \rho_2(\text{Id})v_2 + \rho_1(\text{Id})v_1 \otimes d\rho_2(\dot\gamma(0))v_2 =$$

$$= d\rho_1(X)v_1 \otimes v_2 + v_1 \otimes d\rho_2(v_2).$$

4.3 V rep. of $sl_2(\mathbb{C})$, $V \neq 0$.

We can find $0 \neq W \subset V$ irreducible rep. of $sl_2(\mathbb{C})$.

From the classification thm we know $W \cong V(m)$ for some $m \geq 0$.

$V(m)$ has a basis $\{v, fv, f^2v, \dots, f^m v\}$

$$\begin{array}{cccc} v, & fv, & \dots, & f^m v \\ \cap & \cap & & \cap \\ V_m & V_{m-2} & & V_{-m} \end{array}$$

so if m even $\quad f^{\frac{m}{2}}v \in V(m)_0 \subset W_0 \subset V_0 \neq 0$

if m odd $\quad f^{\frac{m-1}{2}}v \in V(m)_1 \subset W_1 \subset V_1 \neq 0.$

4.4 $so(3)$ has a basis $E_1 = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix}$, $E_3 = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 0 \end{pmatrix}$

$\mathbb{R}^3$ has a basis (e_1, e_2, e_3)

since $[E_1, E_2] = E_3$, $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$

the linear map $\varphi : \mathbb{R}^3 \to \mathfrak{so}_{\mathbb{R}}(3)$ induces an isomorphism

on Lie algebras.

2) $\mathfrak{so}_3(\mathbb{R}) \cong (\mathbb{R}^3, \times)$. If $V \subset \mathbb{R}^3$ of dim 2

and v_1, v_2 basis of V, then $v_1 \times v_2 \notin V$ (otherwise

$\left(v_1 \middle| v_2 \middle| v_1 \times v_2 \right) = 0$

So $\mathfrak{so}_3(\mathbb{R})$ has no subalgebras of dim 2.

In particular, it has no ideals of dim 2. Also ideals of dim 1 cannot exist.

Otherwise, if $I = (v)$ ideal of dim 1, then we can always find w s.t.

$\dim \langle w, v \rangle = 2$ and then $w \times v \notin I$.

3) If $\rho : SU_2(\mathbb{C}) \to GL_2(\mathbb{R})$ irr. rep.

then $d\rho : \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{R})$

$\mathfrak{su}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{R})$

but $[\mathfrak{so}_3(\mathbb{R}), \mathfrak{so}_3(\mathbb{R})] = \mathfrak{so}_3(\mathbb{R})$.

so $\operatorname{Im}(d\rho) \subset [\mathfrak{gl}_2(\mathbb{R}), \mathfrak{gl}_2(\mathbb{R})] = \mathfrak{sl}_2(\mathbb{R})$.

Since $\mathfrak{so}_3(\mathbb{R})$ is simple, then $d\rho = 0$ or $d\rho$ is isomorphism.

But $\mathfrak{so}_3(\mathbb{R}) \not\cong \mathfrak{sl}_2(\mathbb{R})$ because $\mathfrak{sl}_2(\mathbb{R})$ has a subalgebra of dim 2

(namely $\mathfrak{b} = \langle h, e \rangle$)

hence $d\rho = 0$.

But ρ irreducible $\Leftrightarrow$ $d\rho$ irreducible (because $SU_2(\mathbb{C})$ connected).

but $d\rho$ can't be irreducible if $d\rho = 0$