5.1 Haar measure on $\mathbb{C}^*$.

$\mathbb{C}^* \cong \mathbb{R}^* \times S^1$, so the product measure will work as a Haar measure on $\mathbb{C}^*$

The Haar measure on $\mathbb{R}^*$ is $\frac{dx}{|x|}$, the Haar measure on S^1 is $\frac{1}{2\pi} d\theta$

so if $g = (r, \theta)$ $\quad d\theta = \frac{dr}{r} d\theta$

For any $f : \mathbb{C}^* \to \mathbb{C}$ we have

$$2\pi \int_{\mathbb{C}^*} f(g)\,dg = \int_0^{2\pi}\left(\int_{-\infty}^{\infty} f(r,\theta)\,\frac{dr}{r}\right) d\theta$$

Let's show that dg is left invariant

$h = (r', \theta')$

$$2\pi \int_{\mathbb{C}^*} f(hg)\,dg = \int_0^{2\pi}\left(\int_{-\infty}^{\infty} f(r'r, \theta'\theta)\,\frac{dr}{|r|}\right) d\theta =$$

$$= \int_0^{2\pi}\left(\int_{-\infty}^{\infty} f(r, \theta'\theta)\,\frac{dr}{|r|}\right) d\theta = \int_0^{2\pi}\int_{-\infty}^{\infty} f(r,\theta)\,\frac{dr\,d\theta}{|r|}$$

By left invariance of $\frac{dr}{|r|}$ on $\mathbb{R}^*$

by left invariance of $d\theta$ on S^1.

$$= 2\pi \int_{\mathbb{C}^*} f(g)\,dg.$$

5.2 V irr. real repr. of G

$V^* = \mathrm{Hom}(V, \mathbb{R})$ is a G-representation via $g \cdot \lambda(v) = \lambda(g^{-1}v)$ $\forall \lambda \in V^*, g \in G, v \in V$.

We have $\Phi : \mathrm{Bil}_k(V) \xrightarrow{\sim} \mathrm{Hom}_k(V, V^*)$

$$b \xrightarrow{\Phi} (v \mapsto b(v,-))$$

$$b(v,w) = f(v)(w) \xleftarrow{\Phi^{-1}} f$$

Now let $\mathrm{Bil}_G(V) \subset \mathrm{Bil}_k(V)$ be the subspace of G-invariant bilinear form.

If $b \in \mathrm{Bil}_G(V)$, then $\Phi(b)$ is a morphism of G-reps.

In fact, $\forall v, w \in V$ $\quad \Phi(b)(g \cdot v)(w) = b(g \cdot v, w)$

$$g \cdot \Phi(b)(v)(w) = \Phi(b)(v)(g^{-1}w) = b(v, g^{-1}v).$$

and since $b(gv, w) = b(g \cdot v, gg^{-1}w) = b(v, g^{-1}w)$

we get $\Phi(b)(gv) = g\,\Phi(b)(v)$.

Now, if V is irreducible then also V^* is irreducible because if $W \subset V^*$ is subrep, then $W^{\perp} = \{v \in V \mid \lambda(v) = 0 \, \forall \lambda \in W\}$ would be a subrep. of V.

Since V, V^* are irreducible, we have

$\dim \mathrm{Hom}_G(V, V^*) \leq 1$, hence $\dim \mathrm{Bil}_G(V) \leq 1$.

<u>5.3</u> We need to show that dg is left invariant and that $\int_G 1\,dg = 1$.

- $$\int_G 1\,dg = \frac{1}{|G|}\sum_{g\in G} 1 = 1 \quad \checkmark.$$

- Fix $h \in G$. $f: G \to \mathbb{C}^*$

$$\int_G f(hg)\,dg = \frac{1}{|G|}\sum_{g\in G} f(hg) = \frac{1}{|G|}\sum f(g) = \int f\,dg$$

Since $l_h : G \to G$ is a bijection.

Since G finite, $L^2(G) = \{ f: G \to \mathbb{C} \}$, so

$$\dim L^2(G) = |G|.$$

Peter-Weyl thm $\Rightarrow$ $$L^2(G) \cong \bigoplus_{\substack{V_\xi \text{ irred.} \\ \text{rep. of } G}} V_\xi^{\oplus \dim V_\xi}$$

$$\Rightarrow |G| = \dim L^2(G) = \sum_{\substack{V_\xi \text{ irred.} \\ \text{rep. of } G}} \dim\left(V_\xi^{\oplus \dim V_\xi}\right) =$$

$$= \sum \dim V_\xi \dim V_\xi = \sum \dim^2 V_\xi.$$

5.4 1) Let G be a Lie group and $g \in G^\circ$, $g \neq e$ (G° is the connected cpt. of $e \in G$)

Assume, by contradiction, that $\rho_1(g) = \mathrm{Id}$ $\forall \rho_1$ fin. dim. rep. of G

$$PW \Rightarrow L^2(G) \cong \widehat{\bigoplus} V_\xi^{\oplus \dim V_\xi}.$$

In particular, $L^2(G)$ is the closure of the smallest subspace containing all its fin. dim. subrepresentations.

So for every $f \in L^2(G)$ there exists a sequence of fin. dim. rep. V_n of G, $v_n \in V_n$, $\varphi_n : \mathrm{Hom}_G(V_n, L^2(G))$ such that $\varphi_n(v_n) \xrightarrow{n \to \infty} f$.

We are assuming that g acts trivially on any fin. dim. rep.

So $$g \cdot f = \lim_{n \to \infty} g \cdot \varphi_n(v_n) = \lim_{n \to \infty} \varphi_n(g \cdot v_n) = \lim \varphi_n(v_n) = f$$

But certainly there exists $f \in L^2(G)$ such that $g \cdot f \neq f$.

(for example we can construct a continuous function which is $\neq 0$ only in a small nbhd U of $e \in G$ with $g \notin U$

then $g \cdot f(g) = f(g^{-1}g) \neq 0 = f(g)$.)

this shows the first point.

2) $\mathrm{Ker}(\rho_1)$ is a closed subgroup of G and $g \notin \mathrm{Ker}\,\rho_1$.

If $\dim \mathrm{Ker}(\rho_1) = \dim G$ they would have the same Lie algebra hence the same connected component.

But $g \in G^\circ$ and $g \notin (\mathrm{Ker}\,\rho_1)^\circ$. $\Rightarrow$ $\dim \mathrm{Ker}\,\rho_1 < \dim G$.

3) Let $g_2 \in (\ker \rho_1)^\circ$, we can find (as in 1) a f.d. rep. ρ_2 such that

$\rho_2(g_2) \neq id \Rightarrow$ $\ker(\rho_1 \oplus \rho_2) \subset \ker(\rho_1)$ and

$\dim \ker(\rho_1 \oplus \rho_2) < \dim \ker \rho_2$.

Continuing like this we find $\rho_1, \dots, \rho_N$ s.t.

$$\dim \ker(\rho_1 \oplus \dots \oplus \rho_n) = 0, \text{ so } \ker(\rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_N) = \{e, h_1, \dots, h_M\} \subset G.$$

4)

For every i we can find using 1) ρ_{N+i} fin. dim rep. such that $\rho_{N+i}(h_i) \neq id$.

$$\Rightarrow \ker(\rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_{N+M}) = \{e\}$$

Let $\rho := \rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_{N+M}$. This is a fin. dim. faithful rep. of G. So $\rho : G \longrightarrow GL(V) \cong GL_n(\mathbb{R})$ is injective and

and G can be realized as a closed subgroup of $GL_n(\mathbb{R})$ for some n big enough.