Supplementary Notes for the lecture course "Noncommutative Algebra and Symmetry"

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These are supplementary notes for the lecture course "Noncommutative Algebra and Symmetry" held at the University of Freiburg during the Wintersemester 2021/22. This material complements the main lecture notes [1]. A good reference for most of this material is [3]. Another source I have used are the lecture notes [5].

1 Tensor product over a field

1.1 Tensor product of vector spaces over a field

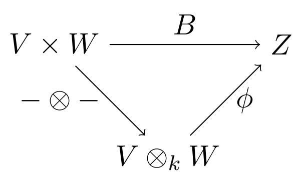
Let k be a field and let V, W be finite dimensional vector spaces over k.

If $\{v_1 \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ are bases of V and W, we can define $V \otimes_k W$ simply as the vector space with basis $\{v_i \otimes w_j\}$. However, this is not very satisfactory as is not clear how the tensor product depends from the chosen bases, so we give a more intrinsic way to define the tensor product of vector spaces, in terms of a universal property.

Definition 1.1. Let V, W and Z be vector spaces over k. A *k-bilinear map* $B : V \times W \to Z$ is a map that satisfies

- For any $v \in V$ the map $B(v, -) : W \to Z$ is k-linear.
- For any $w \in W$ the map $B(-, w) : V \to Z$ is k-linear.

Definition 1.2. Let V, W be vector spaces over k. The tensor product, denoted $V \otimes_k W$ (or simply $V \otimes W$ if the field is clear from context) is a vector space together with a bilinear map $- \otimes - : V \times W \to V \otimes W$ such that for any $B : V \times W \to Z$ bilinear there exists a unique linear map $\phi : V \otimes W \to Z$ such that $B = \phi \circ (- \otimes -)$.



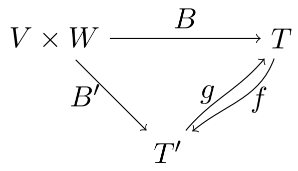
Tensor product can be thought as a tool to transform bilinear maps into linear maps. In fact, we have

$$\{\text{bilinear maps } V \times W \to Z\} \cong \{\text{linear maps } V \otimes W \to Z\}.$$

We need to show that tensor product exists and are unique up to a unique isomorphism. As usual, when an object is defined by means of an universal property the uniqueness is rather easy to show.

Lemma 1.3. *Let T and T' be tensor products of V and W, and let $B : V \times W \to T$ and $B' : V \times W \to T$ be the corresponding linear maps. Then, there exists a unique isomorphism $f : T \to T'$ with $B = f \circ B'$.*

Proof. By the universal property there exists unique $f : T \to T'$ and $g : T' \to T$ such that $B = f \circ B'$ and $B' = g \circ B$. We want to show that f and g are inverse to each other.



For any $v \in V$ and $w \in W$ we have

$$B(v, w) = g(B'(v, w)) = g(f(B(v, w))$$

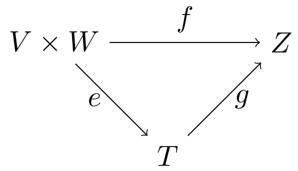
so $g \circ f$ is the unique map such that $B = (g \circ f) \circ B$. Since also $B = Id_T \circ B$ we have $Id_T = g \circ f$. Similarly, $f \circ g = Id_{T'}$ and f and g are inverse to each other. $\square$

Showing the existence is a more tedious task.

Lemma 1.4. *Let V and W be k-vector spaces. Then there exists a tensor product $V \otimes_k W$.*

Proof. Let T be a vector space with basis $\{e_{v,w}\}_{v \in W, w \in W}$. We have

$$\{\text{maps } V \times W \to Z\} \cong \{\text{linear maps } T \to Z\}.$$



So for any $f : V \times W \to Z$ there exists g making the diagram commute. If f is bilinear, then we have

$$0 = f(v + v', w) - f(v, w) - f(v', w) = g(e_{v+v',w} - e_{v,w} - e_{v',w})$$

and similarly

$$g(e_{v,w+w'} - e_{v,w} - e_{v,w'}) = 0$$

$$g(\lambda e_{v,w} - e_{\lambda v,w}) = 0$$

$$g(\lambda e_{v,w} - e_{v,\lambda w}) = 0$$

for all $v, v' \in W$, $w, w' \in W$ and $\lambda \in k$. Let U be the subspace of T generated by all the elements of the form $e_{v+v',w} - e_{v,w} - e_{v',w}$, $e_{v,w+w'} - e_{v,w} - e_{v,w'}$, $\lambda e_{v,w} - e_{\lambda v,w}$, $\lambda e_{v,w} - e_{v,\lambda w}$ for all $v, v' \in V$, $w, w' \in W$ and $\lambda \in k$. Then g factors through T/U in a unique way. Hence, any bilinear map $f : V \times W \to Z$ factors uniquely through T/U, hence T/U is a tensor product. $\square$

Exercise 1.5. We have $V \otimes_k k \cong V$, where the isomorphism is given by $v \otimes \lambda \mapsto \lambda v$. Similarly, $k \otimes_k V \cong V$.

We write $v \otimes w$ for the image of (v, w) in $V \otimes W$. An element of the form $v \otimes w$ is called a *pure tensor*. Pure tensors span $V \otimes W$, but in general there are elements not of this form.

Example 1.6. Let $V = W = \mathbb{C}^2$ with basis $\{e_1, e_2\}$. Then $e_1 \otimes e_1 + e_2 \otimes e_1$ is pure (and equals $e_1 \otimes (e_1 + e_2)$ while $e_1 \otimes e_1 + e_2 \otimes e_2 \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is not pure.

Tensor product is associative, i.e.

$$V \otimes (V' \otimes V'') \cong (V \otimes V') \otimes V''$$

via the unique isomorphism which sends $v \otimes (v' \otimes v'')$ to $(v \otimes v') \otimes v''$. Moreover, we have

$$V \otimes_k k \cong V$$

via the unique isomorphism which sends $(v \otimes \lambda)$ to λv.

Exercise 1.7. Tensor product commutes with direct sum but not with direct products.

We can now show that our naive definition was correct.

Lemma 1.8. *Let V and W be vector spaces over k with bases $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$. Then $\{v_i \otimes w_j\}_{i \in I, j \in J}$ is a basis of $V \otimes W$.*

Proof. Let U be the vector space generated by all the $v_i \otimes w_j$. Each bilinear map $B : V \otimes W \to Z$ factors in a unique way through U, so U is a tensor product and the inclusion $i : U \hookrightarrow V \otimes W$ is the unique linear map commuting with $\otimes$, so by uniqueness of the universal property we have $i = Id_{V \otimes W}$ and $U = V \otimes W$.

It remains to show that the elements $v_i \otimes w_j$ are linearly independent. Assume there is a linear dependency

$$\sum a_{i,j} v_i \otimes w_j = 0.$$

For $i_0 \in I$ and $j_0 \in J$ let $\delta_{i_0,j_0} : V \otimes W \to k$ the bilinear map such that

$$\delta_{i_0,j_0}(v_i, w_j) = \begin{cases} 1 & \text{if } i = i_0 \text{ and } j = j_0 \\ 0 & \text{otherwise} \end{cases}$$

Then δ_{i_0,j_0} is bilinear and factors through $\phi : V \otimes W \to k$. Let $D := \sum a_{i,j} v_i \otimes w_j$. We have

$$\phi(D) = \sum a_{i,j} \delta_{i_0,j_0}(v_i, w_j) = a_{i_0,j_0}.$$

So D cannot be 0 if at least one of the $a_{i,j} \neq 0$. □

In particular, we have

$$\dim_k V \cdot \dim_k W = \dim_k (V \otimes_k W).$$

Moreover, if $\{w_j\}$ is a basis of W, we can always write any element of $V \otimes W$ as $\sum v_j \otimes w_j$, for some $v_j \in V$.

For a vector space V we denote by V^* its dual vector space.

Proposition 1.9. *Let V be finite dimensional over k. We have $V^* \otimes V \cong \operatorname{End}_k(V)$ via $v^* \otimes v \mapsto v^*(-)v$.*

Proof. Both spaces have dimension $(\dim V)^2$, so it is enough to show the surjectivity. Let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{\delta_1, \ldots, \delta_n\}$ be the dual basis. Then if $f : V \to V$ a linear map, we have $f = \sum \delta_i(-) f(v_i)$, hence f is the image of $\sum \delta_i \otimes f(v_i)$. □

1.2 Tensor product of algebras

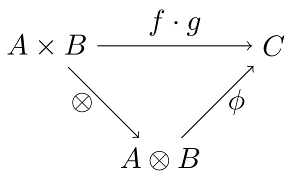
Let A and B be k-algebras. Then the tensor product $A \otimes_k B$ is in a natural way a k-algebra, where the product is defined as

$$(a \otimes b) \cdot (a' \otimes b') = (aa' \otimes bb')$$

on pure tensors and extending by linearity to the whole $A \otimes_k B$. (One can check that this is well defined by constructing in the usual way bilinear map from $A \times B$. For example, right multiplication with $a' \otimes b'$ is induced by the bilinear map which sends (a, b) to $aa' \otimes bb'$.)

Example 1.10. Let G and H be groups. Then $kG \otimes_k kH \cong k(G \times H)$. In fact, the bilinear map $- \otimes - : kG \times kH \to kG \otimes kH$ factors through $f : k(G \times H) \to kG \otimes kH$, where f is defined by $f(g, h) = g \otimes h$. We can find the inverse of f using the universal property of the tensor product.

We can regard A and B as subalgebras of $A \otimes_k B$ via $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$. Notice that the images of A and B in $A \otimes B$ commute. In fact, the tensor product of algebras can also be defined by means of a universal property. Let C be a k-algebra and let $f : A \to C$ and $g : B \to C$ be k-algebra isomorphism. Then, if $f(a)g(b) = g(b)f(a)$ for all $a \in A$ and $b \in B$ there exists a unique k-algebra morphism $\phi : A \otimes B \to C$ such that the following diagram commute.



Example 1.11. Let V and W be finite dimensional vector spaces. Then $\mathrm{End}_k(V) \otimes \mathrm{End}_k(W) \cong \mathrm{End}_k(V \otimes W)$. In fact, if $A \in \mathrm{End}_k(V)$ and $B \in \mathrm{End}_k(W)$, the map

$$(A \odot B)(v \otimes w) = (Av \otimes Bw)$$

defines an endomorphism of $V \otimes W$. The map $(A, B) \mapsto (A \odot B)$ is bilinear, hence this induces a linear map $\Phi : \mathrm{End}_k(V) \otimes \mathrm{End}_k(W) \to \mathrm{End}_k(V \otimes W)$. One can check that Φ is a morphism of algebras, for example using the universal property above. Moreover, Φ is an isomorphism: after we fix bases $\{v_i\}$ and $\{w_j\}$ of V and W, it sends the basis $\{E_{ij} \otimes E_{kh}\}$ to the basis $\{E_{(i,k),(j,h)}\}$, where

$$E_{(i,k),(j,h)}(v_{i'} \otimes w_{k'}) = \begin{cases} v_j \otimes w_h & \text{if } i = i' \text{ and } k = k' \\ 0 & \text{otherwise.} \end{cases}$$

1.3 Tensor product of modules

Let A be a k-algebra and let V and W be two A-modules. In general, the tensor product $V \otimes_k W$ is a module over $A \otimes_k A$ but there is no natural way to define a structure of A-module on $V \otimes W$.

If V is a A-module and W is a B-module, then $V \otimes_k W$ is in a natural way a $A \otimes_k B$ module, where the action is given by

$$(a \otimes b) \cdot (v \otimes w) = (a \cdot v) \otimes (b \cdot w).$$

Theorem 1.12 ([1, Satz 1.7.3]). *Let A and B be algebras over an algebraically closed field $k = \overline{k}$. There is a bijection*

$$\mathrm{Irr}_k^{f.d.}(A) \times \mathrm{Irr}_k^{f.d.}(B) \xrightarrow{\sim} \mathrm{Irr}_k^{f.d.}(A \otimes B),$$

where $\mathrm{Irr}_k^{f.d.}(A)$ denotess the set of isomorphism classes of irreducible finite dimensional A-modules.

In the proof we need to use the following result, which is proved in [1, Korollar 1.6.6]

Proposition 1.13 (Wedderburn's theorem). *Let $k = \overline{k}$. Let A be an algebra over k and let V be an A-module. Then V is simple if and only if the corresponding map $\rho : A \to \mathrm{End}_k(V)$ is surjective.*

Proof of Theorem 1.12. Let V and W be simple modules over A and B respectively of finite dimension over k. Then, the image of $A \otimes B$ in $\mathrm{End}_k(V) \otimes \mathrm{End}_k(W) = \mathrm{End}_k(V \otimes W)$. Hence, $V \otimes W$ is simple by Proposition 1.13.

Let T be a $A \otimes B$-module finite dimensional over k. We can regard it as a A-module by restriction. It contains a simple A-module $E \subset T$. Notice that $\mathrm{Hom}_A(E, T)$ is a B-module. In fact, since the action of B on T commutes with A, for $\phi \in \mathrm{Hom}_A(E, T)$ we have

$$b \cdot \phi(ae) = b \cdot (a \cdot \phi(e)) = (a \otimes b) \cdot \phi(e) = a \cdot (b \cdot \phi(e))$$

hence $b \cdot \phi \in \mathrm{Hom}_A(E, T)$.

This makes $E \otimes_k \mathrm{Hom}_A(E, T)$ a $A \otimes_k B$-module and we have an inclusion of $A \otimes B$-modules

$$\Phi : E \otimes_k \mathrm{Hom}_A(E, T) \hookrightarrow T$$

which sends $e \otimes \phi$ to $\phi(e)$. Assume we know for the moment that Φ is injective (we postpone its proof to Lemma 1.14). Then, if T is simple we have $T = E \otimes_k \mathrm{Hom}_A(E, T)$, so T is a tensor product of a simple A-module E and a B-module $\mathrm{Hom}_A(E, T)$. However, if $\mathrm{Hom}_A(E, T)$ must be simple, since any B-submodule $F \subset \mathrm{Hom}_A(E, T)$ induces a submodule $E \otimes_k F \subset T$. □

As promised, we now show that Φ is injective.

Lemma 1.14. *Let T be an A-module and E a simple submodule such that $\mathrm{End}_A(E) = k$. Then*

$$\Phi : E \otimes_k \mathrm{Hom}_A(E, T) \hookrightarrow T$$

is injective.

Notice that in this Lemma we can neglect the B-action on B. It can be easily shown that the image of Φ is the *isotypic component* T_E of E as a A-module, which is the sum of all the simple submodule of T isomorphic to E.

$$T_E := \sum_{F \subset T, F \cong E} F$$

Proof. Assume that $\Phi(D) = 0$ for some $D \in E \otimes_k \mathrm{Hom}_A(E, T)$. We can write D as $D = \sum_{i=1}^n v_i \otimes \phi_i$ in a way that all the ϕ_i's are all linearly independent over k. We have a morphism of A-modules

$$\bigoplus_{i=1}^n \phi_i : E^n \to T.$$

Since $\sum \phi_i(v_i) = 0$, the morphism $\bigoplus \phi_i$ is not injective. Since E^n is a semisimple A-module, the kernel contains a simple summand isomorphic to E. So there exist a n-uple of $\lambda_i \in \mathrm{End}_A(E) = k$ such that the composition

$$E \xrightarrow{\bigoplus \lambda_i} E^n \xrightarrow{\bigoplus \phi_i} T$$

is 0, i.e. we have $\sum \lambda_i \phi_i = 0$, which contradicts the linear independence of the ϕ's. □

Remark 1.15. Lemma 1.14 can be generalized without the assumption that $\mathrm{End}_A(E) = k$. If $\mathrm{End}_A(E) = D$ we have

$$\Phi : E \otimes_{D^{op}} \mathrm{Hom}_A(E,T) \xrightarrow{\sim} T_E.$$

(see Section 2 for the definition of the tensor product over arbitrary rings)

1.4 Tensor product of representations

We have seen that if V and W are representations of G, then $V \otimes_k W$ is a $kG \otimes kG$-module, i.e. a representation of $G \times G$. However, using the diagonal morphism $\Delta : G \to G \times G$ we can also regard $V \otimes_k W$ as a representation of G. At the level of the group algebra kG, this corresponds to the following structure.

Definition 1.16. Let A be a k-algebra. A *comultiplication* is a morphism of k-algebras $\Delta : A \to A \otimes_k A$. If the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow \mathrm{Id}_A \otimes \Delta \\ A \otimes A & \xrightarrow[\Delta \otimes \mathrm{Id}_A]{} & A \otimes A \otimes A \end{array}$$

we say that Δ is *coassociative*

Using the comultiplication Δ, we can consider $V \otimes_k W$ as a module of A. The *coassociativity* of Δ implies that if V_1, V_2 and V_3 are A-modules, then $V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$.

Exercise 1.17. Check directly that the map $\Delta : kG \to kG \otimes_k kG$ defined by $\Delta(\sum a_g g) = \sum a_g (g \otimes g)$ is a coassociative comultiplication of kG.

If G is a group, the representation of G on $V \otimes W$ can be simply defined as

$$g \cdot (v \otimes w) = gv \otimes gw.$$

Lemma 1.18. *Let V and W be finite dimensional representations of G, with characters χ_V and χ_W. Then, for any $g \in G$ we have*

$$\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g).$$

Proof. Choose basis $\{v_i\}$ and $\{w_j\}$ of V and W. If $g \cdot v_i = \sum a_{ik} v_k$, then $\chi_V(g) = \sum a_{ii}$. Similarly, if $g \cdot w_j = \sum b_{jh} w_h$, then $\chi_W(g) = \sum b_{jj}$. Now, we have

$$g \cdot (v_i \otimes w_j) = \sum_{k,h} a_{ik} b_{jh} (v_k \otimes w_h),$$

hence

$$\chi_{V \otimes W}(g) = \sum_{i,j} a_{ii} b_{jj} = \chi_V(g)\chi_W(g).$$ □

Example 1.19. We know the character table of the group S_4.

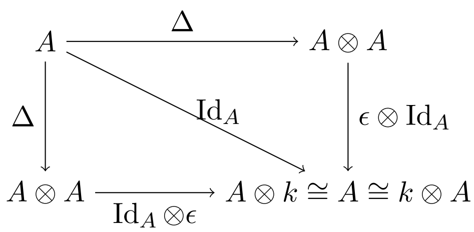
S_4	$\emptyset$	(12)	(123)	(1234)	(12)(34)
triv	1	1	1	1	1
sign	1	-1	1	-1	1
W	3	1	0	-1	-1
W'	3	-1	0	1	-1
V	2	0	-1	0	2

The representation W occurs in the natural representation of S_4 on $\mathbb{C}^4$. From the characters we know that $W' \cong W \otimes (sign)$. So if $\rho : S_4 \to GL(W)$ is the action of S_4 on W, defining $\rho'(g) = \rho(g)\,\mathrm{sgn}(g)$ we obtain the action on W'. Notice that $V \cong V \otimes (sign)$. Moreover, V can be obtained as a summand of $W \otimes W$.

Exercise 1.20. Let V, W be representations with W of dimension 1. Show that $V \otimes W$ is simple if and only if V is simple.

Show that if $\dim V \geq 2$, then $V \otimes V$ is never simple.

Definition 1.21. Let A be an algebra with a coassociative comultiplication Δ. A morphism of algebras $\epsilon : A \to k$ is called a *counit* if $(\mathrm{Id}_A \otimes \epsilon) \circ \Delta = \mathrm{Id}_A = (\epsilon \otimes \mathrm{Id}_A) \circ \Delta$, i.e. if the following diagram commutes



If ϵ is a counit, it induces a structure of A-module on k. The commutativity of the diagram implies that for any A-module V, we have $V \otimes k \cong V \cong k \otimes V$ as A-modules.

Exercise 1.22. Regard k as the trivial representation of a group G and let $\epsilon : kG \to \mathrm{End}_k(k) \cong k$ be the corresponding map. Show that ϵ is a counit of kG.

2 Tensor product over an arbitrary ring

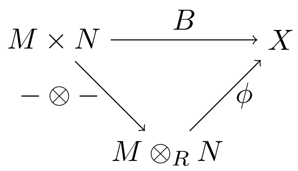
We can generalize the construction of the tensor product. Let R be a ring, not necessarily commutative. Let M be a right R-module and N a left R-module. Let X be an abelian group.

Definition 2.1. A *balanced map* $B : M \times N \to X$ is a map such that

- $B(m, n + n') = B(m, n) + B(m, n')$
- $B(m + m', n) = B(m, n) + B(m', n)$
- $B(mr, n) = B(m, rn)$

for any $m, m' \in M$, $n, n' \in N$ and $r \in R$.

Definition 2.2. The tensor product $M \otimes_R N$ is an abelian group, together with a balanced map $-\otimes- : M \times N \to M \otimes_R N$ such that for any X abelian group and any $B : M \times N \to X$ balanced map, there exists a unique homomorphism of abelian group $\phi : M \otimes_R N \to X$ making the following diagram commute



Proposition 2.3. *Let M be a right R-module and N a left R-module. Then the tensor product $M \otimes_R N$ exists and it is unique up to isomorphism.*

Proof. The uniqueness directly follows by the universal property by a standard argument as in Lemma 1.3. Also the proof of the existence is similar to Lemma 1.4.

Let T be the free abelian group with basis $\{e_{m,n}\}_{m \in M, n \in N}$. Take the subgroup $U \subset T$ generated by all the elements of the form

- $e_{v+v',w} - e_{v,w} - e_{v',w}$
- $e_{v,w+w'} - e_{v,w} - e_{v,w'}$
- $e_{vr,w} - e_{v,rw}$

for anyl $v, v' \in M$, $w, w' \in N$ and $r \in R$. Then T/U is a tensor product. In fact, the map $M \times N \to T$ which sends (m, n) to $e_{m,n}$ is balanced and every balanced map $B : M \times N \to X$ factors in a unique way through T/U. □

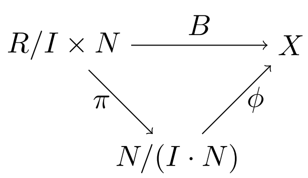
Example 2.4. If N is a free R-module, i.e. $N \cong R^I$, then $M \otimes_R R^I \cong M^I$.

Lemma 2.5. *If $M = R/I$, where I is a right ideal in R, then $R/I \otimes_R N \cong N/(I \cdot N)$*

Proof. Let $B : R/I \times N \to X$ be a balanced map. Then

$$B(r + I, n) = B((1 + I)r, n) = B(1 + I, rn)$$

If we define $\pi : R/I \times N \to N/(I \cdot N)$ as $\pi(r + I, n) = rn + IN$ then π is balanced and the diagram



commutes, where $\phi(n+IN) = B(1+I, n)$. Moreover, ϕ is the unique such map, so N/IN is a tensor product. □

Example 2.6. If $p, q \in \mathbb{N}$ are primes, then $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} \cong (\mathbb{Z}/q\mathbb{Z})/p \cdot (\mathbb{Z}/q\mathbb{Z})$. If $p = q$ then $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$, while if $p \neq q$ then $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = 0$.

Remark 2.7. Assume that R is a commutative ring. Then we do not need to distinguish anymore between left and right modules and $M \otimes_R N$ is also an R-module itself. In fact, the product

$$r \cdot (m \otimes n) = (rm \otimes n) = (m \otimes rn)$$

defines a module structure on $M \otimes_R N$.

In particular, if R is a field, the two definitions of tensor product coincide (one can use the universal properties to construct isomorphisms between the two objects).

More generally, let L be a ring and let M be a (L, R)-bimodule (i.e. it is at the same a left S-module and a right R-module, or in other words it is a $L \otimes_{\mathbb{Z}} R^{op}$-module). Then, for any left R-module N, multiplication on the left by L induces a L-module structure on $M \otimes_R N$.

$$x \cdot (m \otimes n) = (xm \otimes n) \text{ for all } x \in L$$

3 Induction and restriction of modules

Let G, H be groups and let $f : H \to G$ be a homomorphism of groups. The most relevant case will be when f is an inclusion of a subgroup.

Definition 3.1. Let $\rho : G \to GL(V)$ be a representation of G. Then we can regard V as a representation of H by precomposing with f. We call the *restriction* of V the representation so obtained and we denote it by $\mathrm{res}_G^H(V)$.

We want to go now in the other direction, i.e. we want to construct representation of G starting with a representation of H. For any field k this makes kG a right kH-module, where

$$g \cdot h = gf(h) \quad \text{for any } g \in G \text{ and } h \in H.$$

In particular, we can regard kG as a (kG, kH)-bimodule.

Definition 3.2. For any representation V of H we can define a representation of G, called the *induced representation* of V as

$$\mathrm{coind}_H^G(V) := kG \otimes_{kH} V$$

Example 3.3. If $H = \{1\}$ and $V = k$, then $\mathrm{coind}_1^G(k)$ is the regular representation kG.

Exercise 3.4. If $G = \{1\}$, then kG is isomorphic as a kH-module to kH/I, where $I = \{\sum a_h h \in kH \mid \sum a_h = 0\}$. Then

$$\mathrm{coind}_H^1(V) = k \otimes_{kH} V = (kH/I) \otimes_{kH} V = V/(I \cdot V)$$

The vector space $V/(I \cdot V)$ is called the *coinvariants* of V.

Assume now that H is a subgroup of G and f is the inclusion. Recall that G/H denote the left cosets of H in G.

$$G = \bigsqcup_{gH \in G/H} gH.$$

We can choose a set of representatives $[G/H]$ for G/H. Then $[G/H]$ also gives a basis of kG as a free right kH-module, i.e. we have

$$kG = \bigoplus_{g \in [G/H]} g \cdot kH$$

Let V be a representation of H. Then

$$\mathrm{coind}_H^G(V) = \bigoplus_{g\in[G/H]} g\cdot kH \otimes_{kH} V = \bigoplus_{g\in[G/H]} g\otimes V$$

where $g\otimes V := \{g\otimes v \mid v\in V\} \subset kG\otimes_{kH} V$. It follows that

$$\dim_k(\mathrm{coind}_H^G(V)) = |G/H|\cdot \dim_k(V).$$

The action of G on $\mathrm{ind}_H^G(V)$ permutes the vector spaces $g\otimes V$ and moreover it is transitive. Notice that the stabilizer of $1\otimes V$ is H, while in general the stabilizer of $g\otimes V$ is gHg^{-1}.

Example 3.5. If $H\subset G$ and k is the trivial representation of H, then $\mathrm{coind}_H^G(k)$ is the representation of G on $k(G/H)$ induced by the action on G on the set of cosets.

Exercise 3.6. Let $H\subset G$ and k be the trivial representation of H. Then $\mathrm{coind}_H^G(k) = kG(\sum_{h\in H} h)$. (This is for example the case of the representation $M(Y)$ which was used in the study of representation theory of S_n.)

Remark 3.7. If $H\subset G$ and G/H is finite, there is another more geometric way in which we can think of the induction.

$$\mathrm{coind}_H^G(V) = \{f : G\to V \mid f(gh^{-1}) = h\cdot f(x) \text{ for all } h\in H,\ x\in g\}.$$

If f is such a function, we can define the action of g by $g\cdot f(x) = f(g^{-1}x)$. In fact, we have $g\cdot f(xh^{-1}) = f(g^{-1}xh^{-1}) = h\cdot f(g^{-1}x) = h\cdot(g\cdot f(x))$.

It can be showed that the two definitions coincide. Every function $f\in \mathrm{coind}_H^G(V)$ is the determined by the value on g, for $g\in[G/H]$. So we send

$$f\mapsto \sum_{g\in[G/H]} g\otimes f(g)\in kG\otimes_{kH} V.$$

The inverse is induced by the balanced map

$$kG\times V\to \mathrm{coind}_H^G(V)$$

$$(x,v)\mapsto (a\mapsto (a^{-1}x)v)$$

(If G/H is not finite, one should define $\mathrm{coind}_H^G(V)$ as the set of functions which are non-trivial only on finitely many left H-cosets.)

There is another natural way to construct representation of G starting with a representation of H.

Definition 3.8. Consider kG as a left kH-module. Let V a representation of H. Then

$$\mathrm{ind}_H^G(V) := \mathrm{Hom}_H(kG, V)$$

where the action of G is given by $g\cdot f(x) = f(xg)$. (This defines an action: $hg\cdot f(x) = h\cdot f(xg) = f(xhg)$ for all $g,h\in G$.)

Example 3.9. If $H=\{1\}$, then $\mathrm{ind}_1^G(k) = kG^*$ is the dual of the regular representation.

Exercise 3.10. If $G=\{1\}$, then $\mathrm{ind}_H^1(V) = \mathrm{Hom}_H(k,V) = V^H$, the invariants of V.

Proposition 3.11. *Assume that $H \subset G$ and G/H is finite. Then* $\operatorname{coind}_H^G(V) \cong \operatorname{ind}_H^G(V)$.

Proof. The balanced map $kG \times V \to \operatorname{Hom}_H(kG, V)$ defined by

$$(x, v) \mapsto (a \mapsto (ax)v)$$

where $(ax)v = 0$ if $ax \notin kH$ and $(ax) \cdot v$ if $ax \in kH$. This balanced map induces always a homomorphism

$$\Phi : \operatorname{coind}_H^G(V) \to \operatorname{ind}_H^G(V).$$

If G/H is finite, we can define

$$\Psi : \operatorname{ind}_H^G(V) \to \operatorname{coind}_H^G(V)$$

$$\Psi(\theta) = \sum_{g \in [G/H]} g \otimes \theta(g^{-1}).$$

We need to show that these maps are inverse to each other. We have

$$\Phi(\Psi(\theta))(a) = \Phi\left(\sum_{g \in [G/H]} g \otimes \theta(g^{-1})\right)(a) = \sum_g (ag)\theta(g^{-1}).$$

By linearity, it is enough to show the claim for $a \in G$. However, $(ag)\theta(g^{-1}) \neq 0$ only if $ag \in H$, i.e. if $g \in a^{-1}H$, so only for a single element in $[G/H]$. Let $g \in [G/H]$ be the representative for $a^{-1}H$, so $g = a^{-1}h$ for some $h \in H$. We obtain

$$\Phi(\Psi(\theta))(a) = (ag)\theta(g^{-1}) = aa^{-1}h\theta(h^{-1}a) = \theta(hh^{-1}a) = \theta(a).$$

In the other direction, we compute

$$\Psi(\Phi(x \otimes v)) = \sum_{g \in [G/H]} g \otimes (g^{-1}x)v.$$

By linearity, it is enough to show the claim for $x \in G$. But $(g^{-1}x)v \neq 0$ only when $g^{-1}x \in H$, i.e. only for the representative of xH. In this case we have $g = xh$ for some h and we obtain

$$\Psi(\Phi(x \otimes v)) = xh \otimes h^{-1} \cdot v = x \otimes v.$$ □

Example 3.12. In the case $H = \{1\}$ and G finite, we obtain an isomorphism $kG \cong kG^*$.

Assume now that G/H is finite. Given a representation of H we want to compute the character of the induced representation.

Definition 3.13. Let ϕ be a class function of H (i.e. a function $\phi : H \to k$ which is invariant on conjugacy classes). We define the induction of ϕ as

$$\operatorname{ind}_H^G(\phi)(g) = \frac{1}{|H|} \sum_{x \in G} \phi(x^{-1}gx)$$

where we extend ϕ to 0 on $G \setminus H$.

Remark 3.14. Since ϕ is constant on H conjugacy classes, we have

$$\operatorname{ind}_H^G(\phi)(g) = \sum_{x \in [G/H]} \phi(x^{-1}gx).$$

Note that this formula makes sense even when H is infinite but G/H is finite.

Proposition 3.15. *We have*
$$\chi_{\mathrm{ind}_H^G(V)} = \mathrm{ind}_H^G(\chi).$$

Proof. Let $x \in G$. We want to compute the trace of x on $kG \otimes_{kH} V$. Recall that x permutes the subspaces $g \otimes V$, for $g \in [G/H]$. To compute the trace, it is enough to look at the $g \otimes V$ which are fixed by x, i.e. such that $x \in gHg^{-1}$, or in other words that $x = ghg^{-1}$ for some $h \in H$, i.e. $h = g^{-1}xg \in H$.

We want to compute the trace of x on $g \otimes V$. We have

$$x \cdot (g \otimes v) = xg \otimes v = gh \otimes v = g \otimes (h \cdot v) \in kG \otimes_{kH} V.$$

Therefore, the trace of x on $g \otimes V$ is the same as $\chi_V(h) = \chi_V(g^{-1}xg)$. Hence,

$$\chi_{\mathrm{ind}_H^G(V)}(x) = \sum_{g \in [G/H], g^{-1}xg \in H} \chi_V(g^{-1}xg) = \sum_{g \in [G/H]} \chi_V(g^{-1}xg) = \mathrm{ind}_H^G(\chi_V)$$

where in the second equality we have extended χ_V to 0 on $G \setminus H$. □

Example 3.16. Let V the standard representation of S_3 with character given by

S_3	$\emptyset$	(12)	(123)
V	2	0	-1

We regard S_3 as the subgroup of S_4 of permutation fixing 4. We can choose the following set of representatives
$$[S_4/S_3] = \{id, (14), (24), (34)\}.$$
Let $\phi = \mathrm{ind}_{S_3}^{S_4}(\chi_V)$. Then

- $\phi(id) = |S_4/S_3|\chi_V(id) = 4 \cdot 2 = 8$
- $\phi((12)) = 0$ since $\chi_V(t) = 0$ for all transpositions t.
- $\phi((123)) = \chi_V((123)) = -1$, because all other conjugates are not in S_3.
- $\phi((1234)) = \phi((12)(34)) = 0$ because all conjugates are not in S_3.

One can easily check that we have

$$\mathrm{ind}_{S_3}^{S_4}(L(\ydiagram{1,2})) = L(\ydiagram{1,3}) \oplus L(\ydiagram{2,2}) \oplus L(\ydiagram{1,1,2})$$

In general, the coinduction is bigger then the induction, but they coincide if G/H is finite.

Exercise 3.17. If G is finite then the map $V \to V$ defined by $v \mapsto \sum_g g \cdot v$ induces a map $V_G \to V^G$. Show that if $\mathrm{char}\, k = 0$, this map is an isomorphism.

Theorem 3.18 (Frobenius reciprocity). *Let N be a H-module and M a G-module. Then, we have natural isomorphisms of abelian groups*

$$\mathrm{Hom}_{kH}(N, \mathrm{res}_G^H(M)) \cong \mathrm{Hom}_{kG}(\mathrm{coind}_H^G(N), M).$$

$$\mathrm{Hom}_{kH}(\mathrm{res}_G^H(M), N) \cong \mathrm{Hom}_{kG}(M, \mathrm{ind}_H^G(N)).$$

This means that induction is right adjoint to restriction, while coinduction is left adjoint to restriction.

Frobenius reciprocity is a consequence of a more general statements about modules of rings.

Theorem 3.19 (Tensor-Hom adjunction)**.** *Let M be a left R-module and N be a left S-module. Let X a (S,R)-bimodule. Then we have a natural isomorphism of abelian groups*

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(X, N)) \cong \mathrm{Hom}_S(X \otimes_R M, N).$$

Proof. Let $\phi : M \to \mathrm{Hom}_S(X, N)$ a morphism of R-modules. This induces a R-balanced map $X \times M \to N$ defined by

$$(x, m) \mapsto \phi(m)(x).$$

So, by the universal property of the tensor product, we obtain a map $\widetilde{\phi} : X \otimes_R M \to N$. It's easy to check that $\widetilde{\phi}$ is a map of S-modules. We claim that $\phi \mapsto \widetilde{\phi}$ gives the desired isomorphism. In fact, we can define an inverse by taking $f : X \otimes_R M \to N$ to $f' : M \to \mathrm{Hom}_S(X, N)$ defined as $f'(m)(x) = f(x \otimes m)$. □

Proof of the Frobenius reciprocity. We can deduce the desired statement since $\mathrm{res}^H_G(M) \cong kG \otimes_{kG} M$ if we think of kG as (kH, kG)-module and $\mathrm{res}^H_G(M) \cong \mathrm{Hom}_{kG}(kG, M)$ where we regard kG as a (kG, kH)-bimodule. □

Example 3.20. Assume we are in characteristic 0. Let $H \subset G$ be finite groups. Let L be an irreducible representation of H and M an irreducible representation of G. Then M occurs inside $\mathrm{ind}^G_H(L)$ if and only if L occurs inside M

In particular, if k is the trivial representation of H, an irreducible representation M of G occurs in $\mathrm{ind}^G_H(k)$ if and only if $M^H \neq 0$.

Example 3.21. Let Y be a Young diagram. Then $M(Y) = \mathrm{ind}^{S_Y}_C(k)$ where C is the column stabilizer and k is the trivial representation of C (cf. Exercise 3.6). It follows that the irreducible representations L which are summands of $M(Y)$ are precisely those for which $L^C \neq 0$.

If $\mathrm{char}\, k = 0$, we can rephrase Frobenius reciprocity in terms of characters. If $H \subset G$, and χ_V is a character of G, we define $\mathrm{res}^H_G(\chi_V)$ to be the restriction of χ_V to H. By definition, we have

$$\chi_{\mathrm{res}^H_G(V)} = \mathrm{res}^H_G(\chi_V).$$

Corollary 3.22. *Let M be a kH-module and N a kG-module, with* $\mathrm{char}\, k = 0$. *Then*

$$\langle \mathrm{ind}^G_H(\chi_M), \chi_N \rangle = \langle \chi_M, \mathrm{res}^H_G(\chi_N) \rangle$$

Proof. Let V and W be representations of a group G, which decompose into simple representations as $V \cong \bigoplus L_i^{a_i}$ and $W \cong \bigoplus L_i^{b_i}$. Then the orthogonality relations tell us that

$$\langle \chi_V, \chi_W \rangle = \sum_i a_i b_i = \dim \mathrm{Hom}(V, W).$$

The corollary follows from Theorem 3.18. □

Remark 3.23. From Frobenius reciprocity we get

$$\mathrm{Hom}_{kH}(N, \mathrm{res}^H_G(\mathrm{coind}^G_H(N))) \cong \mathrm{Hom}_{kG}(\mathrm{coind}^G_H(N), \mathrm{coind}^G_H(N))),$$

hence the identity of $\mathrm{coind}^G_H(N)$ gives a natural morphism

$$N \to \mathrm{res}^H_G(\mathrm{coind}^G_H(N))$$

$$n \mapsto 1 \otimes n$$

Similarly, we get a natural morphisms

$$\text{coind}_H^G(\text{res}_G^H(M)) \to M$$

$$M \to \text{ind}_H^G(\text{res}_G^H(M))$$

$$\text{res}_G^H(\text{ind}_H^G(N)) \to N$$

4 Clifford theory

In this section we assume that G is a finite group and that $N \subset G$ is a normal subgroup. Clifford theory allows us to link the representation theory of G to the representation theory of N using the restriction and induction functors.

Definition 4.1. Let V be a representation of N defined as $\rho : N \to GL(V)$. Then we denote by V^g the representation of $\rho^g : N \to GL(V)$ defined by $\rho^g(n) = \rho(g^{-1}ng)$.

Notice that V is simple if and only if V^g is simple. In fact, if $W \subset V^g$ is a subrepresentation, then also $W^{g^{-1}} \subset V$ is. Then there is a G-action on the set of finite dimensional irreducible representation of $\text{Irr}_k^{f.d.}(N)$ by $g \cdot \rho = \rho^g$. In fact, we have $h \cdot g \cdot \rho = (\rho^g)^h \cong \rho^{hg}$.

Remark 4.2. We have already seen that $\text{ind}_N^G(W)$ decomposes as a direct sum of $g \otimes W$, for $g \in [G/N]$. Recall that $g \otimes W$ is a gNg^{-1}-representation, so a N-representation since N is normal. Moreover, $g \otimes W \cong W^g$ as N-representations, with the isomorphism given by $g \otimes w \mapsto w$. In fact $n(g \otimes w) = g(g^{-1}ng) \otimes w = g \otimes (g^{-1}ng)w$

Theorem 4.3 (Weak form of Clifford's theorem.)**.** *Let V be a finite dimensional simple G-representation over k. Then $\text{res}_G^N(V)$ is semisimple.*

Proof. Since V is finite dimensional, we can find a simple N-subrepresentation $W \subset \text{res}_G^N(V)$. For any $g \in G$ also gW is a subrepresentation of N. In fact, we have

$$n \cdot gW = g(g^{-1}ng) \cdot W = gW.$$

Moreover, gW is simple: in fact it isomorphic to W^g. To show that $\text{res}_G^N(V)$ it remains to show that $V = \sum_g gW$. But $\sum_g gW$ is a G-representation, and since V is simple, we must have $V = \sum_g gW$. □

Theorem 4.4 (Strong form of Clifford's theorem)**.** *Let N be a normal subgroup of G and V a simple G-representation. Let*

$$\text{res}_G^N(V) = S_1^{a_1} \oplus S_2^{a_2} \ldots \oplus S_m^{a_m}$$

the decomposition $\text{res}_G^N(V)$ into isotypical N-representations (i.e. S_i are simple and $S_i \ncong S_j$ if $i \neq j$). Then the following holds:

1. *G acts transitively on the set of N-isotypical components.*
2. *We have $a_1 = a_2 = \ldots = a_m$ and $\dim_k(S_1) = \dim_k(S_2) = \ldots = \dim_k(S_m)$*

Proof. Since $\mathrm{res}_G^N(V)$ is semisimple we can write it as $S_1^{a_1} \oplus S_2^{a_2} \ldots \oplus S_m^{a_m}$. ($S_i^{a_i}$ can be defined as the sum of all the submodules isomorphic to S_i.)

For $g \in G$ we have that $g(S_i^{a_i})$ is a sum of a_i copies of gS_i, which is simple and isomorphic to S_i^g. So we have $gS_i \cong S_j$ for some j, and we deduce $a_i \leq a_j$. Similarly, since $g^{-1}S_j \cong S_i$, we must also have $a_j \leq a_i$, hence $a_i = a_j$. Moreover, $\sum_{g\in G} g(S_i^{a_i})$ is stable under the G-action, so

$$\sum_{g\in G} g(S_i^{a_i}) = S$$

and we see that the action is transitive, i.e. for any j exists $g \in G$ with $gS_i^{a_i} \cong S_j^{a_j}$, so (2) follows. □

If V is a simple representation of G, then the irreducible representation occurring in $\mathrm{res}_G^N(V)$ form a single G-orbit in $\mathrm{Irr}_k(N)$.

If $V \in \mathrm{Irr}_k(N)$, we denote by G_V the subgroup of $g \in G$ for which $V^g \cong V$. Clearly, $N \subset G_V$ for any $V \in \mathrm{Irr}_k(N)$.

Definition 4.5. The group G_V is called the *inertia group* of V.

Let V and $S_1^{a_1}$ be as in Theorem 4.4. Then we know that for any $g \in G$ we have $g(S_1^{a_1}) = S_1^{a_1}$ or $g(S_1^{a_1}) \cap S_1^{a_1} = 0$. In particular, we have $g(S_1^{a_1}) = S_1^{a_1}$ if and only if $S_1 \cong S_1^g$. In particular, the group G_{S_1} is the stabilizer of $S_1^{a_1}$.

Lemma 4.6. *Let V and $S_1^{a_1}$ be as in Theorem 4.4. Then $S_1^{a_1}$ is simple as a representation of $G_1 := G_{S_1}$ and we have*

$$V \cong \mathrm{coind}_{G_1}^G(S_1^{a_1}).$$

Proof. We can define a homomorphism

$$\Phi : \mathrm{coind}_{G_1}^G(S_1^{a_1}) = kG \otimes_{kG_1} S_1^{a_1} \to V$$

$$g \otimes v \mapsto g \cdot v$$

which is a morphism of kG-modules. We have a decomposition

$$kG \otimes_{kG_1} S_1^{a_1} = \bigoplus_{g\in[G/G_1]} g \otimes S_1^{a_1}.$$

Similarly, we have $V = \bigoplus_{g\in[G/G_1]} g(S_1^{a_1})$, since the action of G is transitive and we conclude since Φ induces isomorphisms

$$\Phi : g \otimes S_1^{a_1} \cong g(S_1^{a_1})$$

for any $g \in [G/G_1]$.

This also shows that $S_1^{a_1}$ is simple for G_1. Otherwise, any subrepresentation $W \subset S_1^{a_1}$ would induce a subrepresentation $\mathrm{coind}_{H_1}^N(W)$ of V. □

For $\chi \in \mathrm{Irr}_k(N)$, let G_χ be the stabilizer of χ, i.e. the inertia subgroup of χ. Let $\mathrm{Irr}_k^\chi(G_\chi)$ be the set of isomorphism classes of irreducible representations of G_χ such that when restricted to N decompose as a direct sum of χ.

Definition 4.7. Let

$$Par(G, N) := \{(\chi, W) \mid \chi \in \mathrm{Irr}_k(N),\ W \in \mathrm{Irr}_k^\chi(G_\chi)\}$$

There is an action of G on the set $Par(G,N)$. In fact, for $(\chi, W) \in Par(G,N)$, we can define $g \cdot (\chi, W) := (\chi^g, W^g)$. where W^g is the representation of

$$G_{\chi^g} = \{h \in G \mid \chi^g \cong \chi^{hg}\} = \{h \mid g^{-1}hg \in G_\chi\} = gG_\chi g^{-1}$$

defined by $gxg^{-1} \cdot w = xw$.

Theorem 4.8 (Clifford's correspondence)**.** *We have a bijection*

$$\mathrm{Irr}_k(G) \cong Par(G,N)/G$$

where $Par(G,N)/G$ is the set of G-orbits in $Par(G,N)$.

Proof. We define a map $F : \mathrm{Irr}_k(G) \to Par(G,N)/G$ by sending $V \in \mathrm{Irr}_k(G)$ to $(S_i, S_i^{a_i})$ as in Theorem 4.4. Since all the $(S_i, S_i^{a_i})$ are in the same orbit, this is well defined. We can now construct

$$G : Par(G,N)/G \to \mathrm{Irr}_k(G)$$
$$(\chi, W) \mapsto \mathrm{coind}_{G_\chi}^G(W).$$

We have already seen in Lemma 4.6 that $GF(V) \cong V$ for all $V \in \mathrm{Irr}_k(G)$.

In the other direction, assume $(\chi, W) \in Par(G,N)$. We want to show that $FG(\chi, W) \cong (\chi, W)$. Let $V := \mathrm{coind}_{G_\chi}^G(W)$. It decomposes as direct sum of $g \otimes W$ for $g \in [G/G_\chi]$, so all the components in $\mathrm{res}_G^N(V)$ are of the form χ^g. Let V_χ be the isotypic component of χ in V. Frobenius reciprocity induces a morphism

$$W \to \mathrm{res}_G^{G_\chi}(V) = kG \otimes_{kH} W$$
$$w \mapsto 1 \otimes w$$

which induces an isomorphism $W \to 1 \otimes W = V_\chi$. In fact, $g \otimes W \not\cong W$ for $g \notin G_\chi$. □

4.1 Applications of Clifford's theory: Subgroups of index two

Assume that $N \subset G$ is a subgroup of index 2, i.e. $G/N = \{1, h\}$ has two elements. In this case N is automatically normal.

We can always define a representation ϵ of dimension 1 by pulling back the non-trivial representation of $G/N \cong \mathbb{Z}/2\mathbb{Z}$. In other words,

$$\epsilon(g) = \begin{cases} 1 & \text{if } g \in N \\ -1 & \text{if } g \notin N \end{cases}$$

Proposition 4.9. *Let k be a field of characteristic 0. Let L be an irreducible representation of G. Then we have one of the following possibilities:*

- $L \cong L \otimes \epsilon$ *and* $\mathrm{res}_G^N(L) \cong S \oplus S^h$ *with S irreducible, $S \not\cong S^h$ and $G_S = N$*
- $L \not\cong L \otimes \epsilon$ *and* $\mathrm{res}_G^N(L)$ *is irreducible and $G_S = G$*

Proof. Notice that since $\epsilon(g) = 1$ for all $g \in N$ we have $\mathrm{res}_G^N(L \otimes \epsilon) = \mathrm{res}_G^N(L)$. Moreover, we have $L \cong L \otimes \epsilon$ if and only if their characters are equal, i.e.

$$L \cong L \otimes \epsilon \iff \chi_L(g) = \chi_L(g)\epsilon(g) \text{ for all } g \in G \iff \chi_L(g) = 0 \text{ for all } g \in G \setminus N$$

We have a decomposition

$$\mathrm{res}_G^N(L) = S_1^a \oplus \ldots \oplus S_k^a.$$

Let G_1 the stabilizer of S_1^a. Then we have either $G_1 = N$ or $G_1 = G$.

Assume $G_1 = N$. Then, S_1^a is simple over $G_1 = N$, in particular $a = 1$ and $\mathrm{ind}_N^G(S_1) = L$. Hence, $\dim L = 2\dim S_1$ and $\mathrm{res}_G^N(L) = S_1 \oplus S_1^h$ for $h \notin N$. By Frobenius reciprocity, we also obtain $\mathrm{ind}_N^G(S_1) \cong L \otimes \epsilon$, so we must have $L \cong L \otimes \epsilon$.

Assume now $G_1 = G$, so that $\mathrm{res}_G^N(L) = S^a$. By Frobenius reciprocity we have that L^a is a summand of $\mathrm{ind}_N^G(S)$. So

$$a \dim L \leq 2 \dim S = 2\frac{\dim L}{a},$$

so $a^2 \leq 2$ and $a = 1$

This implies that

$$1 = (\mathrm{res}_G^N(\chi_L), \mathrm{res}_G^N(\chi_L)) = \frac{1}{|N|}\sum_{n\in N} \chi_L(n)\overline{\chi_L(n)} \neq \frac{1}{|N|}\sum_{g\in G} \chi_L(g)\overline{\chi_L(g)} = 2. \qquad (1)$$

From the inequality in (1) follows that exists $g \in G \setminus N$ such that $\chi_L(g) \neq 0$, hence $L \ncong L \otimes \epsilon$. By Frobenius reciprocity, $\mathrm{ind}_N^G(S)$ contains L and $L \otimes \epsilon$ as summands, and by dimension considerations we have

$$\mathrm{ind}_N^G(S) \cong L \oplus (L \otimes \epsilon). \qquad \square$$

So in this case we have a bijection between

$$\mathrm{Irr}_k(N)/(V \sim V^h) \cong Irr_k(G)/(W \sim W \otimes \epsilon)$$

An important example is the alternating subgroup $A_n \subset S_n$. In this case ϵ is the sign representation. Recall that the irreducible representations of S_n, are the Specht modules $L(Y)$, for Y a Young diagram. We have $L(Y) \otimes \epsilon = L(Y^t)$, so $L(Y) \cong L(Y) \otimes \epsilon$ if and only if Y is symmetric with respect to the diagonal. So if Y is symmetric, then $\mathrm{res}_{S_n}^{A_n}(L(Y))$ decomposes as two irreducible representations of A_n, while if Y is not symmetric than $\mathrm{res}_{S_n}^{A_n}(L(Y)) = \mathrm{res}_{S_n}^{A_n}(L(Y^t))$ is irreducible.

Exercise 4.10. Show that the number of symmetric Young diagram for S_n is equal to the number of conjugacy classes in S_n that split into two classes in A_n

4.2 Applications of Clifford theory: Representations of p-groups

Theorem 4.11. *Let G be a group with $|G| = p^a$. Then, every irreducible representation of G is the induction of a representation of dimension 1 of some subgroup $H \subset G$.*

Before we start with the proof, we need the following Lemma about p-groups.

Lemma 4.12. *Let G be a non-abelian p-group. Then G has an abelian normal subgroup A such that $A \not\subset Z(G)$.*

Proof. Recall that any p-group has a non-trivial center. Since G is not abelian, the group $G/Z(G)$ is not trivial. In particular, $Z(G/Z(G)) \neq 0$, so we can take $id \neq x \in G$ such that $xZ(G)$ is central in $G/Z(G)$.

We can construct A as the group generated by $Z(G)$ and x. The group A is abelian, since $Z(G)$ is abelian and commutes with x. Moreover, it is normal: for any $g \in G$ we have $gxg^{-1} \in gZ(G)xZ(G)g^{-1}Z(G) = xZ(G)$, so $gxg^{-1} \in A$. $\square$

Proof. If G is abelian the statement is trivial since all irreducible representations of G have dimension 1. So we can assume $a > 1$ and G not abelian. We show the theorem by induction on a.

Let S be a simple representation of G, and let $\rho_S : G \to GL(S)$ be the corresponding group homomorphism. Let $N = \mathrm{Ker}(\rho_S)$ be the subgroup of elements acting trivially on S. If N is not trivial, then S descends to a representation of G/N and we can conclude by induction. In fact, we have

$$S \cong \mathrm{res}^G_{G/N}(S) \cong \mathrm{res}^G_{G/N}(\mathrm{ind}^{G/N}_{H/N}(V)) \cong \mathrm{ind}^G_H(V),$$

where H is a subgroup containing N and V is a one-dimensional representation of H.

Assume now that N is trivial, i.e. that ρ_S is injective. There exists an abelian normal subgroup A of G which is not central (we show this in Lemma 4.12). By Clifford's theorem, we have a decomposition

$$\mathrm{res}^A_G(S) = S_1^{a_1} \oplus S_2^{a_2} \oplus \ldots \oplus S_n^{a_n}.$$

Let G_1 be the stabilizer of $S_1^{a_1}$, so $S_1^{a_1}$ is a simple G_1-representation and $S = \mathrm{ind}^G_{G_1}(S_1^{a_1})$. If $G_1 \ncong G$, then we conclude by induction, $S_1^{a_1} \cong \mathrm{ind}^{G_1}_H(V)$ for some subgroup H.

If $G_1 = G$, then $\mathrm{res}^A_G(S) \cong S_1^{a_1}$. But A is abelian, so S_1 is a one-dimensional representation of A. This means that on S, the subgroup A simply acts as multiplication by a scalar. In particular $\rho_S(A)$ commute with $\rho_S(G)$ and since ρ_S is injective we have $A \subset Z(G)$, against the assumption. □

Example 4.13. Let V the unique irreducible2-dim representation of D_4. Let $R = \langle r \rangle$ be the subgroup of D_4 of rotations. Then $\mathrm{res}^R_{D_4}(V) = \epsilon_i \oplus \epsilon_{-i}$, where ϵ_a is the rep. of R which sends r to a. Then $\mathrm{ind}^{D_4}_R(\varepsilon_i) \cong \mathrm{ind}^{D_4}_R(\varepsilon_{-i}) \cong V$

5 The Jacobson radical

Let R be a ring, not necessarily commutative. For a R-module M, we say that $x \in R$ acts trivially on M if $r \cdot m = 0$ for all $m \in M$.

Definition 5.1. The *Jacobson radical* $J(R)$ is the subset of elements of R which act trivially on any simple R-module.

The Jacobson radical is the intersection of all the kernels of the ring homomorphisms

$$f : R \to \mathrm{End}_R(M)$$

for M simple, so it is a two-sided ideal.

Lemma 5.2. *The Jacobson radical $J(R)$ is the intersection of all the maximal left ideals of R.*

Proof. All the simple modules are of the form R/I, for I maximal left ideal, so $J(R)$ is contained in the intersection.

On the other direction, for any simple module M and any $0 \neq m \in M$ the kernel of $f : R \to M$, $f(r) = rm$ is a maximal left ideal. So if $a \in R$ is in all left maximal ideals, then a acts trivially on any m. □

Example 5.3. If k is a field, and $R = k[x]/(x^2)$, then $J(R) = (x)$.

If k is of char p, then $kC_p = k[x]/(x^p - 1) = k[x]/(x-1)^p$ and $J(R) = (x-1)$.

An element of a ring R is invertible (or unit) if it has a left and right inverse. If $x \in R$ is invertible, then its left and right inverse coincide.

Lemma 5.4. *An element $x \in R$ is in $J(R)$ if and only if $1 - rxs$ is invertible for all $r, s \in R$.*

Proof. Assume that $1-rxs$ is always invertible and that $x \notin I$ for some maximal left ideal I. Then $Rx + I = R$, so $1 = rx + a$ for some $r \in R$ and $a \in I$. But a cannot be invertible and we get a contradiction.

Let now $x \in J(R)$. Also $xs \in J(R)$. If $1-rxs$ is not left invertible, then it is contained in a maximal left ideal I, which cannot contain xs as $1 = 1 - rxs + r \cdot xs \notin I$. But $xs \in I$ by Lemma 5.2, so $1 - rxs$ is left invertible. Let u be its left inverse, so $u(1-rxs) = 1$. We have $u = 1 - (-ur)(xs)$ is also left invertible, so u is invertible and $(1 - rxs)u = 1$. □

Example 5.5. Let k be a field. Then $J(k[x]) = 0$. In fact, if $1 - f$ is invertible, then f is a constant.

Lemma 5.6 (Nakayama's Lemma)**.** *Let M be a finitely generated R-module. If $J(R)M = M$ then $M = 0$.*

Proof. Assume $M \neq 0$ and let $m_1, \dots, m_n$ be a minimal set of generators. We have $m_n = \sum_i r_i m_i$ with $r_i \in J(R)$. Hence

$$(1 - r_n)m_n = \sum_{i=1}^{n-1} r_i m_i.$$

Since $(1 - r_n)$ is invertible, this contradicts minimality. □

6 Criterium of semisimplicity

Lemma 6.1. *Let R be a ring of finite length as a module over itself. Then $J(R)$ is a nilpotent ideal, i.e. $J(R)^N = 0$ for some N.*

Proof. The series of ideal $R \supset J(R) \subset J(R)^2 \subset \dots$ must be finite. So there exists N such that $J(R)^{N+1} = J(R)^N$ and we can apply Nakayama's lemma. □

In particular, $J(R)$ consists of nilpotent element.

Lemma 6.2. *Let I be a left ideal consisting of nilpotent elements. Then I acts trivially on any simple module. In particular, $J(R)$ is maximal among left ideals consisting of nilpotent elements.*

Proof. Let L be a simple R-module. Assume that $IL \neq 0$. Then there exists $m \in L$ such that $Im \neq 0$. But Im is a R-submodule, so $Im = L$. Hence, there exists $x \in I$ such that $xm = m$. But this contradicts the fact that x is nilpotent. □

Lemma 6.3. *Let R be a ring of finite length as a module over itself. Then $R/J(R)$ is semisimple.*

Proof. Notice that $R/J(R)$ is semisimple as $R/J(R)$-module if and only if it is semisimple as R-module.

Since R is of finite length, $J(R)$ is the intersection of finitely many maximal left ideals (otherwise, we would get an infinite series). So $J(R) = \bigcap_{i=1}^r M_i$. We have an injective morphism of R-modules

$$R/J(R) \hookrightarrow R/M_1 \oplus R/M_2 \oplus \ldots \oplus R/M_r.$$

The RHS is semisimple, and any submodule of a semisimple module is semisimple. □

In particular, if $J(R) = 0$ if and only if R is semisimple. In fact, if R is semisimple we have a decomposition $R = J(R) \oplus M$, then $1 = x + m$ with $x \in J(R)$ and $m \in M$. But $m \in 1 - x$ is invertible by Lemma 5.4. So $M = R$ and $J(R) = 0$.

Consider now a field k and a finite dimensional k-algebra A.

Definition 6.4. We can define a k-bilinear form on A by

$$(a, b)_{tr} = Tr((a\cdot) \circ (b\cdot) : A \to A).$$

We call this the *trace form.*

Clearly, $(a, b)_{tr} = (b, a)_{tr}$ and $(a, b)_{tr} = Tr((ab\cdot))$, so for any $a, x, b \in A$. As usual, we assume A to be associative. It follows that $(ax, b) = (a, xb)$.

Lemma 6.5. *The radical of the trace form is a two-sided ideal containing the Jacobson radical.*

Proof. Let $R(A)$ be the radical. If $a \in R(A)$, then also $(ax, b) = (a, xb) = 0$, so $ax \in R(A)$. Similarly, if $b \in R(A)$ also xb does.

Since A is finite dimensional, it is of finite length. Hence any element in $J(A)$ is nilpotent, and the trace of a nilpotent operator is 0. So $J(A) \subset R(A)$. □

It follows that if the trace form is non-degenerate, i.e. $R(A) = 0$, then A is semisimple. In characteristic 0 also the inverse holds.

Theorem 6.6. *Let A be a finite dimensional k-algebra, where k is a field of char 0. Then $J(A) = R(A)$. In particular, A is semisimple if and only if the trace form is non-degenerate.*

Proof. An endomorphism f of a vector space is nilpotent if and only if $Tr(f^m) = 0$ for all $m \geq 0$ (this is because in char 0, the polynomials $p_i = x_1^i + x_2^i + \ldots + x_n^i$ generate all symmetric functions, so the characteristic polynomial of f must be T^m).

Let now $a \in R(A)$. Then for all $m \geq 0$, we have $(a, a^m)_{tr} = 0$, so the element a is nilpotent. It follows that $R(A)$ is a left ideal of nilpotent elements, so $R(A) \subset J(A)$ by Lemma 6.2 □

Example 6.7. If $A \in M_n(k)$ then $Tr(A\cdot) = nTr(A)$. If A is nilpotent and $A \neq 0$, then we can put A in a triangular form, with 0 on the diagonal and $A_{12} = 1$. Notice that $E_{21}A = E_{11}$ and $Tr(E_{11}) = 1$, so $A \notin R(A)$ and $R(A) = J(A) = 0$.

Consider now the algebra T of upper triangual matrices. If x is nilpotent, and $y \in T$ then also xy is nilpotent. Hence the nilpotent matrices are precisely the elements of $R(T)$.

7 Local rings and Krull–Schmidt Theorem

Let R be a ring, not necessarily commutative.

Definition 7.1. We say that R is *local* if the non-units in R form a two-sided ideal.

If R is commutative then R local if and only if R has a maximal ideal.

Lemma 7.2. *A ring R is local if and only if it has a unique maximal left ideal.*

Proof. Let $N := R \setminus R^{\times}$. Every proper left ideal is contained in N.

In the other direction, if M is the maximal left ideal, then $M = J(R)$ is the Jacobson radical. Recall that $J(R)$ is two-sided. If $x \notin J(R)$, we have $R = Rx + J(R)$, so there exists $r \in R$ and $c \in J(R)$ such that $1 = rx + c$. By Lemma 5.4, $rx = 1 - c \in R^{\times}$, and so x has a left inverse l with $lx = 1$. Now, $l \notin J(R)$, otherwise also $lx \in J(R)$. Hence $R = Rl + J(R)$ and also l has a left inverse y with $yl = 1$. Since $y = y(lx) = (yl)x = x$ we get $lx = xl = 1$ and $x \in R^{\times}$. □

Remark 7.3. If R is local, then $R \setminus R^{\times}$ is the Jacobson ideal of R and $R/J(R)$ is a division ring. In fact, if $x \notin J(R)$, exists $r \in R$ such that $1 = rx + J(R)$.

Lemma 7.4 (Fitting's Lemma)**.** *Let M be an indecomposable R-module with a composition series. Then $f \in \mathrm{End}_R(M)$ is either invertible or nilpotent.*

Proof. We can consider the two series of submodules

$$M \supset f(M) \supset f^2(M) \supset f^3(M) \subset \dots$$

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(f^2) \subset \mathrm{Ker}(f^3)$$

By the Jordan–Hölder theorem, both series can contain only finitely many submodules, hence they both stabilize.

Let N such that $\mathrm{Im}(f^k) = \mathrm{Im}(f^N)$ and $\mathrm{Ker}(f^k) = \mathrm{Ker}(f^N)$ for all $k \geq N$. We claim that $M = \mathrm{Im}(f^N) \oplus \mathrm{Ker}(f^N)$. If $y = f^N(x)$ and $f^N(y) = 0$ then $f^{2N}(x) = 0$. So $x \in \mathrm{Ker}(f^{2N}) = \mathrm{Ker}(f^N)$ and $y = 0$. So the intersection is trivial. Take now $z \in M$. There exists $x \in M$ such that $f^{2N}(x) = f^N(z)$. Then we have

$$z = (z - f^N(x)) + f^N(x) \in \mathrm{Ker}(f^N) + \mathrm{Im}(f^N).$$

Since M is indecomposable then $M = \mathrm{Im}(f^N)$ and $\mathrm{Ker}(f^N) = 0$, or $f^N = 0$. In the first case f is bijective, in the second is nilpotent. □

Corollary 7.5. *Let M be a R-module with a composition series. Then M is indecomposable if and only if $\mathrm{End}_R(M)$ is local.*

Proof. Assume that M is indecomposable. Then, any element in $\mathrm{End}_R(M)$ is either nilpotent or invertible. We show that the set of nilpotent elements N are a two-sided ideal (so $\mathrm{End}_R(M)$ is local).

Let $x, y \in N$. If $r \in R$ then rx and xr cannot be invertible, otherwise also x is. We want to show that $x + y \in N$. If $x + y$ is invertible, then exists $z = (x + y)^{-1}$. But xz and yz are nilpotent, and

$$xz = 1 - yz = (1 + (yz) + (yz)^2 + \ldots + (yz)^N)^{-1},$$

so xz should also be invertible. This is a contradiction! So $x + y \in N$ and N is a two-sided ideal.

Assume now M is decomposable, so we have $M = M_1 \oplus M_2$. Then $id_{M_1}, id_{M_2} \in \mathrm{End}_R(M)$ are not invertible. Since $id_{M_1} = 1 - id_{M_2}$ we get that $id_{M_2} \notin J(\mathrm{End}_R(M))$. It follows that $\mathrm{End}_R(M) \setminus \mathrm{End}_R(M)^\times \neq J(\mathrm{End}_R(M))$, so by Remark 7.3 $\mathrm{End}_R(M)$ is not local. □

Theorem 7.6 (Krull–Schmidt theorem)**.** *Let M be a R-module with a composition series. Assume we can write M as*

$$M \cong M_1 \oplus M_2 \oplus \dots M_k \cong M'_1 \oplus M'_2 \oplus \dots M'_h$$

with M_i and M'_j indecomposable R-modules. Then $k = h$ and there exists a permutation σ such that $M_i \cong M'_{\sigma(i)}$ for any i.

Remark 7.7. If M has a composition series, we can always write M as direct sum of indecomposable modules: If M is indecomposable we are done, otherwise write $M = M_1 \oplus M'_1$ and if M_1 is not indecomposable $M_1 = M_2 \oplus M'_2$, but this has to eventually finish.

Before we prove Krull–Schmidt theorem, we need some preliminary Lemmas.

Lemma 7.8. *Let M, N be R-modules. Then M is a summand of N if and only if there exists $f : M \to N$ and $g : N \to M$ morphism of R-modules such that $gf \in \mathrm{End}_R(M)$ is invertible.*

Proof. Since gf is bijective, then f is injective and $\mathrm{Im}(f) \cong M$. We claim that $N = \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$.

Their intersection is trivial: if $x \in \mathrm{Im}(f) \cap \mathrm{Ker}(g)$, then $x = f(y)$ for some y and $g(x) = gf(y) \neq 0$. Let $\phi = (gf)^{-1}$. Then we can write $x \in N$ as

$$x = f\phi g(x) + (x - f\phi g(x)) \in \mathrm{Im}(f) \oplus \mathrm{Ker}(g).$$

In fact, $g(x - f\phi g(x)) = g(x) - gf(gf)^{-1}g(x) = g(x) - g(x) = 0$. □

Lemma 7.9. *Let M and N be R-modules both having a composition series and such that $\ell(M) = \ell(N)$. Let $f : M \to N$ be an injective morphism of R-modules. Then f is an isomorphism*

Proof. Let $0 \subset M_1 \subset \dots \subset M_n = M$ be a composition series of M with $n = \ell(M)$. Then $f(M_1) \subset \dots \subset f(M_n) \subseteq N$ is also a series of submodules of N, and $f(M_i) \subsetneq f(M_{i+1})$ since f is injective. Since $\ell(N) = n$, we deduce that $f(M_n) = N$ and f is also surjective. □

Proof. Let ϕ_i, ϕ'_j be the inclusion of the summands M_i and M'_j and π_i, π'_j the corresponding projections. Let $\rho_{i,j}$ be the composition

$$M_i \xrightarrow{\phi_i} M \xrightarrow{\pi'_j} M'_j \xrightarrow{\phi'_j} M \xrightarrow{\pi_i} M_i \in \mathrm{End}_R(M_i)$$

We have $\sum_j \rho_{i,j} = Id_{M_i}$. By Fitting's lemma, at least one of the $\rho_{i,j}$ must be invertible, say $\rho_{1,1}$. Notice that this implies that M_1 is a summand of M'_1 by Lemma 7.8, hence $M_1 \cong M'_1$.

One can check that also $(\sum_{j\geq 2} \pi'_j) \circ (\sum_{i \geq 2} \phi_i)$ is also an isomorphism and we can conclude by induction. In fact, they have the same length since $M_1 \cong M'_1$ and it is enough to show injectivity by Lemma 7.9.

Assume that $(\sum_{j\geq 2}\pi'_j)\circ(\sum_{i\geq 2}\phi_i)(0,a_2,\dots,a_n)=0$ for some $a_2\in M_2,\dots,a_n\in M_n$. Then $(\sum_{i\geq 2}\phi_i)(0,a_2,\dots,a_n)$ is contained in the summand M'_1 but the projection to M_1 of $(0,a_2,\dots,a_n)$ is zero. In other words, we have

$$\pi_1\phi'_1\pi'_1\left(\sum_{i\geq 2}\phi_i\right)(0,a_2,\dots,a_n)=\pi_1\left(\sum_{i\geq 2}\phi_i\right)(0,a_2,\dots,a_n)=0,$$

but $\pi_1\phi'_1$ is an isomorphism, hence $\pi'_1\left(\sum_{i\geq 2}\phi_i\right)(0,a_2,\dots,a_n)=0$. □

Example 7.10. Consider $R=M_n(k)$ as a module over itself. Then $M_n(k)$ decomposes as $R=RE_{11}\oplus RE_{22}\dots\oplus\dots RE_{nn}$, where E_{ii} is the matrix with 1 on the ith entry in the diagonal and zero everywhere else. The idempotents E_{ii} are orthogonal to each other and $1=\sum E_{ii}$. The decomposition is not unique.

If $n=2$ then $R=R\begin{pmatrix}0&1\\0&1\end{pmatrix}\oplus R\begin{pmatrix}1&-1\\0&0\end{pmatrix}$. However, Krull–Schmidt theorem assures that the summands in the different decomposition are isomorphic (in this case they are all k^2.

7.1 Representation of a cyclic group in characteristic p

Let $G=C_p=\langle g\mid g^p\rangle$. Then $kG\cong k[x]/(x^p-1)\cong k[y]/(y^p)$ where $y=x-1$. So a kG-module M is the same as a vector space M together with an endomorphism $f:M\to M$ such that $f^p=0$. We can put f in the Jordan form. Then all the Jordan blocks of f have eigenvalue 0. The condition $y^p=0$ implies that all the blocks have dimension at most n. The decomposition into Jordan blocks induces a decomposition of M into submodules.

So the indecomposable kG-module are $V_1,\dots,V_p$. On V_k, g acts as the Jordan block of dimension k

$$\begin{pmatrix}1&1&&&\\&1&1&&\\&&\ddots&\ddots&\\&&&1&1\\&&&&1\end{pmatrix}$$

Thanks to the Krull–Schmidt theorem, we know that every kG-module can be written in a unique way as a sum of the V_i. Notice that we have morphism $V_i\to V_j$ and $V_j\to V_i$ even when $i\neq j$.

Remark 7.11. Cyclic groups are the only p-groups for which we have finitely many classes of indecomposable. In fact, if G is not cyclic, it contains a subgroup isomorphic to $C_p\times C_p$. A representation of $C_p\times C_p$ is basically a pair of commuting endomorphism f,g such that $f^p=g^p=0$.

For example, we can define an indecomposable representation as follows. Let V_{2n+1} be a vector space with basis $v_0,\dots,v_n,w_1,\dots,w_n$ and let $f(w_i)=v_{i-1}$, $f(v_i)=0$, $g(w_i)=v_i$ and $g(v_i)=0$. Then $f^2=g^2=0$ and $fg=gf=0$.

Exercise 7.12. Show that V_{2n+1} is an indecomposable representation of $C_p\times C_p$

8 Projective modules and idempotents

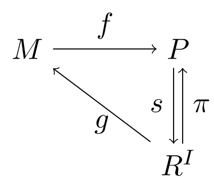
Let R be a ring.

Definition 8.1. A R-module P is said *projective* if it is a summand of a free R-module.

Proposition 8.2. *A R-module P is projective if and only if every surjective morphism $f : M \to P$ splits, i.e. there exists $g : P \to M$ such that $fg = Id_P$.*

Proof. Assume that every surjective map to P splits. There exists a free module R^k and a surjective map $\pi : R^I \to P$ (just take $\{\pi(x_i)\}_{i \in I}$ to be a set of generators of P). Then π splits, so P is projective.

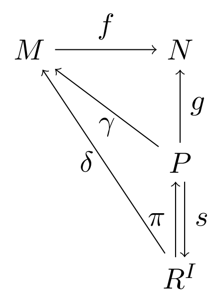
Assume now that P is projective, so exists a free module R^I such that $R^I \cong P \oplus P'$. Let $\pi : R^I \to P$ and $s : P \to R^I$ be the projection and the inclusion of P. Let $f : M \to P$ be a surjective morphism.



Then we can define a morphism $g : R^I \to M$ making the diagram commute. If $\{x_i\}_{i \in I}$ is a basis of R^I, then we define $g(x_i) = m_i$ with $m_i \in f^{-1}(\pi(x))$. We have $fg(x_i) = f(m_i) = \pi(x_i)$, so $fg = \pi$. Now we can find a section s' of f by defining $s'" = fgs$. In fact, $s'(p) = fgs(p) = \pi s(p) = p$. □

Lemma 8.3. *Let M, N, P be R-modules with P projective. Let $f : M \to N$ and $g : P \to N$ be morphisms, with f surjective. Then there exists $\gamma : P \to M$ such that $f\gamma = g$*

Proof. Since P is a summand of a free module R^I with basis $\{x_i\}_{i \in I}$, we have maps $\pi : R^I \to P$ and $s : P \to R^I$ and we can define a morphism $\delta : R^I \to M$ by $\delta(x_i) = m_i$ with $m_i \in f^{-1}(g(\pi(x_i)))$.



So we can define $\gamma : P \to M$ as $\gamma := \delta s$. For $p \in P$ we have

$$f\gamma(p) = f\delta s(p) = g\pi s(p) = g(p).$$ □

Corollary 8.4. *If P is projective, the functor* $\operatorname{Hom}(P, -)$ *is exact. In other words, if $N \subset M$ is a submodule, then*

$$\operatorname{Hom}(P, M/N) \cong \operatorname{Hom}(P, M)/\operatorname{Hom}(P, N)$$

Proof. Composition induces two maps

$$\operatorname{Hom}(P, N) \xrightarrow{f} \operatorname{Hom}(P, M) \xrightarrow{g} \operatorname{Hom}(P, M/N) \qquad (2)$$

Clearly f is injectve, and g is surjective by Lemma 8.3. Moreover, $\phi \in \operatorname{Hom}(P, M)$ belongs to $\operatorname{Ker}(g)$ if and only if $\operatorname{Im}(\phi) \subset N$, i.e. if $\phi \in \operatorname{Im}(f)$. It follows that $\operatorname{Hom}(P, M/N) \cong \operatorname{Hom}(P, M)/\operatorname{Hom}(P, N)$. □

A sequence as in Equation (2), where f is injective, g is surjective and $\mathrm{Ker}(g) = \mathrm{Im}(f)$ is called a *short exact sequence*. An *exact functor* is a functor which sends short exact sequences to short exact sequences.

Definition 8.5. An element $0 \neq e \in R$ is called *idempotent* if $e^2 = e$.

Two idempotents e_1, e_2 are called *orthogonal* if $e_1 e_2 = 0 = e_2 e_1$.

Remark 8.6. $1 \in R$ is an idempotent. If $e \in R$ is idempotent, then $(1-e)$ is an idempotent orthogonal to e.

If e_1 and e_2 are orthogonal, then also $e_1 + e_2$ is an idempotent.

Lemma 8.7. *Let $e \in R$ be an idempotent. Then Re is a projective module. All the projective modules which are summand of R are of this form.*

Proof. We have $R = Re \oplus R(1-e)$. In fact $1 = e + (1-e) \in Re + R(1-e)$ and if $x \in Re \cap R(1-e)$ then $x = xe = xe(1-e) = 0$. So Re is projective.

If P is a summand of R, then $R = P \oplus P'$ for some module P'. We have $1 = e + e'$ with $e \in P$ and $e' \in P'$. We have $P = Re$. In fact, if $p \in P$ then $p = pe + pe'$, so $p = pe \in Re$ and pe'. In particular, $e = e^2$ so e is an idempotent. □

Proposition 8.8. *Let R be a ring. Then we have a bijection*

$$\left\{ \begin{array}{c} \textit{sets of orthogonal} \\ \textit{idempotents } \{e_1, \ldots, e_k\} \\ \textit{with } e_1 + \ldots + e_k = 1 \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{decompositions of } R\textit{-modules} \\ R \cong L_1 \oplus \ldots \oplus L_k \end{array} \right\}$$

$$\{e_1, \ldots, e_k\} \mapsto R \cong Re_1 \oplus \ldots \oplus Re_k$$

Proof. Let $\{e_1, \ldots, e_k\}$ be a set of orthogonal idempotents with $\sum e_i = 1$. Then

$$Re = Re_1 + \ldots Re_k \ni 1.$$

We need to show that the sum is direct. If $Re_1 \cap (\sum_{i>2} Re_i) \neq 0$ we have

$$x_1 e_1 = \sum_{i>1} x_i e_i$$

then multiplying by e_1 on the right we get $x_1 e_1 = 0$.

The map is injective: if $Re_1 \oplus \ldots \oplus Re_k$ and $Re'_1 \oplus \ldots \oplus Re'_k$ are the same decomposition, then $Re_1 = Re'_j$ for some j. In particular,

$$\sum_i e'_i e_1 = 1 \cdot e_1 = e_1 = e_1 \cdot 1 = \sum_i e_1 e'_i$$

and $e_1 e'_i \in Re'_i$. Since $e_1 \in Re'_j$, we must have $e_1 e'_i = 0$ if $i \neq j$ and $e_1 = e_1 e'_j$. Similarly, we have $e'_i e_1 = 0$ if $i \neq j$ and $e'_j e_1 = e'_j$. It follows that $e_1 = e'_j$.

It remains to prove the surjectivity. Consider a decomposition

$$R = L_1 \oplus \ldots \oplus L_k.$$

We can write $1 = \sum e_i$ with $e_i \in L_i$. Then $e_i = e_i \cdot 1 = \sum e_i e_j \in L_i$ and $e_i e_j \in L_j$. It follows that $e_i e_j = 0$ if $i \neq j$ and $e_i^2 = e_i$. So $\{e_1, \ldots, e_k\}$ are orthogonal idempotents. □

Corollary 8.9. *Let $e \in R$ and idempotent and $L = Re$. Then we have a bijection*

$$\left\{ \begin{array}{c} \textit{sets of orthogonal} \\ \textit{idempotents } \{e_1, \ldots, e_k\} \\ \textit{with } e_1 + \ldots + e_k = e \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{decompositions} \\ L \cong L_1 \oplus \ldots \oplus L_k \end{array} \right\}$$

$$\{e_1, \ldots, e_k\} \mapsto Re \cong Re_1 \oplus \ldots \oplus Re_k$$

Proof. If $\{e_1, \ldots, e_k\}$ are orthogonal idempotents with $\sum e_i = e$, then

$$e_i(1-e) = e_i(1 - \sum_i e_i) = e_i - e_i^2 = 0,$$

so also $\{e_1, \ldots, e_k, 1-e\}$ are orthogonal.

On the other hand, we have $R = Re \oplus R(1-e)$, so if $L_1 \oplus \ldots \oplus L_{k+1}$ is a decomposition of Re, then $L_1 \oplus \ldots \oplus L_{k+1} \oplus R(1-e)$ is a decomposition of R.

So we conclude by Proposition 8.8. □

Definition 8.10. An idempotent is called *primitive* if it is not the sum of two orthogonal idempotents.

An idempotent is called *central* if $e \in Z(R)$.

Corollary 8.11. *The following are equivalent:*

1. *The module Re is indecomposable*
2. *e is primitive*
3. *The unique idempotent of eRe is e.*

Proof. We have already seen in Corollary 8.9 that 1 and 2 are equivalent.

Recall that $\mathrm{Hom}_R(Re, Re) \cong (eRe)^{op}$. If $Re = M \oplus M'$, then Id_M and $Id_{M'}$ gives two idempotents of $(eRe)^{op}$, so (3) $\implies$ (1).

Viceversa, if $e \neq e' \in eRe$, so $e' = exe$ for some x, we have $e'e = exe = ee'$, so e' and $e - e'$ are orthogonal idempotents and $Re = Re' \oplus R(e-e')$. It follows that (1) $\implies$ (3). □

9 Radical of a module

Let R be a ring and M be a R-module.

Definition 9.1. The *radical* $\mathrm{rad}(M)$ of M is the intersection of all the maximal submodules of M.

Example 9.2. If R is regarded as a module over itself, we have $\mathrm{rad}(R) = J(R)$.

Proposition 9.3. *Let R be a ring of finite length over itself and M an R-module. We have $\mathrm{rad}(M) = J(R)M$ and $\mathrm{rad}(M)$ is the smallest submodule such that $M/\mathrm{rad}(M)$ is semisimple.*

Proof. $M/J(R)M$ is a module over $R/J(R)$. Recall by Lemma 6.3 that $R/J(R)$ is semisimple, so $M/J(R)M$ is also semisimple.

Assume that N is a submodule of M such that M/N is semisimple. Recall that by Definition 5.1, $J(R)$ acts trivially on simple modules, so acts trivially on M/N. Hence $J(R)M \subset N$. It follows that $J(R)M$ is the smallest submodule such that $M/J(R)M$ is semisimple.

It remains to show that $J(R)M = \operatorname{rad}(M)$. Let N be a maximal submodule of M. Then M/N is simple, hence $J(R)M \subset N$. So $J(R)M$ is contained in all the maximal submodules, hence $J(R)M \subset \operatorname{rad}(M)$. On the other hand, since $M/J(R)M$ is semisimple, the intersection of maximal submodules of $M/J(R)M$ is zero, so $J(R)M = \operatorname{rad}(M)$. □

Definition 9.4. The *head* $\operatorname{hd}(M)$ of a module M is $M/\operatorname{rad}(M)$, the largest semisimple quotient of M.

Example 9.5. Let G be a p-group and k be a field of characteristic p. Then $J(kG) = IG = \{\sum a_g g \mid \sum a_g = 0\}$. In fact, k is the unique simple kG-module and $J(kG) = IG$ is the set of elements acting trivially on k.

Moreover, kG is indecomposable as a module over itself. If $kG \cong M \oplus N$ with $M, N \neq 0$, then $J(kG) = \operatorname{rad}(M) \oplus \operatorname{rad}(N)$. But this would imply that $\dim J(kG) \leq \dim kG - 2$.

In particular, if G is C_p, then $kG \cong V_p$ (cf. Section 7.1).

10 Essential morphisms and projective covers

Definition 10.1. A morphism $f : M \to N$ is *essential* if it is surjective and for any proper submodule $M' \subset M$ we have $f(M') \neq N$.

A *projective cover* of M is an essential morphism $f : P \to M$ with P projective.

Lemma 10.2. *Let M be a finitely generated R-module. If R is of finite length, then $M \to M/\operatorname{rad}(M)$ is essential.*

Proof. In this case, we have $\operatorname{rad}(M) = J(R)M$ by Proposition 9.3. Assume there is a submodule $N \subset M$ such that $N + J(R)M = M$. By Nakayama's Lemma (Lemma 5.6) we have

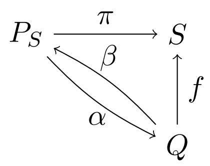
$$J(R) \cdot M/N = (J(R)M + N)/N = M/N \implies M/N = 0$$

and so $M = N$. □

Proposition 10.3. *Let $\pi : P_S \to S$ be a projective cover of S. Assume that $f : Q \to S$ is a surjective morphism with Q projective. Then P_S is a summand of Q and f factors through the projection on P_S.*

In particular, a projective cover, if exists, it is unique up to isomorphism.

Proof. By Lemma 8.3, there exists $\beta : Q \to P_S$ such that $\pi\beta = f$ and $\alpha : P_S \to Q$ such that $f\alpha = \pi$.



We have $\pi\beta\alpha = \pi$ and since π is essential we have $\beta\alpha$ is surjective, so also β is surjective. Since P_S is projective β splits and P_S is a summand of Q.

Assume now that $f = \pi\beta$ is also essential. Since $\pi\beta$ is already surjective when restricted to the summand of Q isomorphic to P_S, we have $Q \cong P_S$. □

Remark 10.4. Projective covers are unique up to isomorphism, but this isomorphism is not unique. This is not an universal property!

In the rest of this section, we assume that A is a finite dimensional k-algebra. In this case, projective covers always exist and indecomposable projectives are in bijection with the simple modules.

Lemma 10.5. *Let A be a finite dimensional algebra over a field k. Let P be a finitely generated projective A-module. Then P is indecomposable if and only if $P/\operatorname{rad}(P)$ is simple.*

Proof. Since A is finite dimensional we have $\operatorname{rad}(P) = J(A)P$. If $P = P_1 \oplus P_2$, then $\operatorname{rad}(P) = J(A)P = J(A)(P_1 \oplus P_2) = \operatorname{rad}(P_1) \oplus \operatorname{rad}(P_2)$. It follows that $P/\operatorname{rad}(P) \cong P_1/\operatorname{rad}(P_1) \oplus P_2/\operatorname{rad}(P_2)$ is not simple.

Assume now that P is indecomposable. Since $P/\operatorname{rad}(P)$ is semisimple, it suffices to show that $P/\operatorname{rad}(P)$ is indecomposable, or equivalently, that $\operatorname{End}_A(P/\operatorname{rad}(P))$ is local.

Let $\phi : P \to P$ be a morphism. Then $\phi(\operatorname{rad}(P)) = \phi(J(A)P) = J(A)\phi(P) \subset \operatorname{rad}(P)$. So ϕ induces a morphism $\overline{\phi} : P/\operatorname{rad}(P) \to P/\operatorname{rad}(P)$ and the map $\theta : \operatorname{End}_A(P) \to \operatorname{End}_A(P/\operatorname{rad}(P))$ defined by $\theta(\phi) = \overline{\phi}$ is a morphism of k-algebras. Moreover, θ is surjective. In fact, if $\psi : P/\operatorname{rad}(P) \to P/\operatorname{rad}(P)$, then by Lemma 8.3 we find $\phi : P \to P$ such that the following diagram commute

$$\begin{array}{ccc} P & \xrightarrow{\quad\phi\quad} & P \\ \pi\big\downarrow & & \big\downarrow\pi \\ P/\operatorname{rad}(P) & \xrightarrow{\ \psi\ } & P/\operatorname{rad}(P) \end{array}$$

so that $\theta(\phi) = \psi$. Since P is finitely generated A-module, it is finite dimensional over k, so it has a composition series. By Lemma 7.4, any $\phi \in \operatorname{End}_A(P)$ is either nilpotent or invertible. Let $\psi \in \operatorname{End}_A(P/\operatorname{rad}(P))$ and let $\phi \in \operatorname{End}_A(P)$ such that $\theta(\phi) = \psi$. Then ψ is invertible if ϕ is, and is nilpotent if ϕ is. So all the element of $\operatorname{End}_A(P/\operatorname{rad}(P))$ are invertible or nilpotent, hence by Corollary 7.5, the ring $\operatorname{End}_A(P/\operatorname{rad}(P))$ is local. □

Proposition 10.6. *Let A be a finite dimensional algebra over a field k. Let S be a simple A-module. Then*

1. *there exists a projective cover P_S of S such that S is the only simple quotient of P_S, i.e. we have $P_S/\operatorname{rad}(P_S) \cong S$,*
2. *we have $P_S = Af$ for some idempotent $f \in A$*
3. *we have $fS \neq 0$ and $fT = 0$ for all T simple modules not isomorphic to S.*

Proof. We can decompose $A = P_1 \oplus \ldots \oplus P_k$ into indecomposable projective A-module. Since S is simple, for any $0 \neq s \in S$, the morphism $\Phi : A \to S$ defined by $\Phi(a) = as$ is surjective. Moreover, we have $\Phi(P_i) = S$ for some i. So there exists a surjective morphism $\Phi : P_i \to S$ with P_i indecomposable projective. Moreover $\Phi(\operatorname{rad}(P_i)) = J(A)S = 0$ because the Jacobson radical acts trivially on simple modules. So Φ factors through $P_i/\operatorname{rad}(P_i)$. Notice that P_i is finitely generated, so by Lemma 10.5, we have that $P_i/\operatorname{rad}(P_i)$ is simple, so $\Phi : P_i/\operatorname{rad}(P_i) \cong S$ is an isomorphism. Since $P_i \to P_i/\operatorname{rad}(P_i)$ is a projective cover, we deduce that P_i is the projective cover P_S of S. This shows the first part.

Since $P_S \cong P_i$ is a summand of A, we have $P_S = Af$ for some idempotent $f \in A$. Moreover, we have $Afs = S$, so $fs \neq 0$. If T is a simple module, and $ft \neq 0$ for some $t \in T$, then $af \mapsto aft$ defines a surjective morphism $\phi : P_S \to T$. Since $\operatorname{rad}(P_S) = J(A)P_S$, we have $\phi(J(A)P_S) = J(A)T = 0$. Hence, ϕ factors through $P_S/\operatorname{rad}(P_S)$, and it induces a surjective morphism $S \cong P_S/\operatorname{rad}(P_S) \cong T$. It follows that $S \cong T$. □

Theorem 10.7. *Let A be a finite dimensional algebra over a field k. We have a bijection*

$$\Psi : \left\{ \begin{matrix} \textit{projective indecomposable} \\ A\textit{-modules} \end{matrix} \right\}_{/\cong} \xrightarrow{\sim} \left\{ \begin{matrix} \textit{simple} \\ A\textit{-modules} \end{matrix} \right\}_{/\cong}$$
$$P \mapsto P/\operatorname{rad}(P)$$

Moreover, all projective indecomposable A-modules are summand of A and we have a decomposition of A-modules

$$A \cong \bigoplus_{S \in \operatorname{Irr}_k(A)} P_S^{\dim S}.$$

Proof. Let P be an indecomposable projective A-module. Notice that we cannot apply directly Lemma 10.5, because we do not know a fortiori that P is finitely generated.

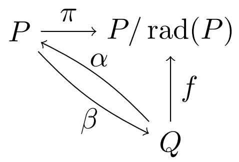
Since A is finite dimensional we have $\operatorname{rad}(P) = J(A)P$. Then $P/\operatorname{rad}(P)$ is a module over the semisimple ring $A/J(A)$, so it splits into simple modules. In particular, it has a simple quotient S. Let now P_S be the projective cover of S. It follows by Proposition 10.3, that P_S is a summand of P. Since P is indecomposable, we have $P \cong P_S$. Then $P/\operatorname{rad}(P) \cong S$. It follows that Ψ is well-defined. Moreover, it is injective by Proposition 10.3 and surjective by Proposition 10.6.

Moreover, if P is indecomposable then $P \cong P_S$ for $S \cong P/\operatorname{rad}(P)$, so it is a summand of A by Proposition 10.6. Recall that $A/J(A)$ is semisimple, so by the Artin–Wedderburn theorem we have

$$A/J(A) \cong \bigoplus_{S \text{ simple}} S^{\dim_k S}.$$

Both A and $\bigoplus_{S \text{ simple}} P_S^{\dim_k S}$ are projective cover of $A/J(A)$, so they are isomorphic. □

Remark 10.8. With a similar argument, one can show that any projective module P is a direct sum of indecomposable projectives. In fact, if $P/\operatorname{rad}(P) \cong \bigoplus_{i \in I} S_i$, consider the module $Q := \bigoplus_{i \in I} P_{S_i}$. We have a surjective morphism $f : Q \to P/\operatorname{rad}(P)$, and we can find lifts $\alpha : Q \to P$ and $\beta : P \to Q$ as in the following commutative diagram.



Now, $\Psi := \operatorname{Id}_P - \alpha\beta \in \operatorname{End}_A(P)$ and for any $x \in P$ we have

$$\pi(\Psi(x)) = \pi(x - \alpha\beta(x)) = \pi(x) - f\beta(x) = \pi(x) - \pi(x) = 0,$$

so $\operatorname{Im}(\Psi) \subset \operatorname{rad}(P) = J(A)P$. So $\operatorname{Im}(\Psi^N) = \Psi(\operatorname{Im}(\Psi^{N-1}) \subset \Psi(J(A)^{N-1}P) = J(A)^N P$ by induction, so Ψ is nilpotent. It follows that and $\alpha\beta = (\operatorname{Id}_P - \Psi) = (\operatorname{Id}_P + \Psi + \Psi^2 + \ldots)^{-1}$ is an automorphism of P.

We have $\operatorname{rad}(Q) = J(A)Q = \bigoplus_i \operatorname{rad}(P_{S_i})$. Similarly, we have $\operatorname{Im}(\operatorname{Id}_Q - \beta\alpha) \subset \operatorname{rad}(Q)$, so $\beta\alpha$ is also an automorphism of Q.

It follows that α is both surjective and injective, so $P \cong Q$.

11 Projectives for the group algebra

In this section we specialize to the case $A = kG$ and discuss some examples.

Example 11.1. Let G be a finite p-group and k be a field of characteristic p. Then the trivial representation is the unique simple kG-module. It follows that kG is the unique indecomposable projective module. In particular, the only idempotent in kG is 1 and all projective kG-modules are free.

Example 11.2. Let k be an algebraically closed field of characteristic 2, for example $k = \overline{\mathbb{F}_2}$ (but also $k = \mathbb{F}_4$ would work.) We want to find the indecomposable projective kS_3-modules. Let N be the subgroup generated by a 3-cycle. Then kN is semisimple, and we have $kN \cong k^3$. More precisely, we have

$$kN = kNe_1 \oplus kNe_2 \oplus kNe_3$$

with

$$e_1 = 1 + (123) + (132),\ e_2 = 1 + \omega(123) + \omega^2(132) \text{ and } e_3 = 1 + \omega^2(123) + \omega(132),$$

where $\omega \in k$ is a primitive third root of unity. (These can be obtained via the inverse Fourier transform: notice that $|N|^{-1} = 1$ in k).

So $kS_3 = kS_3e_1 \oplus kS_3e_2 \oplus kS_3e_3$ is a decomposition of kS_3 into projectives. Since $S_3 = N \cup (12)N$ we have $kS_3e_i = kNe_i + k(12)Ne_i = ke_i + k(12)e_i$, and $\dim_k(kS_3e_i) = 2$ for all i. One can show that $kS_3e_2 \cong kS_3e_3$. In fact,

$$\mathrm{Hom}_{kS_3}(kS_3e_2, kS_3e_3) = e_2(kS_3)e_3$$

Since $e_2 = (12)e_3(12)$, we have $(12)e_3 = (12)e_3(12)(12)e_3 \in e_2(kS_3)e_3$, so we have a map $f : kS_3e_2 \to kS_3e_3$ which sends xe_2 to $x(12)e_3$, and f is an isomorphism. (If $f(xe_2) = x(12)e_3 = 0$, then also $x(12)e_3(12) = xe_2 = 0$).

One-dimensional representation of S_3 factor through $S_3/N \cong C_2$. So the trivial representation is the only irreducible representation of kS_3 of dimension one. Since the projective cover of k can only occur once in kS_3, we have the kS_3e_1 is the projective cover of k, while $kS_3e_2 \cong kS_3e_3$ are irreducible. There are two simple representations of S_3 over k: the trivial representation and a simple representation of dimension 2 (constructed as in characteristic 0).

In kS_3e_1 we have the submodule $k(\sum_{g\in S_3} g) = k((12) + 1)e_1$, and its quotient is again isomorphic to the trivial representation.

Example 11.3. Let G be the direct product $G = H \times N$, where H is a p-group and $|N|$ is not divisible by p. Then $kG \cong kH \otimes_k kN$. Since kN is semisimple, we have

$$kN \cong \bigoplus_S S^{\dim S}$$

where the sum runs over all simple kN-modules S. On the other hand, kH is indecomposable, so

$$kG \cong \bigoplus_S (kH \otimes_k S)^{\dim S}.$$

So $kH \otimes S$ is a projective kG-module. Moreover, each $kH \otimes S$ occurs $\dim S$-times, and so they are indecomposable with each $kH \otimes S$ being the projective cover of the simple module $k \otimes S$.

Projective covers are useful because they can be use to compute the composition factor of a module.

Let M be an A-module with a composition series and S a simple A-module. We denote by $[M : S]$ the number of times S occurs as a factor in a composition series of M. From the Jordan–Hölder theorem we know that $[M : S]$ is well defined.

Proposition 11.4. *Let $\pi : P_S \to S$ be a projective cover of S. Let M be an A-module of finite length. Then*

$$\dim_k \operatorname{Hom}_A(P_S, M) = [M : S] \dim_k \operatorname{End}_A(S).$$

In particular, if k is algebraically closed we have $\dim_k \operatorname{Hom}_A(P_S, M) = [M : S]$.

Proof. The proof is by induction on the length $\ell(M)$. If $\ell(M) = 1$ then M is simple. Then $f : P_S \to M$ factors through $P_S/\operatorname{rad}(P_S) \cong S$ and f is either an isomorphism if $M \cong S$ or 0.

Assume now the statements for modules N with $\ell(N) < \ell(M)$. We have a composition series $0 \subset M_1 \subset \ldots \subset M_{\ell-1} \subset M_\ell = M$. Then by Corollary 8.4 we have

$$\operatorname{Hom}_A(P_S, M)/\operatorname{Hom}_A(P_S, M_{\ell-1}) \cong \operatorname{Hom}_A(P_S, M/M_{\ell-1})$$

Since $\ell(M_{\ell-1}) < \ell(M)$ and $M/M_{\ell-1}$ is simple, the first claim follows by induction. The second claim follows from Schur's lemma. □

Definition 11.5. For S, T simple modules, we define

$$c_{ST} := [P_T : S] = \dim_k \operatorname{Hom}_A(P_S, P_T)/\dim_k \operatorname{End}_A(S).$$

The matrix $(c_{ST})_{S,T}$ is called the *Cartan matrix* of G

Example 11.6. If G is a p-group, then the Cartan matrix of G has a single entry, which is $|G|$.

Example 11.7. If $G = S_3$ and k is of characteristic 2, then there are two classes of simple kG-modules: the trivial module and kS_3e_2. The Cartan matrix is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 11.8. *Let S, T be simple modules of kG. Assume that k is splitting for G. Then we have*

$$\dim_k \operatorname{Hom}_{kG}(P_S, P_T) = \dim_k \operatorname{Hom}_{kG}(P_T, P_S).$$

In particular, if k is a splitting field for G. Then for any S, T we have $c_{ST} = c_{TS}$. In other words, the Cartan matrix fo G is symmetric.

Proof. We postpone the proof. □

Let k be a field of characteristic p. Let G be a finite group.

Lemma 11.9. *Let H be a subgroup of G. Then res^H_G and ind^G_H preserve projectives.*

Proof. If P is a projective kG-module, then exists I with $kG^I \cong P \oplus P'$. Then $\operatorname{res}^H_G(kG) \cong \bigoplus_{g \in [G/H]} kHg$ is a free kH-module, and so is $\operatorname{res}^H_G(kG^I)$. So $\operatorname{res}^H_G(P)$ is a summand of a free module and is projective.

If Q is a projective kH-module, then Q is a summand of kH^I. Since $\operatorname{ind}^G_H(kH) \cong kG \otimes_{kH} kH \cong kG$ is free, also $\operatorname{ind}^G_H(Q)$ is a summand of a free module. □

Proposition 11.10. *If p^a divides $|G|$, then p^a divides $\dim_k P$, for any projective P.*

Proof. Let H be the p-Sylow subgroup of G. If P is projective, then $\mathrm{res}_G^H(P)$ is free, so $P \cong (kH)^n$ and $\dim P = p^a n$. $\square$

Definition 11.11. We denote by $O_p(G)$ the unique maximal normal p-subgroup of G.

Notice that $O_p(G)$ always exists. In fact, if H and K are normal p-subgroups, then also HK is a normal p-group.

Lemma 11.12. *The irreducible representations of G over k are in correspondence with the irreducible representations of $G/O_p(G)$, with the bijection given by the pullback.*

Proof. Let S be a simple kG-module. Then $\mathrm{res}_G^{O_p(G)}(S)$ is also semi-simple by Clifford's theorem. Since the trivial module is the only simple $kO_p(G)$-module, it follows that $O_p(G)$ acts trivially on S. Hence S factors to a simple representation of $G/O_p(G)$. Vice versa, the pullback of a simple $G/O_p(G)$-module always give a simple G-module. $\square$

Example 11.13. In this example we study representation of the group A_4 in characteristic p. If $p \neq 2, 3$, then kA_4 is semisimple and the representation theory is the same as in characteristic 0.

Assume $\mathbf{p = 2}$. Let k be a splitting field of A_4 of characteristic 2 (for example, we can take $k = \mathbb{F}_4$ or $k = \overline{\mathbb{F}_2}$.) In this case the 2-Sylow subgroup, given by the Klein subgroup $K = \{id, (12)(34), (13)(24), (14)(23)\}$ is normal and the quotient A_4/K is a cyclic group of order 3. By Lemma 11.12, we know that the simple kA_4-module can be obtained as the pullback of the simple $kA_4/K \cong kC_3$-modules. The algebra kC_3 is semisimple. In particular, there are three simple modules all of dimension one. We call them k_1, k_ω and k_{ω^2}, where $\omega \in k$ is a third root of unity, where k_i is the module on which (123) acts as i.

We also have a section of $A_4 \to A_4/K$, i.e. there is a subgroup $C_3 = \langle (123) \rangle$ of A_4 such that the composition

$$C_3 \hookrightarrow A_4 \twoheadrightarrow A_4/K$$

is an isomorphism (this happens because A_4 is the *semidirect product* of K and C_3). Since k_1, k_ω and k_{ω^2} are simple kC_3-modules and kC_3 is semisimple, they are also indecomposable projective kC_3-modules. By Theorem 10.7, also $P_1 := \mathrm{ind}_{C_3}^{A_4}(k_1)$, $P_\omega := \mathrm{ind}_{C_3}^{A_4}(k_\omega)$ and $P_{\omega^2} := \mathrm{ind}_{C_3}^{A_4}(k_{\omega^2})$ are projective. Moreover, by Frobenius reciprocity we have

$$\dim_k \mathrm{Hom}_{kA_4}(P_1, k_1) \cong \dim_k \mathrm{Hom}_{kC_3}(k_1, k_1) = 1$$

and similarly $\mathrm{Hom}_{kA_4}(P_1, k_\omega) = \mathrm{Hom}_{kA_4}(P_1, k_{\omega^2}) = 0$. So, after decomposing P_1 into indecomposable projective modules, which are the projective cover of k_1, k_ω and k_{ω^2} by Theorem 10.7. we see that P_1 is the projective cover of k_1. Similarly, P_ω is the projective cover of k_ω and P_{ω^2} is the projective cover of k_{ω^2}. We have

$$kA_4 \cong P_1 \oplus P_\omega \oplus P_{\omega^2}.$$

We compute now the Cartan matrix. By Frobenius reciprocity, we have

$$\dim_k \mathrm{Hom}_{kA_4}(P_1, P_1) \cong \dim_k \mathrm{Hom}_{kC_3}(k_1, \mathrm{res}_{A_4}^{C_3} P_1)$$

Since $|C_3|$ is coprime with 2, we can easily compute $\mathrm{res}_{A_4}^{C_3}(P_1)$ by looking at the characters. For $g \in C_3$ we have

$$\chi_{\mathrm{res}_{A_4}^{C_3}(P_1)}(g) = \chi_{\mathrm{res}_{A_4}^{C_3}\mathrm{ind}_{C_3}^{A_4}(k_1)}(g) = \sum_{x \in [A_4/C_3]} \chi_1(x^{-1}gx) = |\{x \in [A_4/C_3] \mid x^{-1}gx \in C_3\}|.$$

Since $K = \{id, (12)(34), (13)(24), (14)(23)\} = [A_4/C_3]$, it is easy to see that $\chi_{\mathrm{res}^{C_3}_{A_4}(P_1)}(1) = 4$ while $\chi_{\mathrm{res}^{C_3}_{A_4}(P_1)}((123)) = \chi_{\mathrm{res}^{C_3}_{A_4}(P_1)}((132)) = 1$. In the same way, we can compute the following character table.

C_3	1	(123)	(132)
$\mathrm{res}(P_1)$	4	1	1
$\mathrm{res}(P_\omega)$	4	ω	ω^2
$\mathrm{res}(P_{\omega^2})$	4	ω	ω^2

We obtain $\mathrm{res}^{C_3}_{A_4}(P_1) \cong k_1^2 \oplus k_\omega \oplus k_{\omega^2}$, $\mathrm{res}^{C_3}_{A_4}(P_\omega) \cong k_1 \oplus k_\omega^2 \oplus k_{\omega^2}$ and $\mathrm{res}^{C_3}_{A_4}(P_1) \cong k_1 \oplus k_\omega \oplus k_{\omega^2}^2$. Hence, the Cartan matrix of A_4 is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Assume now $\mathbf{p = 3}$. Let k be a field of characteristic 3. In this case, the 3-Sylow is not normal and $O_3(A_4) = 1$. However, $K \cong C_2 \times C_2$ is a normal subgroup so we can apply Clifford theory to K. Recall from Theorem 4.8 that the simple modules of kA_4 are in bijection with $Par(A_4, K)/A_4$. We want to compute the set $Par(A_4, K)$. Its elements are the pairs (χ, W) where χ is an irreducible representation of K and W is an irreducible representation of the inertia group G_χ such that $\mathrm{res}^K_{G_\chi}(W) \cong \chi^a$ for some a.

Notice that kK is semisimple, and that the simple kK-modules are $k_{1,1}, k_{1,-1}, k_{-1,1}$ and $k_{-1,-1}$, where $k_{i,j}$ is the one-dimensional module on which $(12)(34)$ acts by i and $(13)(24)$ acts by j. The inertia subgroup of $k_{1,1}$ is A_4. The inertia subgroup of $k_{-1,1}$ is the centralizer of $(12)(34)$. This contains K and is not A_4, so it must be K. It follows that the orbit of $k_{-1,1}$ has $|A_4/K| = 3$ elements, so $k_{1,-1}, k_{-1,1}$ and $k_{-1,-1}$ are all in the same orbit.

If $\chi = k_{1,-1}, k_{-1,1}$ or $k_{-1,-1}$, then $G_\chi = A_4$ and $W = \chi$. It follows that $Q := \mathrm{ind}^{A_4}_K(k_{1,-1})$, which has dimension 3 is both simple and projective.

If $\chi = k_{1,1}$, then W is an irreducible representation of A_4 such that $\mathrm{res}^K_{A_4}(W) = k_{1,1}^a$. One possibility is $W = k$ is the trivial representation and $a = 1$. By Frobenius reciprocity, we have

$$a = \dim \mathrm{Hom}_{kK}(\mathrm{res}^K_{A_4}(W), k_{1,1}) = \dim \mathrm{Hom}_{kA_4}(W, \mathrm{ind}^{A_4}_K(k_{1,1})).$$

Hence the W-isotypic component of $\mathrm{ind}^{A_4}_K(k)$ is isomorphic to W^a and $a \dim W = a^2 \leq 3$ by Lemma 1.14. It follows that $a = 1$ and W is the trivial representation. So there are only 2 G-orbits in $Par(A_4, K)$, hence 2 irreducible representations of A_4. Moreover, by Lemma 4.6, these are the trivial representation k and $Q = \mathrm{ind}^{A_4}_K(k_{1,-1})$, which has dimension 3. Since induction preserves projective modules, Q is also projective. Let P_k be the projective cover of k. Then, we have

$$kA_4 \cong P_k \oplus Q^3,$$

hence P_k has dimension 3. Then, we must have $P_k = \mathrm{ind}^{A_4}_K(k_{1,1})$. In fact, by Frobenius, we have a surjective map $\mathrm{ind}^{A_4}_K(k_{1,1}) \to k$, hence by Proposition 10.3, P_k is a summand of $\mathrm{ind}^{A_4}_K(k_{1,1})$. So they are the same because $\dim \mathrm{ind}^{A_4}_K(k_{1,1}) = 3$. Finally, we compute the Cartan matrix. We have

$$\dim \mathrm{Hom}_{kA_4}(P_k, P_k) = \dim \mathrm{Hom}_{kK}(\mathrm{res}(P_k), k_{1,1}) = \dim \mathrm{Hom}_{kK}(k_{1,1}^3, k_{1,1}) = 3$$

and $\dim_{kA_4}(P_k, Q) = \dim_{kA_4}(Q, P_k) = 0$ and $\dim_{kA_4}(Q, Q) = 1$ because Q is irreducible. Hence, the Cartan matrix of A_4 in characteristic 3 is

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case, we also have a decomposition as rings as

$$kA_4 \cong \mathrm{End}(P_k)^{op} \times Mat(3, \mathrm{End}(Q)^{op}) \cong \mathrm{End}(P_k)^{op} \times Mat(3, k).$$

This is an example of a block decomposition.

12 p-modular systems and decomposition matrices

We want to connect the representation theory in characteristic p of G to the representation theory in characteristic 0, which is much more well understood. For this reason, the main tool is to use a $(0, p)$-ring $\mathcal{O}$, i.e. a commutative local ring $\mathcal{O}$ with a maximal ideal $\mathfrak{m}$ such that the field of fractions $K = \mathrm{Quot}(\mathcal{O})$ is of characteristic 0 and the residue field $F = \mathcal{O}/\mathfrak{m}$ is a field of characteristic p.

Remark 12.1. If $\mathcal{O}$ is a commutative ring, then $\mathcal{O}$ is local if and only if it has a maximal ideal by Lemma 7.2.

If N is a $\mathcal{O}$-module, then $N/\mathfrak{m}N$ is a F-module and $N \otimes_{\mathcal{O}} K$ is a K-module. Similarly, if G is a group and N is a $\mathcal{O}G$-module, we can construct from N a FG-module $N/\mathfrak{m}N$ and a KG-module $N \otimes_{\mathcal{O}} K$.

Example 12.2. The ring $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \nmid b\} \subset \mathbb{Q}$ is the localization of $\mathbb{Z}$ at (p). It is local, its maximal ideal is $J(\mathbb{Z}_{(p)}) = p\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \mid a\}$. The fraction field is $\mathbb{Q}$, and the residue field $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{F}_p$.

Definition 12.3. We say that a commutative ring is a *principal ideal domain* (PID for short) if it is an integral domain such that every ideal is principal, i.e. every ideal $I = (a)$ for any ideal I.

Example 12.4. The ring $\mathbb{Z}$ is a PID. In fact, all its ideals are of the form (n), for some $n \in \mathbb{N}$. The ring $\mathbb{Z}_{(p)}$ is also a PID. Its ideals are all of the form (p^k), for some $k \geq 0$.

We recall the following fundamental fact about PID

Theorem 12.5 (Structure theorem for modules over a PID)**.** *Let R be a PID. Then a finitely generated R-module M is isomorphic to a direct sum of $R/(a_j)$ for some elements $a_j \in R$, i.e. we have*

$$M \cong \bigoplus_j R/(a_j).$$

Proof. See for example [2, Satz 2.4.5]. □

Lemma 12.6. *Let $\mathcal{O}$ be a PID with field of fractions K. Let V be a K-vector space. Then, any finitely generated $\mathcal{O}$-submodule M that contains a basis of V is a* full lattice*, i.e. $M \cong \mathcal{O}^n$ with $n = \dim_K V$.*

Proof. The submodule M is a finitely generated $\mathcal{O}$-modules. Since M is torsion free, by Theorem 12.5, it must be isomorphic to $\mathcal{O}^k$ for some k. Let $m_1, \ldots, m_k$ be a basis of M over $\mathcal{O}$. Since M contains a basis of V, then $m_1, \ldots, m_k$ generate V over K and we have $k \geq n$. Assume we have a linear combination of the form

$$c_1 m_1 + c_2 m_2 + \ldots + c_k m_k = 0$$

with $c_i \in K$. After multiplying by the denominators, we can assume that m_i are linearly dependent over $\mathcal{O}$, which is a contradiction. □

Proposition 12.7. *Let $\mathcal{O}$ be a PID with field of fractions K. Let G be a finite group and let V be a KG-module. Then there exists a full $\mathcal{O}$-lattice M which is stable under G (equivalently, M is a $\mathcal{O}G$-module).*

Proof. Let $v_1, \ldots, v_n$ be any K-basis of V. Then, we consider $M = \sum_i \mathcal{O}G \cdot v_i$. Then M is a finitely generated $\mathcal{O}$-module which contains a basis of V, so it is a full lattice by Lemma 12.6. □

This proposition gives a way to construct representations of G in characteristic p starting from representations of G in characteristic zero. For example, if V is a $\mathbb{Q}G$-module, we can always find a full lattice $V_0 \subset V$ that is a $\mathbb{Z}G$-module. Finally, for every prime p we can reduce mod p by considering the $\mathbb{F}_p G$-module V_0/pV_0.

Definition 12.8. A *discrete valuation ring* (or *DVR*, for short) is a PID R which has a unique non-zero maximal ideal. In other words, it is a local PID which is not a field.

Definition 12.9. A *p-modular system* is a triple $(F, \mathcal{O}, K)$ where $\mathcal{O}$ is a discrete valuation ring with maximal ideal $\mathfrak{m}$ generated by π, $K = \mathrm{Quot}(\mathcal{O})$ is a field of characteristic zero and $F = \mathcal{O}/\mathfrak{m}$ is a field of characteristic p.

Example 12.10. $(\mathbb{F}_p, \mathbb{Z}_{(p)}, \mathbb{Q})$ is a p-modular system.

For any p-modular system, starting with a KG-module V we can construct a FG-module. First we take a $\mathcal{O}$-lattice V_0 stable under G, and we construct the FG-module $V_0/\mathfrak{m}V_0$ where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}$. As the next example shows, in general different choices for the lattice V_0 can induce not isomorphic modules $V_0/\mathfrak{m}V_0$.

Example 12.11. Let $G = C_2 = \{1, g\}$ be the cyclig group with two elements. Then, in the regular representation $\mathbb{Q}G$ we can choose the lattice $L_1 = \mathbb{Z}_{(2)}1 \oplus \mathbb{Z}_{(2)}g$. We have $L_1/2L_1 \cong \mathbb{F}_2 1 \oplus \mathbb{F}_2 g \cong \mathbb{F}_2 G$, so it is indecomposable by Example 11.1.

We can also choose the lattice $L_2 = \mathbb{Z}_{(2)}(g-1) \oplus \mathbb{Z}_{(2)}(g+1)$. In this case, we have $L_2/2L_2 \cong \mathbb{F}_2(g-1) \oplus \mathbb{F}_2(g+1)$. But both $\mathbb{F}_2(g-1) \cong \mathbb{F}_2(g+1)$ are isomorphic to the trivial representation of G, so $L_2/2L_2$ is not indecomposable and $L_2/2L_2 \ncong L_1/2L_1$.

However, at least the composition factors of V_0/pV_0 are uniquely determined.

Theorem 12.12 (Bauer–Nesbitt's theorem)**.** *Let $(F, \mathcal{O}, K)$ be a p-modular system and G a finite group. Let V be a KG-module and let L_1, L_2 be two full $\mathcal{O}$-lattices which are stable under G. Then $L_1/\pi L_1$ and $L_2/\pi L_2$ are two FG-modules with the same composition factors and the same multiplicities.*

Proof. Notice that $L_1 + L_2$ is also a full $\mathcal{O}$-lattice, so it is enough to prove the theorem in the case $L_1 \subset L_2$. Since L_1 and L_2 are free $\mathcal{O}$-module of the same rank, the quotient L_2/L_1 is a torsion module. This implies that L_2/L_1 has finite length as a $\mathcal{O}$-module, hence also as a $\mathcal{O}G$-module. By induction, it is enough to show the theorem when L_1 is a maximal $\mathcal{O}G$-submodule of L_2. In other words, we assume the $\mathcal{O}G$-module L_2/L_1 to be simple. Since L_2/L_1 is simple, by Nakayama's lemma we have $\mathfrak{m}(L_2/L_1) = (\pi L_2 + L_1)/L_1 = 0$, hence $\pi L_2 \subset L_1$. Consider the chain of $\mathcal{O}$-lattices

$$\pi L_1 \subset \pi L_2 \subset L_1 \subset L_2.$$

We want to compare the composition series of $L_1/\pi L_1$ and $L_2/\pi L_2$. Both contain the composition series of $L_1/\pi L_2$, so we conclude by showing that L_2/L_1 and $\pi L_2/\pi L_1$ are isomorphic. In fact, we have a morphism of $\mathcal{O}G$-modules

$$L_2 \to \pi L_2/\pi L_1$$

$$x \mapsto \pi x + \pi L_1$$

which is surjective with kernel L_1. □

Thanks to Brauer–Nesbitt's theorem, the composition factors of the FG-module $L/\pi L$ do not depend on the chosen lattice L.

Definition 12.13. Let $(F, \mathcal{O}, K)$ be a p-modular system and G a finite group. The *decomposition matrix* is a matrix $D = (d_{TS})$ with columns indexed by the simple representation of G over F and rows indexed by the simple representations of G over K. The coefficient d_{TS}, corresponding to a simple FG-module S and a simple KG-module T, is the multiplicity $[L/\pi L : S]$, where L is any $\mathcal{O}$-lattice of T.

Example 12.14. Consider the group $G = S_3$ and a splitting 2-modular system $(F, \mathcal{O}, K)$ for G. (Note that $(\mathbb{F}_2, \mathbb{Z}_{(2)}, \mathbb{Q})$ is not splitting, but one can take for example the modular system $(\mathbb{F}_2, \mathbb{Z}[\omega]_{(2)}, \mathbb{Q}(\omega))$, where ω a third root of the unit). In Example 11.2 we computed the projective and simple FS_3-modules. The simple FS_3-modules are the trivial representation and the 2-dimensional module $FS_3e_2 \cong FS_3e_3$, with $e_2 = 1 + \omega(123) + \omega^2(132)$.

Notice that also KS_3e_2 is a simple 2-dimensional representation. (In fact, we have $KS_3 = KS_3e_1 \oplus KS_3e_2 \oplus KS_3e_3$ and KS_3e_1 contains the trivial and the sign representation). Then, $\mathcal{O}S_3e_2$ is a lattice inside KS_3e_2, and its reduction to a FS_3-module returns precisely FS_3e_2. On the other hand, both the trivial and the sign representations over K returns the trivial FS_3-module. Hence, the decomposition matrix is

$$\begin{array}{c} \\ triv \\ sgn \\ V \end{array} \begin{array}{c} \begin{array}{cc} triv & FS_3e_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Example 12.15. Consider now the group $G = A_4$ as in Example 11.13. Let $(F, \mathcal{O}, K)$ be a splitting p-modular system for G.

Assume p=2. We have 3 simple FS_3-module, F_1, F_ω and F_{ω^2}, all of dimension 1. There are 4 irreducible representations in characteristic 0, the standard representation V of dimension 3 and 3 one-dimensional representations (which we denote by K_1, K_ω and K_{ω^2}) which are obtained from pullbacking the representations of the kernel $A_4/K \cong C_3$.

All the representations $F_1, F_\omega, F_{\omega^2}, K_1, K_\omega$ and K_{ω^2} come from representations of A_4/K, so it is clear that the corresponding minor of the decomposition matrix is the identity matrix. It remains to consider the standard representation V.

Recall the regular representation KA_4 splits as $V^{\oplus 3} \oplus K_1 \oplus K_\omega \oplus K_{\omega^2}$. On the other hand, we have the decomposition $FA_4 = P_1 \oplus P_\omega \oplus P_{\omega^2}$. From this, and the computation of Example 11.13, we know that $[FA_4 : K_1] = [FA_4 : K_\omega] = [FA_4 : K_{\omega^2}] = 4$. Since $\mathcal{O}A_4$ is a lattice of KA_4 with quotient FA_4 we obtain

$$d_{V,F_1} = \frac{[FA_4 : F_1] - d_{F_1,K_1} - d_{F_1,K_\omega} - d_{K_{\omega^2},F_1}}{[KA_4 : V]} = \frac{4-1}{3} = 1.$$

Similarly, we also get $d_{V,F_\omega} = d_{V,F_{\omega^2}} = 1$. The decomposition matrix in this case is

$$\begin{array}{c} \\ K_1 \\ K_\omega \\ K_{\omega^2} \\ V \end{array} \begin{array}{c} \begin{array}{ccc} F_1 & F_\omega & F_{\omega^2} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

Assume p=3. In this case we have only two simple FA_4-modules, the trivial representation F_1 and the three dimensional representation Q. By dimension consideration, the only composition factors in a reduction mod p of $K_1, K_\omega, K_\omega^2$ is F_1. Looking at the composition factors of KA_4 and FA_4 we also deduce that the only composition factor of a reduction mod 3 of V is Q. Hence, the decomposition matrix is

$$\begin{array}{c} \\ K_1 \\ K_\omega \\ K_{\omega^2} \\ V \end{array} \begin{array}{c} \begin{array}{cc} F_1 & Q \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}.$$

13 Complete splitting p-modular systems

In what follows, we will impose more restrictive hypothesis on the p-modular system, so to ensure a better behaved relation between KG-modules and FG-modules. This is mostly a technical assumption and does not have really consequence on the representation theory (in the same way as representation theory on any splitting field of a given characteristic is basically the same).

Definition 13.1. Let $\mathcal{O}$ be a DVR with maximal ideal $\mathfrak{m}$. A sequence $(a_n)_{n\geq 0}$ of elements of $\mathcal{O}$ is said a *Cauchy sequence* if for any $b \geq 1$ there exists N such that for any $n_1, n_2 \geq N$ we have

$$a_{n_1} - a_{n_2} \in \mathfrak{m}^b$$

Remark 13.2. The condition on Cauchy sequence coincides with the usual condition for metric space if we define the distance

$$d(x,y) = 2^{-k}$$

where k is the maximal integer such that $x - y \in \mathfrak{m}^k$.

Definition 13.3. Let $\mathcal{O}$ be a DVR with maximal ideal $\mathfrak{m}$. We say that $\mathcal{O}$ is *complete* if every Cauchy sequence admits a limit, i.e. if for every $(a_n)_{n\geq 0}$ Cauchy there exists $a \in \mathcal{O}$ such that for any $b \geq 1$ there exists $N > 0$ such that $a_n - a \in \mathfrak{m}^b$ for any $n > N$.

Example 13.4. Let $\mathcal{O} = \mathbb{Z}_{(p)}$. Consider the sequence (a_n) with $a_n = 1+p+p^2+\ldots+p^n$. Notice that if $m, n > N$ then $p^N \mid a_n - a_m$, so (a_n) is Cauchy. Assume that the sequence has a limit $a = \lim a_n$, then for any $N \geq 1$ we have

$$p^N \mid a - (1 + p + \ldots + p^{N-1}).$$

If a is positive, then it is bigger that p^{N-1} for any N, which is impossible. If a is negative, we have for any $N \geq 0$ that

$$a \leq -p^N + (1 + \ldots + p^{N-1}) = -p^N + \frac{p^N - 1}{p-1} = \frac{-p^N(p-2) - 1}{p-1}.$$

This is impossible if $p > 2$. So (a_n) has not limit if $p > 2$ and $\mathbb{Z}_{(p)}$ is not complete.

However, if $p = 2$ we have $\lim a_n = -1$. To show that $\mathbb{Z}_{(2)}$ is not complete, one can consider the Cauchy sequence $b_n = 1 + 4 + 4^2 + \ldots + 4^n$.

Given a *DVR* $\mathcal{O}$ we can consider the *completion* $\widehat{\mathcal{O}}$ of $\mathcal{O}$ as a metric space. Formally, $\mathcal{O}$ is the set of Cauchy sequences of $\mathcal{O}$ under the equivalence relation $(a_n) \sim (b_n)$ if $\lim(a_n - b_n) = 0$. The completion $\widehat{\mathcal{O}}$ acquires in a natural way the ring structure from $\mathcal{O}$. (For example $(a_n) \cdot (b_n) = (a_n b_n)$ is a Cauchy sequence: $a_n b_n - a_m b_m = a_n(b_n - b_m) + (a_n - a_m)b_m$).

Consider the ideal

$$\widehat{\mathfrak{m}} := \{(a_n) \in \mathcal{O} \mid \overline{a_n} = 0 \in \mathcal{O}/\mathfrak{m} \text{ for } n \gg 0\} \subset \widehat{\mathcal{O}}.$$

Lemma 13.5. *For any $l \geq 0$ the inclusion $\mathcal{O} \hookrightarrow \widehat{\mathcal{O}}$ induces an isomorphism $\mathcal{O}/\mathfrak{m}^l \cong \widehat{\mathcal{O}}/\widehat{\mathfrak{m}}^l$.*

Proof. We have $\widehat{\mathfrak{m}}^l = \{(a_n) \in \mathcal{O} \mid \overline{a_n} = 0 \in \mathcal{O}/\mathfrak{m}^l \text{ for } n \gg 0\}$, so the map $j : \mathcal{O} \to \widehat{\mathcal{O}}/\widehat{\mathfrak{m}}^l$ has $\mathfrak{m}^l$ as kernel. Moreover, the map j is surjective. In fact, if $(a_n) \in \widehat{\mathcal{O}}$, then $(a_n) - a_N \in \widehat{\mathfrak{m}}^l$ for $N \gg 0$, si $(a_n) = j(a_N)$ in $\widehat{\mathcal{O}}/\widehat{\mathfrak{m}}^l$. □

Lemma 13.6. *Let $\mathcal{O}$ be a DVR with maximal ideal $\mathfrak{m} = (\pi)$. Fix a set of representatives $[\mathcal{O}/\mathfrak{m}]$. Then, every element of $x \in \widehat{\mathcal{O}}$ can be written in a unique way as*

$$x = x_0 + x_1\pi + x_2\pi^2 + \ldots$$

with $x_i \in [\mathcal{O}/\mathfrak{m}]$.

Proof. Let $x \in \widehat{\mathcal{O}}$. By Lemma 13.5, since $\mathcal{O}/\mathfrak{m} \cong \widehat{\mathcal{O}}/\widehat{\mathfrak{m}}$ we can find a unique $x_0 \in [\mathcal{O}/\mathfrak{m}]$ such that $x - x_0 \in \widehat{\mathfrak{m}}$. So $x = x_0 + \pi x_0'$ for some $x_0' \in \widehat{\mathcal{O}}$. Reiterating, we get $x_0' = x_1 + \pi x_1'$ with $x_0 \in [\mathcal{O}/\mathfrak{m}]$ and $x_1' \in \widehat{\mathcal{O}}$. The sequence $x_0, x_0 + \pi x_1, x_0 + \pi x_1 + \pi^2 x_2, \ldots$ is Cauchy, and is equivalent to x. □

Corollary 13.7. *The completion of a DVR $\mathcal{O}$ is a complete DVR with the same residue field.*

Proof. We know that $\widehat{\mathcal{O}}$ is complete. Let $x = x_0 + x_1\pi + x_2\pi^2 + \ldots$. The system of equations $(x_0 + x_1\pi + x_2\pi^2 + \ldots)(y_0 + y_1\pi + y_2\pi^2 + \ldots) = 1$ admits a solution in the y_i's if $x_0 \neq 0$. Then x is invertible if and only if $x_0 \neq 0$. Hence, $\mathcal{O}$ is local with maximal ideal generated by π.

Assume $I \subset \widehat{\mathcal{O}}$ is an ideal and let $x = x_0\pi^a + x_1\pi^{a+1} + x_2\pi^{a+2} + \ldots \in I$ with $x_0 \neq 0$. Then $x = \pi^a y$, with y invertible. Let a minimal such that there exists $x \in I$ of this form. Then $I = (\pi^a)$, so $\widehat{\mathcal{O}}$ is a PID.

It follows that $\widehat{\mathcal{O}}$ is a complete DVR which has the same residue field by Lemma 13.5. □

Example 13.8 (The ring of p-adic integers.)**.** The completion of $\mathbb{Z}_{(p)}$ is denoted by $\mathbb{Z}_p$ and it is called the *ring of p-adic integers*. We have

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \dots \mid a_i \in \{0, \dots, p-1\}\}.$$

In $\mathbb{Z}_p$, we have $-1 = \sum_{n=0}^{\infty}(p-1)p^n$.

Definition 13.9. We say that a p-modular system $(F, \mathcal{O}, K)$ is complete if $\mathcal{O}$ is a complete DVR.

Example 13.10. The fraction field of $\mathbb{Z}_p$ is denoted by $\mathbb{Q}_p$ and is called the *field of p-adic numbers*. Notice that it is a field of characteristic 0. The triple $(\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p)$ is a complete p-modular system.

The modular system $(\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p)$ is probably the only complete modular system that we meet explicitly in this course. However, for later applications, it usually convenient to assume that we have a complete p-modular system $(F, \mathcal{O}, K)$ such that both F and K are splitting field for G (or, even better, for all subgroups of G). In this case, we say that the p-modular system is *splitting*.

Theorem 13.11. *For any group G and any characteristic p, there exists a complete splitting p-modular system for G.*

Proof. We omit the proof. See for example [4, Prop. 16.21] □

Concretely, a splitting modular system can be obtained from a modular system by adding a e-th primitive root to $\mathbb{Q}_p$, where e is the exponent of G. However, the details for the construction of a splitting modular system are beyond the scope of this course. Moreover, this does not have direct consequence in representation theory. In fact, the representation theory of G in characteristic p does not depend on the field k, as long as k is splitting.

From now on, for any finite group G to have a splitting complete p-modular system $(F, \mathcal{O}, K)$.

13.1 Lifting of idempotents

The condition that $\mathcal{O}$ is complete is important because it allows to lift idempotents from FG to $\mathcal{O}G$. This implies that projective FG-module can always be obtained as a quotient of a projective $\mathcal{O}G$-module.

We begin with the simpler case of a nilpotent ideal, which will also serve as a guide for the complete case.

Lemma 13.12. *Let R be a ring and I a nilpotent ideal, i.e. $I^N = 0$ for some $N > 0$. Let $e \in R/I$ be an idempotent. Then, there exists $f \in R$ idempotent with $e = f + I$.*

Proof. We start by looking for an idempotent in R/I^2. Take $a \in R$ such that $\overline{a} = e$. We have $\overline{a^2 - a} = 0$ and $a^2 - a \in I$. So $(a^2 - a)^2 \in I^2$ and $(a^2 - a)^2 = 0$ in R/I^2. We define $e_2 := 3a^2 - 2a^3$. We have $\overline{e_2} = 3\overline{a} - 2\overline{a} = e$ and

$$\begin{aligned} e_2^2 - e_2 =& (3a^2 - 2a^3)(3a^2 - 2a^3 - 1) \\ =& a^2(3 - 2a)(a-1)^2(-1-2a) \in I^2. \end{aligned}$$

It follows that e_2 is an idempotent of R/I^2. We reiterate this process: starting with $e_{i-1} \in R$ one constructs an element $e_i \in R$ such that e_i is an idempotent in R/I^i and $\overline{e_i} = \overline{e_{i-1}} \in R/I^{i-1}$. because $R/I^N = R$ for some $N > 0$, we conclude by taking $f = e_N$. □

Basically the same proof works for group algebras over a complete DVR.

Proposition 13.13. *Let $\mathcal{O}$ be a complete DVR with maximal ideal $\mathfrak{m} = (\pi)$ and residue field $F = \mathcal{O}/\mathfrak{m}$. Let G be a finite group. Let $e \in FG$ be an idempotent. Then, there exists $f \in \mathcal{O}G$ idempotent with $e = f + \pi\mathcal{O}G$.*

Proof. We have $FG = \mathcal{O}G/\pi\mathcal{O}G$. As in the proof of Lemma 13.12, we can construct a sequence (e_n) of elements of $\mathcal{O}G$ such that e_n is an idempotent of $\mathcal{O}G/\pi^n\mathcal{O}G$ for any n and $e_n - e_{n-1} \in \pi^{n-1}\mathcal{O}G$. We can write each e_n as

$$e_n = \sum_{g \in G} e_n(g)g$$

for some $e_n(g) \in \mathcal{O}$. It follows that, for any $g \in G$, $e_n(g) - e_{n-1}(g) \in \mathfrak{m}^{n-1}$ and $(e_n(g))$ is a Cauchy sequence which has a limit $f(g) := \lim e_n(g)$. Let $f := \sum_{g \in G} f(g)g$. Then f is also an idempotent and $e = f + \pi\mathcal{O}G$. □

Lemma 13.14. *Let $e \in FG$ and $f \in \mathcal{O}G$ idempotents with $e = \overline{f}$. Then e is primitive if and only if f is primitive.*

Proof. If f is not primitive, then $f = f_1 + f_2$ with f_1, f_2 orthogonal idempotents. But then also $\overline{f_1}$ and $\overline{f_2}$ are orthogonal idempotents. Moreover, $\overline{f_1} \neq 0$. Otherwise, $f_1^k \in \mathfrak{m}^k$ and $f_1 = \lim f_1^k = 0$.

We only sketch the proof of the other direction. If f is primitive, then f is the unique idempotent in $f\mathcal{O}Gf$. Proposition 13.13 can be generalized to the case of any finitely generated $\mathcal{O}$-algebra which is free over $\mathcal{O}$. So every idempotent in $\overline{f}FG\overline{f}$ can be lifted to $f\mathcal{O}Gf$. This means that every idempotent $e' \in \overline{f}FG\overline{f}$ can be lifted to f, and $e' = \overline{f}$. Hence, $\overline{f}$ is the only idempotent in $\overline{f}FG\overline{f}$ and $\overline{f}$ is primitive. □

Corollary 13.15. *Let $(F, \mathcal{O}, K)$ be a complete p-modular system and G be a finite group. Let P be a finitely generated projective module for FG. Then, there exists a projective module $\widehat{P}$ for $\mathcal{O}G$ such that $P \cong \widehat{P}/\pi\widehat{P}$.*

Proof. We can assume that P is indecomposable, so we have $P = (FG)e$ for some primitive idempotent $e \in FG$. Then, we can lift e to an idempotent $f \in \mathcal{O}G$. Then $\widehat{P} := (\mathcal{O}G)f$ is projective. The quotient $\mathcal{O}G \to FG$ restricts to a surjective morphism $\Phi : (\mathcal{O}G)f \to (FG)e$. We have $(\pi\mathcal{O}G)f \subset \mathrm{Ker}(\Phi)$. On the other hand, if $a \in \mathrm{Ker}(\Phi)$, then $a \in \pi\mathcal{O}G$ and $a = af$, so $\mathrm{Ker}(\Phi) \subset (\pi\mathcal{O}G)f$. It follows that Φ induces an isomorphism $\widehat{P}/\pi\widehat{P} \cong P$. □

Remark 13.16. In general, there are FG-modules which do not admit lift to a $\mathcal{O}G$-module. However, this is true for solvable group by the Fong–Swan theorem. So, to find a counterexample we can't look at groups which are too small. The smallest example is A_5 for $p = 2$. In this case, there are two irreducible A_5 representations of dimension 2 which do not have any lift to characteristic 0 (in fact, the dimensions of the irreducible representations in char 0 are $1, 3, 3, 4, 5$). The decomposition matrix in this case is

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

14 Brauer reciprocity

Given a projective FG-module P, we can find a lift to a projective $\mathcal{O}G$-module $\widehat{P}$. Notice that $\widehat{P}$ is free as a $\mathcal{O}$-module (finitely generated projective modules over a PID are always free!) Then, by extending scalars, we obtain a KG-module $\widehat{P}^K := \widehat{P} \otimes_{\mathcal{O}} K$. In principle, different lifts could lead to different $\mathcal{O}G$-modules.

Lemma 14.1. *We have $\pi\mathcal{O}G \subset J(\mathcal{O}G)$.*

Proof. Let V be simple $\mathcal{O}G$-module. Then $\pi V \subset V$ is a submodule, so $\pi V = 0$ or $\pi V = V$.

Assume that $\pi V = V$. Recall that since V is simple, then $V \cong \mathcal{O}Gv$ for some $v \in V$. In particular, V is finitely generated as a $\mathcal{O}$-module. We have $(\pi) = J(\mathcal{O})$, so we can apply Nakayama's Lemma (Lemma 5.6) to deduce that $V = 0$.

Hence, $\pi V = 0$ for any simple module V. So π acts trivially on any $\mathcal{O}G$-simple module and $(\pi) \subset J(\mathcal{O}G)$. □

Proposition 14.2. *Let P be finitely generated projective module for FG. Then the lift $\widehat{P}$ to a $\mathcal{O}G$-module which is free as $\mathcal{O}$-module is unique up to isomorphism. Moreover, $\widehat{P}$ is necessarily projective as a $\mathcal{O}G$-module.*

Proof. We can assume that P is indecomposable, i.e. that $P \cong P_S$ for some S. We know from Corollary 13.15 that there exists a projective $\mathcal{O}G$-module $\widehat{P_S}$ lifting P_S.

Assume that L is another $\mathcal{O}G$-modules such that $L/\pi L \cong P_S$. It follows that $\mathrm{rk}_{\mathcal{O}}(L) = \dim_F(P_S) = \mathrm{rk}_{\mathcal{O}}(P_S)$.

Since $\widehat{P_S}$ is projective, we can find $\theta : \widehat{P_S} \to L$ so that the following diagram of $\mathcal{O}G$-modules is commutative.

$$\begin{array}{ccc} \widehat{P_S} & \longrightarrow & P_S \\ & \overset{\theta}{\searrow} & \uparrow \\ & & L \end{array}$$

Moreover, θ induces an isomorphism $\overline{\theta} : \widehat{P_S}/\pi\widehat{P_S} \to L/\pi L$, so $\theta(\widehat{P_S}) + \pi L = L$. We have $\pi(L/\theta(\widehat{P_S})) = \pi L + \theta(\widehat{P_S})/\theta(\widehat{P_S}) = L/\theta(\widehat{P_S})$, so by Nakayama's Lemma (Lemma 5.6) we obtain $L = \theta(\widehat{P_S})$, or θ is surjective. But a surjective morphism between two free modules of the same rank is also injective, then θ is an isomorphism. □

Given the lift $\widehat{P}$, we can extend scalar and obtain a KG-module $\widehat{P}^K := \widehat{P} \otimes_{\mathcal{O}} K$. The next theorem shows how we can decompose $\widehat{P}^K$ into simple KG-modules.

Theorem 14.3 (Brauer reciprocity)**.** *Let G be a finite group and let be $(F, \mathcal{O}, K)$ be a splitting complete p-system for G. Let $T_1, T_2, \ldots, T_a$ be a complete set of representatives of the isomorphism classes of simple KG-modules and $S_1, \ldots, S_b$ a complete set of representatives of the isomorphism classes of simple FG-modules.*

Let $e_1, \ldots, e_b \in \mathcal{O}G$ be idempotents such that $FG\overline{e_j} \subset FG$ is the projective cover of S_j. Then for any j we have

$$(\widehat{FG\overline{e_j}})^K = KGe_j \cong \bigoplus_{j=1}^{a} d_{ij} T_i,$$

where d_{ij} is the corresponding entry of the decomposition matrix.

Proof. Recall that KG is semisimple, so we can decompose KGe_j into simple modules. We have

$$KGe_j \cong \bigoplus_{i=1}^{a} d'_{ij} T_i.$$

for some $d'_{ij} \in \mathbb{N}$. Since K is splitting, we have

$$d'_{ij} = \dim_K \mathrm{Hom}_{KG}(FGe_j, T_i) = \dim_K(e_j T_i)$$

Let $L_i \subset T_i$ be a full $\mathcal{O}$-lattice which is G-stable. Then $e_j L_i$ is a $\mathcal{O}$-submodule of L_i, so it is free over $\mathcal{O}$ and it is a full lattice of $e_j T_i$. Moreover, we have $\overline{e_j L_i} := e_j L_i / \pi e_j L_i \cong \overline{e_j}(L_i/\pi L_i)$. It follows that:

$$\begin{aligned} d'_{ij} = \dim_K(e_j T_i) =& \mathrm{rk}_{\mathcal{O}}(e_j L_i) \\ =& \dim_F \overline{e_j}(L_i/\pi L_i) \\ =& \dim_F \mathrm{Hom}_{FG}(FG\overline{e_j}, L_i/\pi L_i) = [L_i/\pi L_i : S_j] = d_{ij} \end{aligned}$$

by definition of the decomposition matrix. We conclude that $d_{ij} = d'_{ij}$ for any i and j. □

Corollary 14.4. *Let G be a finite group and let be $(F, \mathcal{O}, K)$ be a splitting complete p-system for G. Let C be the Cartan matrix of FG and D be the decomposition matrix. Then we have $C = D^t D$. In particular, the Cartan matrix is symmetric.*

Proof. Let $S_1, \ldots, S_b$ be a complete set of representatives of the isomorphism classes of simple FG-modules. Let $e_1, \ldots, e_b \in \mathcal{O}G$ be idempotents such that $FG\overline{e_j} \subset FG$ is the projective cover of S_j. Then, we have

$$\begin{aligned} c_{ij} = [FG\overline{e_j} : S_i] =& \dim_F \mathrm{Hom}_{FG}(FG\overline{e_i}, FG\overline{e_j}) \\ =& \dim_F(\overline{e_i} FG \overline{e_j}) \\ =& \mathrm{rk}_{\mathcal{O}}(e_i \mathcal{O}G e_j) \\ =& \dim_K(e_i KG e_j) \\ =& \dim_K \mathrm{Hom}_{KG}(KGe_i, KGe_j) \\ =& \dim_K \mathrm{Hom}_{KG}\left(\bigoplus_r d_{ri} T_r, \bigoplus_{r=1}^{a} d_{rj} T_r\right) \\ =& \sum_{r=1}^{a} d_{ri} d_{rj}. \end{aligned}$$

where $T_1, \ldots, T_a$ are the simple KG-modules. □

Remark 14.5. The Brauer reciprocity can also be stated as

$$[\widehat{P_S}^K : T] = [L/\pi L : S].$$

for any simple FG-module S and any KG-module T. We can also write it as

$$\dim_K \mathrm{Hom}_{KG}(\widehat{P_S}^K, T) = \dim_F \mathrm{Hom}_{FG}(P_S, L/\pi L),$$

which can be generalized to any projective FG-module P and any KG-module V to

$$\dim_K \mathrm{Hom}_{KG}(\widehat{P}^K, V) = \dim_F \mathrm{Hom}_{FG}(P, L/\pi L).$$

The pair of "wannabe functors" $P \to \widehat{P}^K$ and $V \to L/\pi L$ are adjoint with respect of the pairings $\dim_K \mathrm{Hom}_{KG}(-,-)$ and $\dim_F \mathrm{Hom}_{FG}(-,-)$. (They are not real functors, but this statement can be made precise at the level of Grothendieck groups.)

15 Brauer characters

In characteristic 0, the study of representations of a finite simple is made simple by character theory: to compute the decomposition of a representation of irreducible it is enough to know the character table. We want to find an analogue in characteristic p, at least to determine the composition factors of a given representations. The usual character theory does not work. For example, if V is a representations, then V has the same character of $V^{\oplus p+1}$, so we cannot distinguish them using the characters. However, if we restrict to simple representations, this cannot happen and, actually, they still give linear independent functions.

Proposition 15.1. *Characters of simple representations over a splitting field are linearly independent.*

Proof. Let $S_1, \dots, S_n$ be a complete set of isomorphism classes of irreducible representations of FG-modules, where G is a finite group and F is splitting for G.

Let $A = FG/J(FG)$. We know that A is a semisimple algebra. Recall that the Jacobson radical acts trivially on simple modules. Then $S_1, \dots, S_n$ is also a complete set of isomorphism classes of simple A-modules.

Since A is semisimple, by the Artin–Wedderburn theorem, we have

$$A \cong \mathrm{End}_F(S_1) \times \mathrm{End}_F(S_2) \times \ldots \times \mathrm{End}_F(S_n) \tag{3}$$

For $M_i \in \mathrm{End}_F(S_i)$, let $\widehat{M_i}$ the elements corresponding to $(0, \dots, 0, M_i, 0, \dots, 0)$ under the isomorphism (3).

The character $\chi_{S_i} = \mathrm{Tr}(\rho : G \to GL(S_i))$ is a function from G to F. We can extend it by linearity to a function $\widetilde{\chi}_{S_i}$ from FG to F, i.e. if $\sum a_g g \in FG$ we have

$$\widetilde{\chi}_{S_i}\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g \chi_{S_i}(g).$$

Let $\widetilde{M_i}$ be a lift of $\widehat{M_i}$ to FG, i.e. $\widetilde{M_i} + J(FG) = \widehat{M_i}$. Then, we have

$$\widetilde{\chi}_{S_i}(\widetilde{M_j}) = \mathrm{Tr}(\widetilde{M_j}\cdot : S_i \to S_i) = \mathrm{Tr}(\widehat{M_j}\cdot : S_i \to S_i) = \begin{cases} \mathrm{Tr}(M_i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where the second equality is because the Jacobson radical acts trivially on S_i.

Assume now that $(\chi_{S_1}, \dots, \chi_{S_n})$ are linearly dependent, i.e. there exists $c_1, \dots, c_n \in F$ with

$$\sum_{i=1}^{n} c_i \chi_{S_i}(g) = 0 \quad \text{for all } g \in G.$$

Then, also

$$\sum_{i=1}^{n} c_i \widetilde{\chi}_{S_i}(a) = 0 \quad \text{for all } a \in FG.$$

In particular, we have

$$0 = \sum_{i=1}^{n} c_i \widetilde{\chi}_{S_i}(\widetilde{M_j}) = c_i \,\mathrm{Tr}(M_i).$$

We can easily pick $M_j \in \mathrm{End}_F(S_j) \cong Mat(F, \dim S_j)$ of trace 1, so we obtain $c_i = 0$ for all i. □

If we want some character theory which distinguish between V and $V^{\oplus p+1}$ we cannot work in characteristic p. We want to find a lift to K.

Let $\rho : G \to GL(V)$ be a representation of V over F. Recall that all the eigenvalues of $\rho(g)$ are roots of unity. So we have the character is a sum of roots of unity, i.e.

$$\chi_V(g) = \mathrm{Tr}(\rho(g)) = \zeta_1 + \ldots + \zeta_n$$

with $\zeta_i \in F$ roots of unity.

We want to define a lift of χ to K by finding a lift of all the roots of unity.

Definition 15.2. For any $e \geq 1$, we denote by $\mu_e(F) \subset F^\times$ the group of eth roots of unity.

Lemma 15.3. *Let $(F, \mathcal{O}, K)$ be a p-modular system and let e be an integer not divisible by p. Assume that F and K contain all e-roots of unity, i.e. $|\mu_e(F)| = |\mu_e(K)| = e$. Then $\mu_e(K) \subset \mathcal{O}$ and the projection defines an isomorphism*

$$Q : \mu_e(K) \cong \mu_e(F) \tag{4}$$
$$\zeta \mapsto \overline{\zeta}.$$

Proof. Let $\mathfrak{m} = (\pi)$ be the maximal ideal of $\mathcal{O}$. Let $\zeta \in \mu_e(K)$, so $\zeta^e = 1$. Then $\zeta = \frac{a}{b}$ with $a, b \in \mathcal{O}$. We can write $a = \pi^\alpha a'$ and $b = \pi^\beta b'$ with $a', b' \in \mathcal{O}^\times$ invertible. Then $\zeta^e = 1$ implies $\pi^{e\alpha} = \pi^{e\beta} \frac{b'}{a'}$. Hence, $e\alpha = e\beta$ and $\zeta = \frac{a'}{b'} \in \mathcal{O}$. This shows $\mu_e(K) \subset \mathcal{O}$.

Since $\mu_e(K) \subset \mathcal{O}$, we can consider the quotient of any root $\zeta \in \mu_e(K)$ to $\mathcal{O}/\mathfrak{m} = F$. Since $\zeta^e = 1$, also $\overline{\zeta}^e = 1$, so $\overline{\zeta} \in \mu_e(F)$.

In the polynomial ring $\mathcal{O}[x]$ we have

$$x^e - 1 = \prod_{\zeta \in \mu_e(K)} (x - \zeta) \in \mathcal{O}[x]$$

Taking the quotient to $F[x]$, we also get

$$x^e - 1 = \prod_{\zeta \in \mu_e(K)} (x - \overline{\zeta}) \in \mathcal{F}[x]$$

But the polynomial $p(x) = x^e - 1 \in F[x]$ is separable, i.e. all its roots have multiplicity one. It follows that if $\overline{\zeta_1} = \overline{\zeta_2}$ then also $\zeta_1 = \zeta_2$. It follows that Q is injective, and therefore bijective. □

Thanks to Lemma 15.3, we can lift all eth root of unity using $Q^{-1} : \mu_e(F) \cong \mu_e(K)$. We write $\widehat{\zeta} \in \mu_e(K)$ for the image of $\zeta \in \mu_e(F)$.

Definition 15.4. We say that an element $g \in G$ is *p-regular* if its order is not divisible by p. We denote by G_{reg} the subset of p-regular elements.

We call p-exponent the least common multiple of the orders of g, for $g \in G_{reg}$.

Lemma 15.5. *Let $\rho : G \to GL(V)$ be a represetnation of G over F. Let $g \in G_{reg}$. Then $\rho(g)$ is diagonalizable, and the eigenvalues are in $\mu_e(F)$, with e the p-exponent of G.*

Proof. The subgroup generated by g is a cyclic group of order q, with $p \nmid q$. Then $F\langle g \rangle$ is semisimple, and all the simple $F\langle g \rangle$-modules have dimension 1. So $\rho(g)$ is diagonalizable and its eigenvalues satisfy $\xi^e = 1$. □

Let e be the p-exponent of G. Then, for any $g \in G$, we have g^e. For any representation $\rho : G \to GL(V)$ over F, the eigenvalues of $\rho(g)$ is a sum of eth roots of unity. We are know ready to define the correct analogue of characters in characteristic p.

Definition 15.6 (Brauer character)**.** Let $\rho : G \to GL(V)$ be a representation of G over F, i.e. a FG-module. We define the *Brauer character* of G the map

$$\lambda_V : G_{reg} \to K$$

defined by $\lambda_V(g) = \widehat{\zeta_1} + \ldots + \widehat{\zeta_d}$ where $\zeta_1 + \ldots + \zeta_d$ are the eigenvalues of $\rho(g)$.

Notice that Brauer characters have values in $\mathcal{O}$. The Brauer character is only defined on p-regular elements. However, as the following Lemma shows, it will not add any information to compute it for arbitrary elements of G.

Lemma 15.7. *Let $g \in G$. Then, we can write it in a unique way as $g = g_1 g_2$ where g_1 is p-regular, $ord(g_2)$ is a power of p and $g_2 g_1 = g_1 g_2$.*

Moreover, if $\rho : G \to GL(V)$ is a representation of G over F, then $\rho(g)$ and $\rho(g_1)$ have the same eigenvalues.

Proof. We have $ord(g) = p^s m$ with $s \geq 0$ and $(p, m) = 1$. Then, we can find integers a, b with

$$ap^s + bm = 1.$$

Let $g_1 := g^{ap^s}$ and $g_2 = g^{bm}$. We have $g_1 g_2 = g^{ap^s + bm} = g$, and $\text{ord}(g_1) \mid m$ and $\text{ord}(g_2) \mid p^s$. If $g = g_1' g_2'$ is another such decomposition, then $g^{p^s m} = 1$ implies that $\text{ord}(g_1') \mid m$ and $\text{ord}(g_2') \mid p^s$. We have

$$g_1 = g^{ap^s} = g^{1-bm} = (g_1')^{1-bm} (g_2')^{ap^s} = g_1'.$$

Then, also $g_2 = g_2'$.

Let now ρ be a representation of G. Then $\rho(g) = \rho(g_1)\rho(g_2)$. Since $\rho(g_1)$ is diagonalizable and $\rho(g_2)$ commutes with $\rho(g_1)$, then g_2 is triangularizable in the basis of eigenvectors of $\rho(g_1)$. Hence, the eigenvalues of $\rho(g)$ are products of eigenvalues of $\rho(g_1)$ and of $\rho(g_2)$.

Any eigenvalue ω of $\rho(g_2)$ satisfies $\omega^{p^s} = 1$. Since $\text{char}(F) = p$, this implies $\omega = 1$. □

We begin by studying some properties of the Brauer characters.

Proposition 15.8.

1. *The Brauer characters are class functions on the set of p-regular elements.*
2. *If V is a FG-module, then $\lambda_V(1) = \dim_F(V)$.*
3. *If V is a FG-module, we have $\overline{\lambda_V(g)} = \chi_V(g)$ for any $g \in G_{reg}$.*
4. *If $V \subset W$ are FG-modules, then $\lambda_W = \lambda_V + \lambda_{W/V}$.*
5. *Let W be a KG-module. Let L be a full G-stable $\mathcal{O}$-lattice and let $\overline{W} = W/\pi W$. Then $\lambda_{\overline{W}} = \chi_W|_{G_{reg}}$.*

Proof.

1. The matrices $\rho(g)$ and $\rho(h^{-1}gh)$ are conjugated, so they have the same eigenvalues. It follows that $\lambda(g) = \lambda(h^{-1}gh)$ for any $g \in G_{reg}$ and $h \in G$.
2. The only eigenvalue of the action of $1 \in G$ on V is $1_F \in F$ with multiplicity $\dim_F(V)$. We conclude since the lift of $1_F \in F$ is $1_K \in K$.

3. This is clear, because $\hat{\bar{\zeta}} = Q(Q^{-1}(\zeta)) = \zeta$ for any $\zeta \in \mu_e(F)$.

4. We can put $\rho(g)$ in a block triangular form, with blocks corresponding to V and W/V. It follows that the eigenvalues for the action of g on W are the union of the eigenvalues for V and V/W.

5. Let $\rho_W : G \to GL(W)$ be the corresponding homomorphism. Let $w_1, \dots, w_n$ be a K-basis of K contained in L. This induces an isomorphism $GL(W) \cong GL(n, K)$. This identifies $\rho_W(g)$ with an element $M \in GL(n, K)$. Moreover, we have the following commutative diagram

$$\begin{array}{ccccc} GL(W) & \xrightarrow{\sim} & GL(n,K) & \hookleftarrow & GL(n,\mathcal{O}) \\ \rho_W(g) & \longmapsto & M & & \\ & & \downarrow & & \downarrow \\ \rho_{\overline{W}}(g) & \longleftarrow & \overline{M} & & \\ GL(\overline{W}) & & \xrightarrow{\sim} & & GL(n,F) \end{array}$$

where $\overline{M}$ is obtained by restricting the entries of M to $\mathcal{O}/\mathfrak{m}$. The eigenvalues of $\rho_W(g)$ are the same of M, and the eigenvalues of $\rho_{\overline{W}}(g)$ are the same of $\overline{M}$.

Let $g \in G_{reg}$. Recall that M is diagonalizable, and the eigenvalues of M are in $\mu_e(K) \subset \mathcal{O}$. Let $p_M(x) \in \mathcal{O}[x]$ denote the characteristic polynomial. We have

$$p_M(x) = \prod_{\zeta} (x - \zeta) \quad \text{and} \quad \chi_W(g) = \sum_{\zeta} \zeta$$

where ζ runs over the eigenvalues of M. Then $\overline{p_M}(x) = p_{\overline{M}}(x) \in F[x]$ is the characteristic polynomial of $\overline{M}$. It follows, that

$$p_{\overline{M}}(x) = \overline{p_M(x)} = \prod_{\zeta} (x - \bar{\zeta})$$

By Lemma 15.5, also $\overline{M}$ is diagonalizable, so the $\bar{\zeta}$ are all the eigenvalues of $\overline{M}$, or equivalently of $\rho_{\overline{W}}(g)$. So we have

$$\lambda_{\overline{M}}(g) = \sum_{\zeta} \hat{\bar{\zeta}} = \sum_{\zeta} \zeta = \chi_W(g). \qquad \square$$

We can now prove the main theorem on Brauer characters.

Theorem 15.9. *Let $S_1, \dots, S_n$ be a complete set of representatives of isomorphism classes of simple FG-modules. Then, the Brauer characters $\lambda_{S_1}, \dots, \lambda_{S_n}$ form a basis of the K-valued class functions on G_{reg}.*

Proof. We first show that $\lambda_{S_1}, \dots, \lambda_{S_n}$ are linearly independent over K. Suppose not, then there are $c_1, \dots, c_n \in K$ such that

$$\sum_{i=1}^{n} c_i \lambda_{S_i}(g) = 0 \quad \text{for all } g \in G_{reg}. \tag{5}$$

Up to multiplying by the denominators, we can assume $c_i \in \mathcal{O}$. Up to dividing by some power of π, we can assume that there exists c_i with $c_i \notin (\pi)$. We can now restrict (5) to $F = \mathcal{O}/(\pi)$ and obtain, by Proposition 15.8.3,

$$\sum_{i=1}^{n} \overline{c_i}\,\overline{\lambda_{S_i}(g)} = \sum_{i=1}^{n} \overline{c_i} \chi_{S_i}(g) \quad \text{for all } g \in G_{reg}.$$

Moreover, if $g \in G$, we can write $g = g_1 g_2$ with $g_1 \in G_{reg}$ as in Lemma 15.7. Since the eigenvalues of g and g_1 coincide for any representations, we have $\chi_{S_i}(g) = \chi_{S_i}(g_1)$ for any i. It follows that

$$\sum_{i=1}^{n} \overline{c_i}\chi_{S_i}(g) \quad \text{for all } g \in G.$$

But this is a non-trivial combination which contradicts Proposition 15.1. We deduce that $\lambda_{S_1}, \ldots, \lambda_{S_n}$ are linearly independent over K.

Take now a class function $\xi : G_{reg} \to K$. We can extend it to a class function $\xi_! : G \to K$ by defining $\xi(g) = 0$ for any $g \notin G_{reg}$. We know that in characteristic 0, the characters of simple representations form a basis of the space of class functions. So, we can find $a_i \in K$ such that

$$\xi_!(g) = \sum_{i=1}^{m} a_i \chi_{V_i}(g) \quad \text{for all } g \in G,$$

where $V_1, \ldots, V_m$ are the isomorphism classes of irreducible representations of G over K. Restricting to G_{reg}, we get

$$\xi = \sum_{i=1}^{m} a_i \chi_{V_i}|_{G_{reg}}.$$

By Proposition 15.8.5, $\chi_{V_i}|_{G_{reg}}$ is the Brauer character of $\overline{V_i}$, and by Proposition 15.8.4, $\lambda_{\overline{V_i}}$ can be written as a combination of Brauer characters of irreducible representations. It follows that ξ lies in the vector space generated by $\lambda_{S_1}, \ldots, \lambda_{S_n}$. Hence, $\lambda_{S_1}, \ldots, \lambda_{S_n}$ generate the space of K-valued class functions on G_{reg}. □

Definition 15.10. We say that a conjugacy class $C \subset G$ is p-regular if it consists of p-regular elements.

Corollary 15.11. *The number of isomorphism classes of simple FG-module is equal to the number of p-regular conjugacy class of G.*

Proof. Let $S_1, \ldots, S_n$ be a complete set of representatives of isomorphism classes of simple FG-modules. Then the Brauer characters $\lambda_{S_1}, \ldots, \lambda_{S_n}$ form a basis of the K-valued class functions on G_{reg}. So we have

$$n = \dim_K(K\text{-valued class functions on } G_{reg} = |\text{conjugacy classes in } G_{reg}|.$$ □

Similarly to characteristic 0, we can construct a Brauer character table containing all the characters of irreducible representations. Knowing the Brauer character table, allows to find all the composition factors with multiplicities of any arbitrary FG-module.

Remark 15.12. The Brauer characters are often considered as $\mathbb{C}$-valued functions. In fact, we can fix an isomorphism $\mu_e(K) \cong \mu_e(\mathbb{C})$ and then lift the eigenvalues of a representation ρ from F to $\mathbb{C}$. However, the isomorphism $\mu_e(K) \cong \mu_e(\mathbb{C})$ is not unique, and different embedding will lead to different Brauer characters.

Corollary 15.13. *Let V be a FG-module. Let $S_1, \ldots, S_n$ be a complete set of representatives of isomorphism classes of simple FG-modules. Then*

$$\lambda_V = \sum_{i=1}^{n} [V : S_i]\lambda_{S_i}.$$

Proof. This immediately follows from Proposition 15.8.4. □

Example 15.14. We compute the Brauer character table of S_3 for $p = 2$. Recall that the decomposition matrix is

$$\begin{array}{c} \\ triv \\ sgn \\ V \end{array} \begin{array}{c} \begin{array}{cc} triv & FS_3e_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

and the character table in characteristic 0 is

S_3	$\emptyset$	(12)	(123)
triv	1	1	1
sign	1	-1	1
V	2	0	-1

There are 2-regular conjugacy classes in S_3: 1 and (123). In this case, Brauer character are simply the character of *triv* and the standard representation V, that is

S_3	$\emptyset$	(123)
triv	1	1
FS_3e_2	2	-1

(The same argument works similarly for all other cases that we have computed (i.e. S_3 and A_4) because for these groups every irreducible representation has a lift to characteristic 0.)

As another consequence we may show that the Cartan matrix is invertible.

Proposition 15.15. *The decomposition matrix D has full rank and the Cartan matrix C of FG is invertible.*

Proof. Let $T_1, \dots, T_a$ be the simple KG-modules and let $S_1, \dots, S_b$ be the simple FG-modules. We have

$$\chi_{T_i}|_{G_{reg}} = \lambda_{L_i/\pi L_i} = \sum_{j=1}^{b} d_{ij}\lambda_{S_j}$$

where L_i is a $\mathcal{O}G$-lattice in T_i. We know that $(\chi_{T_i})_{1\le i\le a}$ is a basis of the class functions on G, so, after restriction to G_{reg}, they generate the class functions on G_{reg}. It follows that there exists a subset with b-elements $U \subset \{T_1, \dots, T_a\}$ such that $\{\chi_T|_{G_{reg}}\}_{T\in U}$ is a basis of the class functions $G_{reg} \to K$. It follows that the rows of the decomposition matrix corresponding to U are linearly independent, hence D has full rank.

Recall that $C = D^tD$ and both C and D have coefficient in $\mathbb{R}$ (actually in $\mathbb{N}$, to be precise). Let v be a vector $v \in \mathbb{R}^b$ such that $Cv = 0$. Then, $v^tD^tDv = 0$, or $(Dv, Dv) = 0$, where $(-,-)$ is the standard scalar product on $\mathbb{R}^b$. It follows that $Dv = 0$, but since D has full rank, it is injective, so $v = 0$. □

However, Brauer characters do not satisfy the orthogonality relations as ordinary characters do in the semisimple case. To recover similar relations, we also need to look at Brauer characters of the projective covers.

Theorem 15.16. *Let $S_1, \dots, S_b$ be simple FG-modules and let P_i denote the projective cover of S_i. Let λ denote the Brauer character. Then*

$$\langle \lambda_{P_i}, \lambda_{S_j}\rangle_{reg} := \frac{1}{|G|}\sum_{g\in G_{reg}} \lambda_{P_i}(g)\lambda_{S_j}(g^{-1}) = \delta_{ij}$$

Before the proof, we need two preliminary Lemmas.

Lemma 15.17. *Let P be a projective FG-module with lift $\widehat{P}$ to a $\mathcal{O}G$-module. Let $g \in G \setminus G_{reg}$. Then $\chi_{\widehat{P}^K}(g) = 0$.*

Proof. We can write $g = g_1 g_2 = g_2 g_1$ with g_1 p-regular and g_2 p-unipotent. Let H be the subgroup of G generated by g and, for $i \in \{1, 2\}$, let $H_i \subset H$ be the subgroup generated by g_i. We have $g_2 \neq 1$, so H_2 is a non-trivial p-group. Notice that H is generated by g_1 and g_2.

We can diagonalize the action of g_1 on P, so we can decompose

$$P = \bigoplus_{\zeta \in \mu_e(F)} W_\zeta$$

into eigenspaces for g_1. Since g_2 commutes with g_1, then g_2 preserves the eigenspaces W_ζ. In particular, each W_ζ is a FH-module.

Recall that $\mathrm{res}_G^H(P)$ is a projective FH-module. Being a summand of P, also W_ζ is a projective FH-modules. It is enough to prove the claim for $P = W_\zeta$.

Then g_1 acts on W_ζ as multiplication by ζ, so we have $W_\zeta \cong F_\zeta^{\oplus \dim_F W_\zeta}$ as FH_1-modules, where F_ζ is the one-dimensional representation on which g_1 acts by ζ. By uniqueness of the lift, we have

$$\widehat{W_\zeta}^K \cong K_{Q^{-1}(\zeta)}^{\oplus \dim W_\zeta}$$

with Q as in (4). In particular, g_1 acts on $\widehat{W_\zeta}$ as multiplication by $Q^{-1}(\zeta) \in \mu_e(K)$. Hence,

$$\chi_{\widehat{W_\zeta}}(g) = Q^{-1}(\zeta)\chi_{\widehat{W_\zeta}}(g_2).$$

It remains to show that $\chi_{\widehat{W_\zeta}}(g_2) = 0$

The restriction $\mathrm{res}_H^{H_2}(W_\zeta)$ is a projective FH_2-module. Since H_2 is a p-group, then $\mathrm{res}_H^{H_2}(W_\zeta)$ is free, i.e. $W_\zeta \cong (FH_2)^{\oplus n_\zeta}$ as FH_2-modules for some $n_\zeta \in \mathbb{N}$. By uniqueness of the lift, we also have

$$\widehat{W_\zeta}^K \cong (KH_2)^{\oplus n_\zeta}$$

as KH_2-modules. Then we obtain

$$\chi_{\widehat{W_\zeta}}(g_2) = n_\zeta \operatorname{Tr}(g_2 : KH_2 \to KH_2) = 0$$

because $g_2 \neq 0$. □

Proof of Theorem 15.16. Consider the $(b \times b)$-matrix $\Lambda = (\langle \lambda_{P_i}, \lambda_{S_j} \rangle_{reg})_{i,j}$. Our goal is to show that $\Lambda = \mathrm{Id}_n$.

Recall that we can lift P_S to a KG-module $\widehat{P_S}^K$. Let $T_1, \ldots, T_a$ be the isomorphism classes of irreducible KG-module. Let L_i be a G-stable $\mathcal{O}$-lattice inside T_i. We have $[L_i/\pi L_i : S_j] = d_{ij}$. Taking the Brauer characters, we have

$$\chi_{T_i}|_{G_{reg}} = \lambda_{L_i/\pi L_i} = \sum_{j=1}^{b} d_{ij} \lambda_{S_j}$$

On the other hand, by Theorem 14.3, we have

$$\widehat{P_j}^K = \bigoplus_{i=1}^{a} d_{ij} T_i$$

Taking the character we obtain

$$\chi_{\widehat{P}_j^K} = \sum_{i=1}^{a} d_{ij}\chi_{T_i}.$$

From the orthogonality relations for KG-modules we have $\langle \chi_{\widehat{P}_j^K}, \chi_{T_k} \rangle = d_{kj}$. Recall by Lemma 15.17 that $\chi_{\widehat{P_j}^K}(g) = 0$ if $g \notin G_{reg}$. We obtain

$$d_{kj} = \langle \chi_{\widehat{P}_j^K}, \chi_{T_k} \rangle = \langle \chi_{\widehat{P}_j^K}, \chi_{T_k} \rangle_{reg} = \langle \lambda_{P_j}, \lambda_{L_k/\pi L_k} \rangle_{reg} = \sum_{r=1}^{b} d_{kr} \langle \lambda_{P_i}, \lambda_{S_r} \rangle_{reg}.$$

From here, it follows that $D = D\Lambda$. Multiplying on the left by D^t, we obtain $C = C\Lambda$. Since C is invertible, we get $\Lambda = \mathrm{Id}_n$. □

15.1 Brauer character tables of the symmetric group S_4

Recall that the ordinary character table of S_4 in characteristic 0 is

S_4	$\emptyset$	(12)	(123)	(1234)	(12)(34)
triv	1	1	1	1	1
sign	1	-1	1	-1	1
W	3	1	0	-1	-1
W'	3	-1	0	1	-1
V	2	0	-1	0	2

We compute the Brauer character table for $p = 2$ and $p = 3$.

p=2. The are only two 2-regular conjugacy classes: 1 and (123), so there are two irreducible representations in characteristic 2. One is the trivial one. Let $\overline{V}$ be the restriction of V mod 2. Then $\overline{V}$ is irreducible, or $\overline{V}$ is an extension of the trivial representation with itself. But $\lambda_{\overline{V}} = \chi_V|_{G_{reg}} \neq 2\lambda_{triv}$, so $\overline{V}$ must be irreducible. The Brauer character table is

S_4	$\emptyset$	(123)
triv	1	1
$\overline{V}$	2	-1

Looking at Brauer character, we immediately see that $\overline{sign} = triv$, and $\lambda_{\overline{W}} = \lambda_{\overline{W'}} = \lambda_{triv} + \lambda_{\overline{V}}$. Hence, the decomposition and the Cartan matrices are

$$D = \begin{matrix} & triv & \overline{V} \\ triv \\ sign \\ W \\ W' \\ V \end{matrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$$

p=3. The are four 3-regular conjugacy classes: all except (123). $triv$ and $sign$ are well-defined and different in characteristic 3. There are no other representations of dimension 1.

We have $\lambda_{\overline{V}} = \lambda_{triv} + \lambda_{sign}$. Now, consider $\overline{W}$. It is easy to see that $\lambda_{\overline{W}}$ is not a linear combination of λ_{triv} and λ_{sign}. So either $\lambda_{\overline{W}}$ is irreducible, or there exists a representation Z of dimension 2 with $\lambda_Z = \lambda_{\overline{W}} - \lambda_{triv}$ or $\lambda_Z = \lambda_{\overline{W}} - \lambda_{sign}$. We show that there cannot be any representation with such a character.

In the first case, we have $\lambda_Z((1234)) = \lambda_Z((12)(34)) = -2$. This means that both (1234) and (12)(34) act as multiplication by -1 on Z. This is not possible because $(13)(24) = (1234)^2$.

In the second case, we have $\lambda_Z((12)) = 2$, so (12) acts trivially on Z. Since the conjugacy class of (12) generates S_4, then S_4 acts trivially on Z. This is also not possible.

So such a Z cannot exist and $\overline{W}$ is irreducible. Similarly, $\overline{W}$ is also irreducible. We can now compute the decomposition and the Cartan matrix. We have

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1. \end{pmatrix}$$

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