Noncommutative Algebra and Symmetry
WS 2021/22 — Übungsblatt 1

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Exercise 1.1: We say that a representation W of a group G is **cyclic** if there exists $w \in W$ such that $\langle g \cdot w \mid g \in G \rangle = W$.
Consider the representation $V = \mathbb{R}_+ \oplus \mathbb{R}_-$ of $\mathbb{Z}/2\mathbb{Z}$.

1. How many different subrepresentations does V have?
2. Is V cyclic?
3. What about $V' = \mathbb{R}_+ \oplus \mathbb{R}_+ \oplus \mathbb{R}_-$?

Exercise 1.2: Let $C_n = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with n elements.

1. Show that representations of C_n is determined by $A \in End(V)$ such that $A^n = \mathrm{id}$
2. Show that all the irreducible (and indecomposable) representations of C_n over $\mathbb{C}$ are of dimension 1.
3. Find all irreducible representations of C_n over $\mathbb{C}$
4. Let $V = \mathbb{C}^n$ and let ρ be the representation of C_n obtained by cycling the coordinates, that is for any $k \in \mathbb{Z}/n\mathbb{Z}$ we have

   $$\rho(k)(x_1, x_2, \ldots, x_n) = (x_{k+1}, x_{k+2}, \ldots, x_{k+n \pmod n}).$$

   Write the decomposition of V into irreducible representations.

Exercise 1.3: Let p a prime and C_p the cyclic group with p elements.

1. Find all irreducible representations of C_p over $\mathbb{F}_p$.
2. Find all indecomposable representations of C_p over $\mathbb{F}_p$.
3. (*) Let now G a p-group, that is $|G| = p^k$. Show that G has only one irreducible representation over $\mathbb{F}_p$.

   (Hint: every p-group contains a cyclic group C_p in its center $Z(G)$).

Exercise 1.4: Let (V, ρ) be a representation of a group G.

1. Show that

   $$\rho^*(g) : \lambda \mapsto \big(v \mapsto \lambda(\rho(g^{-1})v)\big)\,.$$

   defines a representation of G on V^*. This is called the **dual** representation.

2. Assume that V is finite dimensional. Show that if ρ irreducible, then ρ^* is also irreducible.

Bonus Exercise 1.5: There are n people sitting at a round table, with n an odd number. Each of them has a certain amount of coins. They play a game: at every turn each person gives half of their coins to the person on their left and half to the person on their right.
What is the distribution of the coins after l turns, for $l \mapsto +\infty$?
(Hint: use exercise 1.2.4)