# Noncommutative Algebra and Symmetry
# WS 2021/22 — Ubungsblatt 2

26. October 2021

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**Exercise 2.1:** Let $k$ be a field. Match the monoids on the left with the corresponding monoid algebra on the right (and show that they are isomorphic!)

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|---|---|
| 1. $\mathbb{Z}$ | a $k[x,y]$ |
| 2. $\mathbb{N} \times \mathbb{N}$ | b $k[x,y]/(x^2 - y^3)$ |
| 3. $\{0\} \cup \{n \in \mathbb{N} \mid n \geq 2\}$ | c $k[x,y]/(xy-1)$. |

**Exercise 2.2:** Let $k$ be a field and $C_n$ be the cyclic group with $n$ elements.

1. Show that $kC_n \cong k[x]/(x^n - 1)$.
2. Assume that $(x^n - 1)$ decomposes into linear factors in $k[x]$. Show that all irreducible representations of $C_n$ have dimension 1.
3. Assume further that $n \neq 0$ in $k$. Then show that $kC_n$ is isomorphic as a ring to $\underbrace{k \times k \times \ldots \times k}_{n}$.
4. Show that in this case all representations of $C_n$ are semisimple.

**Exercise 2.3:** Let $m$ be an integer and consider the $\mathbb{Z}$-module $\mathbb{Z}/m\mathbb{Z}$.

1. Write a Jordan-Hölder composition series of the $\mathbb{Z}$-module $\mathbb{Z}/m\mathbb{Z}$.
2. Show that $\mathbb{Z}/m\mathbb{Z}$ is a semisimple $\mathbb{Z}$-module if and only if $m$ is square-free (that is, if $p^2$ does not divide $m$ for any prime $p$).

**Exercise 2.4:** Let $k$ be an algebraic closed field. Let $M$ be a $k[x]$-module which is finite dimensional as a $k$-vector space. Show that $M$ is semisimple if and only if the action of $x$ on $M$ is diagonalizable.

**Exercise 2.5:** Let $k$ be a field and $T_2(k)$ be the algebra of upper triangular $2 \times 2$ matrices over $k$. Find a composition series of $T_2(k)$ as a module over itself. Deduce that there are exactly two classes of isomorphism of simple $T_2(k)$-modules.