6.1 $\mathrm{Lie}(\mathrm{Aut}\,\mathfrak{g}) = \mathrm{Der}\,\mathfrak{g}$.

"$\subseteq$"

$X \in \mathrm{Lie}(\mathrm{Aut}\,\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$.

Let $\gamma : I \to \mathrm{Aut}\,\mathfrak{g}$ curve. with $\dot\gamma(0) = d$

Want to show $d \in \mathrm{Der}\,\mathfrak{g}$.

Let $X, Y \in \mathfrak{g}$

$$\frac{d}{dt}\left([\gamma(t)X, \gamma(t)Y]\right)\Big|_{t=0} = [\dot\gamma(0)X, Y] + [X, \dot\gamma(0)Y]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{d}{dt}\gamma(t)[X,Y] \qquad\qquad [dX, Y] + [X, dY]$$

$$\| \qquad\qquad = $$

$$d([X,Y]) \qquad \Rightarrow d \in \mathrm{Der}\,\mathfrak{g}.$$

In the other direction.

Let $d \in \mathrm{Der}\,\mathfrak{g}$. Want to show that $e^{dt} \in \mathrm{Aut}\,\mathfrak{g}\ \forall t \in \mathbb{R}$

$d \in \mathrm{Der}\,\mathfrak{g} \Rightarrow e^{d} \in \mathrm{Aut}\,\mathfrak{g}$.

by induction.

$X, Y \in \mathfrak{g}$. $\quad d^{m+1}([X,Y]) = d\left(d^m([X,Y])\right) = d\sum \binom{m}{i}[d^i X, d^{m-i} Y]$

$$= \sum_{i=0}^{m}\binom{m}{i}\left([d^{i+1}X, d^{m-i}Y] + [d^i X, d^{m-i+1}Y]\right)$$

$$= \sum_{i=0}^{m+1}\left(\binom{m}{i-1} + \binom{m}{i}\right)[d^i X, d^{m+1-i}Y]$$

$$= \sum_{i=0}^{m+1}\binom{m+1}{i}[d^i X, d^{m+1-i}Y]$$

So $e^{d}([X,Y]) = \sum_{k=0}^{\infty} \frac{d^k}{k!}([X,Y]) =$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \underbrace{\frac{1}{k!}\binom{k}{i}}_{= \frac{1}{i!(k-i)!}} [d^i X, d^{k-i} Y]$$

$$[e^d X, e^d Y] = \sum_{i,j=0}^{\infty} \left[\frac{d^i}{i!} X, \frac{d^j}{j!} Y\right] =$$

$i+j=k$
$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{1}{i!} \frac{1}{(k-i)!} [d^i X, d^{k-i} Y]$$

So $e^d([X,Y]) = [e^d X, e^d Y]$ and $e^d \in \mathrm{Aut}(\mathfrak{g})$. $\square$

6.2 $\mathfrak{g}$ Lie algebra, I ideal.

$X, Y \in I$. Want to show

$$\mathrm{tr}(\mathrm{ad}\,X|_I \,\mathrm{ad}\,Y|_I) = \mathrm{tr}(\mathrm{ad}\,X \,\mathrm{ad}\,Y).$$

Let $f = \mathrm{ad}\,X\,\mathrm{ad}\,Y$. $f: \mathfrak{g} \to \mathfrak{g}$.

Since I is an ideal we have

$f(\mathfrak{g}) \subset I$. We can find a vector space S such that $\mathfrak{g} = I \oplus S$.

Choose basis $\{\underbrace{x_1, \dots, x_i}_{\text{basis of } I}, \underbrace{y_1, \dots, y_j}_{\text{basis of } S}\}$ of $\mathfrak{g}$

in this basis we have

$$f = \left(\begin{array}{c|c} f|_I & * \\ \hline 0 & 0 \end{array}\right), \text{ so } tr(f) = tr(f|_I)$$

$\square$

6.3 $\quad T = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$

$T \cong S^1$, so is a torus. We want to show that it is maximal.

If not, $\exists$ S torus, with $T \subsetneq S \subsetneq \mathfrak{so}_3(\mathbb{R})$

so $\dim S = 2$. ($\mathfrak{so}_3(\mathbb{R})$ is not abelian, so it is not a torus)

But we know from EXERCISE 4.4 that $\mathfrak{so}_3(\mathbb{R})$ does not contain any subalgebra of dim 2.

and if $T \subsetneq S$, S torus then $\mathrm{Lie}\, S \subset \mathfrak{so}_3(\mathbb{R})$ is a subalgebra of dim 2. $\square$

6.4 $T \subset K$ torus $\Rightarrow$ lie T abelian subalgebra (see EXERCISE 3.3).

$\mathfrak{a} \subset$ lie K abelian subalgebras

$\Rightarrow$ the group generated by $\exp \mathfrak{a}$ is abelian

$\Rightarrow$ the group $H := \overline{\langle \exp \mathfrak{a} \rangle}$ is abelian.

H is compact, abelian, connected $\Rightarrow$ H torus

Let T be a max torus.
Assume that lie T is not maximal. Then $\exists\, \mathfrak{a} \supset$ lie T abelian subalgebras and $\overline{\langle \exp \mathfrak{a} \rangle} \supset T$ is a larger torus $\lightning$

Let $\mathfrak{a}$ maximal abelian subalgebra.
and let $H = \overline{\langle \exp \mathfrak{a} \rangle}$. H is a torus, so lie H is abelian. Since $\exp \mathfrak{a} \subseteq$ lie H

we conclude the $\exp \mathfrak{a} =$ lie H,

so $\mathfrak{a}$ is lie algebra of a maximal torus $\square$