Noncommutative Algebra and Symmetry
WS 2021/22 — Ubungsblatt 6

12.11.2021

Exercise 6.1: Let k be a field and A a vector space over k. Let I be a set and for any $i \in I$ let B_i be a vector space over k.

1. Show that the tensor product commutes with direct sums:

$$\bigoplus_{i \in I} A \otimes_k B_i \cong A \otimes_k \left(\bigoplus_{i \in I} B_i \right)$$

2. (*) Tensor products do not commute with direct products! Let k be a field, V a vector space over k and I a set. We denote by $\mathrm{Ens}(I, V)$ the vector space of maps $I \to V$.

   - Show that $\prod_{i \in I} V = \mathrm{Ens}(I, V)$
   - Show that the canonical map

$$V \otimes_k \prod_{i \in I} k \cong V \otimes_k \mathrm{Ens}(I, k) \to \mathrm{Ens}(I, V)$$

   defined by $v \otimes \phi \mapsto (i \mapsto \phi(i)v)$ is not an isomorphism.

   Hint: All the maps in the image are contained in a finite dimen sional subspace of V.

Exercise 6.2: Let G be a group and let V, W be representations of G over k with W of dimension Then $V \otimes_k W$ is simple if and only if V is simple.

Exercise 6.3: Let k be a field with $\mathrm{char}(k) \neq 2$ and V a finite dimensional vector space over k. We have the following two subspaces of $V \otimes V$.

$$\Lambda^2 V = \langle v \otimes w - w \otimes v \mid v, w \in V \rangle \text{ and } S^2 V = \langle v \otimes w - w \otimes v \mid v, w \in V \rangle.$$

1. Compute the dimension of these subspaces and show that we have $V \otimes V = \Lambda^2 V \oplus S^2 V$.

2. Let G a group and V be a G representation over k, . Show that $\Lambda^2 V$ and $S^2 V$ are subrepresentations of $V \otimes V$. In particular, if $\dim_k V \geq 2$, then $V \otimes V$ is not simple as a G-representation.

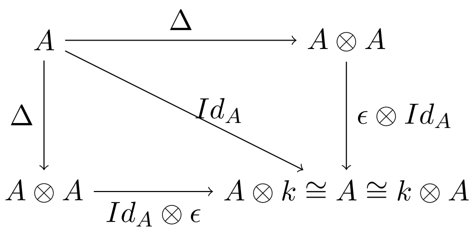
3. Show the following formulas for the character.

$$\chi_{\Lambda^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2} \text{ and } \chi_{\Lambda^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}$$

4. Let $G = S_4$ and let W the standard representation of S_4 (i.e. the 3-dim. irreducible representation contained in $\mathbb{C}^4$. Show that $\Lambda^2 V$ is irreducible.

   (Note: this is actually true for any $n \geq 1$ and any exterior power $\Lambda^k V$ of the standard representation!)

Exercise 6.4: Let A be a k-algebra with a coassociative comultiplication Δ. A morphism of k-algebras $\epsilon : A \to k$ is called a *counit* if $(Id_A \otimes \epsilon) \circ \Delta = Id_A = (\epsilon \otimes Id_A) \circ \Delta$, i.e. if the following diagram commutes



(Note: If ϵ is a counit, it induces a structure of A-module on k. The commutativity of the diagram implies that for any A-module V, we have $V \otimes k \cong V \cong k \otimes V$ as A-modules)
Let G be a group and $A = kG$. Show that

$$\epsilon : kG \to k$$

defined by $\epsilon(g) = 1$ for all $g \in G$ is a counit of kG.