7.1 We show that

$$H = \left\{ \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$$

is a maximal abelian subgroup of $SO_3(\mathbb{R})$

Clearly, H is an abelian subgroup and it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

We need to show that H is maximal.

But if $g \in Z(H)$, then g fixes the eigenspaces of the elements of H

so it is diagonal. But $g \in SO_3(\mathbb{R})$ + g diagonal $\Rightarrow$

$$g = \begin{pmatrix} a_1 & & 0 \\ 0 & a_2 & \\ & & a_3 \end{pmatrix} \text{ with } a_i^2 = 1, a_1 a_2 a_3 = 1$$

so $a_i = \pm 1$, and 0 or 2 of them are -1

$\Rightarrow g \in H$.

7.2 K cpt. Lie group. $Z(K) = \bigcap_{T \text{ max. torus.}} T$

"$\subseteq$" $g \in Z(K) \Rightarrow \exists\, T$ containing g. All the others tori are of the form hTh^{-1}

since $g \in Z(K)$ $hgh^{-1} = g \in T'$ ✓

"$\supseteq$" $g \in \bigcap_{T \text{ max. torus.}} T$. $\forall h \in K$ $\exists\, T$ s.t. $h \in T \Rightarrow h$ and g commute

$\Rightarrow g \in Z(K)$.

__7.3__

$$t_\theta := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A basis of $\mathrm{lie}_{\mathbb{C}}\, SO_3(\mathbb{R})$ is

$$\left\{ \underbrace{\begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}}_{= E_1},\ \underbrace{\begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix}}_{= E_2},\ \underbrace{\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & \end{pmatrix}}_{= E_3} \right\}$$

Also $\{E_1 + iE_2,\ E_1 - iE_2,\ E_3\}$ is a basis.

$$\mathrm{Ad}(t_\theta)\,E_3 = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= E_3}$$

$$\mathrm{Ad}(t_\theta)(E_1 + iE_2) = (i\cos\theta - \sin\theta)\,E_2 + (\cos\theta \ \ i\sin\theta)\,E_1$$
$$= e^{i\theta}(E_1 + iE_2)$$

$$\mathrm{Ad}(t_\theta)(E_1 - iE_2) = e^{-i\theta}(E_1 + iE_2)$$

So we see that $\mathrm{lie}_{\mathbb{C}}\, SO_3(\mathbb{R}) \cong so_3(\mathbb{R})_0 \oplus so_3(\mathbb{R})_\lambda \oplus so_3(\mathbb{R})_{-\lambda}$

where $\lambda : T \to S^1$, $t_\theta \mapsto e^{i\theta}$ $\quad \lambda$ is a generator of $\mathcal{X}(T) \cong \mathbb{Z}$

so $R(k, T) \cong \{\lambda, -\lambda\}$

7.4 X lattice. $X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ dual lattice.

s reflection in X. with root α and coroot $\alpha^\vee$.

this means that $s(v) = v - \langle \alpha^\vee, v \rangle \alpha \quad \forall v \in X$.

Let $s^\vee : X^\vee \to X^\vee$ be defined by $s^\vee(\lambda)(v) = \lambda(s(v))$.

We need to show that $s^\vee$ satisfies

$$s^\vee(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee.$$

For every $v \in X$ we have

$$s^\vee(\lambda)(v) = \lambda(s(v)) = \lambda(v - \langle \alpha^\vee, v \rangle \alpha) = \lambda(v) - \langle \alpha^\vee, v \rangle \lambda(\alpha)$$

$$(\lambda - \langle \alpha, \lambda \rangle \alpha^\vee)(v) = \lambda(v) - \langle \alpha, \lambda \rangle \langle \alpha^\vee, v \rangle$$

In particular, $s^\vee$ is a reflection.

In fact. $(s^\vee)^2 = id$ because

$$(s^\vee)^2(\lambda) = s^\vee(\lambda - \langle \alpha, \lambda \rangle \alpha^\vee) = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee - \langle \alpha, \lambda - \langle \alpha, \lambda \rangle \alpha^\vee \rangle \alpha^\vee =$$

$$= \lambda - \langle \alpha, \lambda \rangle \alpha^\vee - \langle \alpha, \lambda \rangle \alpha^\vee + \langle \alpha, \lambda \rangle \underbrace{\langle \alpha, \alpha^\vee \rangle}_{= 2} \alpha^\vee = \lambda$$

and $\alpha^\vee$ is a root for $s^\vee$.

In fact, for every $\lambda \in X^\vee$ $\quad s^\vee(\lambda) - \lambda = \langle \alpha, \lambda \rangle \alpha^\vee$

and $\langle \alpha, \lambda \rangle \in \mathbb{Z}$.