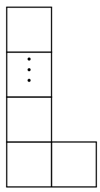
Noncommutative Algebra and Symmetry
WS 2021/22 — Ubungsblatt 7

26.11.2021

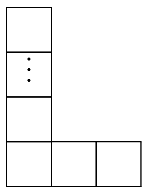
Exercise 7.1: Let Y be an Young diagram and let $L(Y)$ be the corresponding irreducible representation of S_n. Let V_{sgn} be the sign representation. Show that the tableau of $L(Y) \otimes V_{\text{sgn}} \cong L(Y^t)$, where Y^t denotes the transpose Young diagram.

Exercise 7.2: Consider the natural representation of S_n on $\mathbb{C}^n$. We have $\mathbb{C}^n = triv \oplus \theta$, where θ is irreducible (cf. Exercise 3.3 and 3.4) Show that θ is the representation corresponding to the following Young diagram



Find all the standard tableaux of this shape.

Exercise 7.3: Assume now that $G = S_k$ and that V is the representation θ as of dimension $k-1$ in 7.2. Recall from Exercise 6.3 that $\Lambda^2 V$ is irreducible. Show that the Young diagram corresponding to $\Lambda^2 V$ is



and for this diagram find all the corresponding standard tableaux

Exercise 7.4: Let k be a field and let V be a representation of a group G over k. For any $n \geq 0$ we can regard

$$V^{\otimes n} := \overbrace{V \otimes V \otimes \ldots \otimes V}^{n \text{ times}}$$

as a representation of G. Moreover, $V^{\otimes n}$ can also be regarded as a representation of the symmetric group S_n, which acts by permuting the factors.

1. Show that the actions of G and S_n on $V^{\otimes n}$ commute with each other. In particular, G preserves the isotypic components of S_n in $V^{\otimes n}$

2. Let $\operatorname{char} k = 0$. Show that the isotypic component of the trivial representation is

$$S^n V := \left\langle \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(n)} \mid v_1, v_2, \ldots, v_n \in V \right\rangle$$

while the isotypic component of the sign representation is

$$\Lambda^n V := \left\langle \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(n)} \mid v_1, v_2, \ldots, v_n \in V \right\rangle$$

and deduce that these are representations of G.

3. For $n = 2$ and $n = 3$ compute the character $V^{\otimes n}$ as a S_n-representation and use it compute $\dim \Lambda^n V$ and $\dim S^n V$.