Noncommutative Algebra and Symmetry
WS 2021/22 — Übungsblatt 3

29.10.2021

Exercise 3.1: Let D be a division ring and $n \in \mathbb{N}$. Show that

$$End_{Mat(n,D)}(D^n) \cong D^{op}.$$

Exercise 3.2: An *idempotent* is an element e such that $e^2 = e$.

1. Let R be a semisimple ring. Show that every non-zero left ideal of R is generated by an idempotent.

2. If $e \in R$ is an idempotent, show that

$$End_R(Re) \cong (eRe)^{op}$$

Exercise 3.3: For any $n \geq 3$ let D_n be the dihedral group of order 2n. This is the subgroup of $GL(\mathbb{R}^2)$, generated by the rotation r by an angle $2\pi/n$ and a reflection s. The elements satisfy $r^n = id$, $s^2 = id$ and $srs = r^{-1}$. Let $\mathbb{C}D_n$ be the group algebra over the complex numbers.

1. Show that every simple $\mathbb{C}D_n$-module has dimension at most 2. (Hint: If v is an eigenvector of the action of r, then sv is also an eigenvector for the action of r.)

2. Determine all the one-dimensional $\mathbb{C}D_n$-modules (The answer will depend on the parity of n!)

3. Write the Artin–Wedderburn decomposition of the algebra $\mathbb{C}D_n$.

Exercise 3.4: Determine which of the following algebras are isomorphic to the group algebra $\mathbb{C}G$ of some finite group G.

1. $M_3(\mathbb{C})$

2. $\mathbb{C} \times M_2(\mathbb{C})$

3. $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

4. (*) $\mathbb{C} \times M_3(\mathbb{C})$.