Lie Groups
SoSe 2023 — Ubungsblatt 6

21.06.2023

Aufgabe 6.1: Recall from the exercise 5.3 the representation $V_l := \mathbb{R}[X,Y,Z]^l$ representation of $SO(3,\mathbb{R})$. We have $V_{l+2} = P(V_l) \oplus L(2l)$, where $L(2l)$ is the irreducible real representation of $SO(3,\mathbb{R})$ of dimension $2l+1$ and P is given by multiplication by $(X^2+Y^2+Z^2)$.
Let $S^2 = \{x^2+y^2+z^2\} \subset \mathbb{R}^3$. The natural action of $SO(3,\mathbb{R})$ induces a representation of $SO(3,\mathbb{R})$ on the vector space of continuous functions $\mathcal{C}(S^2,\mathbb{R})$. On $\mathcal{C}(S^2,\mathbb{R})$ we have a $SO(3,\mathbb{R})$-invariant scalar product given by [1]

$$\langle f, g\rangle = \int_{S^2} f\bar{g}$$

- Show that the evaluation of polynomials gives an embedding of representations $\phi_l : V_l \subset \mathcal{C}(S^2,\mathbb{R})$.
- Show that if l and m have different parities, the subspaces $\phi_l(V_l)$ and $\phi_m(V_m)$ are orthogonal to each other.
- Let $\mathcal{H}_l$ be the orthogonal of the image of $P(V_{l-2})$. Show that $\mathcal{H}_l \cong L(2l)$ and that

  $$V_l \cong \mathcal{H}_l \oplus P(\mathcal{H}_{l-2}) \oplus \ldots P^{\lfloor \frac{l}{2} \rfloor}(\mathcal{H}_\epsilon),$$

  where ϵ is 0 or 1 depending on the parity of l.

Assume we know by the Stone–Weierstrass theorem that every continuous function on $\mathcal{C}^2(S^2)$ can be approximated by polynomial function, i.e. the image of $\mathbb{R}[X,Y,Z]$ is dense in $\mathcal{C}(S^2,\mathbb{R})$.

- Show that $\bigoplus_{l\geq 0} \mathcal{H}_l$ is dense in $\mathcal{C}(S^2,\mathbb{R})$ and that

  $$\dim \operatorname{Hom}_{SO(3,\mathbb{R})}(L(2l), \mathcal{C}(S^2,\mathbb{R})) = 1$$

  for all $l \in \mathbb{N}$.

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[1] Concretely, this can be defined as

$$\int_0^\pi \int_0^{2\pi} f(\theta,\phi)\overline{g(\theta,\phi)} \sin(\theta) d\theta d\phi,$$

where θ and ϕ are the polar coordinates, i.e. $(x,y,z) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$.

(One can find nice basis of each $\mathcal{H}_l$ by taking the eigenvectors of the subgroup S^1 of rotations along the z-axis. Since $\mathcal{C}(S^2)$ is dense in $L^2(S^2, \mathbb{R})$ we also get $\hat{\bigoplus}_{l \geq 0} \mathcal{H}_l \cong L^2(S^2, \mathbb{R})$. One gets in this way an Hilbert basis of $L^2(S^2)$, which is called the basis of *spherical harmonics*. They are very important in physics to solve equations with spherical symmetry. Here is a picture from Wikipedia.)

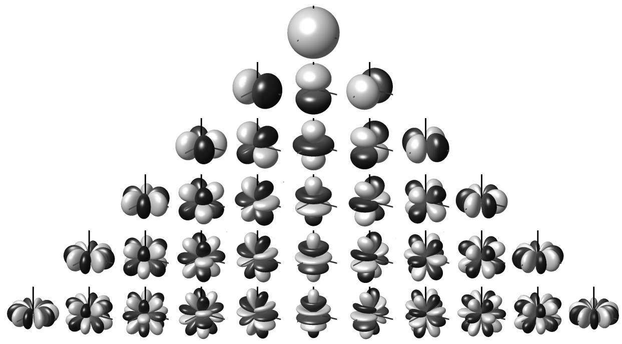

Figure 1: Visual representations of the first few real spherical harmonics $Y_{\ell,m}(\theta,\phi)$. Gray portions represent regions where the function is positive, and black portions represent where it is negative. The distance of the surface from the origin indicates the absolute value of $Y_{\ell,m}(\theta,\phi)$ in angular direction (θ,ϕ).

Aufgabe 6.2: Show that c is a matrix coefficient for G if and only if $\text{span}\langle c \circ (h \cdot) \mid h \in G \rangle$ is finite dimensional.

Aufgabe 6.3: Let G be a compact Lie group and let $\rho : G \to GL(V)$ be a finite dimensional irreducible complex representation of G. Show that the isomorphism $\mathcal{M}(\rho) \cong End_{\mathbb{C}}(V)$ is an isomorphism of $G \times G^{op}$ representations (the action $G \times G^{op}$ is given on both spaces by $g \cdot f \cdot h(x) = f(g^{-1}xh)$. In particular, we have that $\mathcal{M}(\rho)$, as a left G-module, is isomorphic to $V^{\dim_{\mathbb{C}} V}$

Aufgabe 6.4: Let V and W be irreducible complex representations of G. Let χ_V and χ_W denote the trace of $\rho_V(g)$ and $\rho_W(g)$. Show that χ_V and χ_W are matrix coefficients and that

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W. \end{cases}$$