

Applications of model theory to combinatorics

Frank O. Wagner

Université Claude Bernard Lyon 1

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Ultraproducts

Non-principal ultraproducts are a good way to study *asymptotic behaviour*.

Consider a class \mathcal{K} of finite structures, a sequence $(\mathfrak{M}_n : n < \omega)$ in \mathcal{K} with $|M_n| \geq n$, and a non-principal ultrafilter \mathcal{U} on ω . Then the theory T of the ultraproduct $\prod_{\mathcal{U}} \mathfrak{M}_n$ exhibits the asymptotic properties of the sequence $(\mathfrak{M}_n)_n$. In particular:

- A first-order property ϕ has a 0 – 1-law for the sequence $(\mathfrak{M}_n)_n$ iff “ $\phi \in T$ ” does not depend on \mathcal{U} .
 It has a 0 – 1-law for \mathcal{K} iff “ $\phi \in T$ ” does not depend on \mathcal{U} nor on the sequence $(\mathfrak{M}_n)_n$.

Partitioning \mathcal{K} so that every part has a 0 – 1-law for all sentences amounts to classifying the pseudofinite completions. An important example is Ax’ study of pseudofinite fields: Two pseudofinite fields are elementarily equivalent iff they have the same algebraic numbers.

Of course, nothing requires the \mathfrak{M}_n to be finite. Establishing 0 – 1-laws then yields *transfer principles*. The prime example here is the theory ACF of algebraically closed fields (where the characteristic replaces size).

Nor do we have to choose different structures \mathfrak{M}_n . We can simply re-interpret the language. For instance, in Gromov's *asymptotic cones* we consider a metric space (a finitely generated group with the word metric), and obtain \mathfrak{M}_n from \mathfrak{M}_1 by dividing the metric by n .

Given a group $G = \mathfrak{M}_1$ and a subset X , Hrushovski obtains a type-definable subgroup $H = \bigcap_{i < \omega} X_i$ containing X by interpreting X_i in \mathfrak{M}_n as X^{n-i} , for all $n > i$.

Pseudofinite measure

Suppose we are given an ultraproduct $\mathfrak{M} = \prod_{\mathcal{U}} \mathfrak{M}_n$ and two interpretable subsets $\varphi(x)$ and $\psi(y)$ such that $\varphi(\mathfrak{M}_n)$ and $\psi(\mathfrak{M}_n)$ are finite for (almost) all n .

We define the *pseudofinite measure* of φ relative to ψ as

$$\mu_{\psi}(\varphi) = \text{std} \left(\prod_{\mathcal{U}} \frac{|\varphi(\mathfrak{M}_n)|}{|\psi(\mathfrak{M}_n)|} \right) \in \mathbb{R}_0^+ \cup \{\infty\}.$$

This is most often used for $\varphi \subseteq \psi$.

If the \mathfrak{M}_n are (almost) all finite, one usually takes $\psi(y)$ to be $y = y$ in the home sort.

Pseudofinite measure is finitely additive, invariant under definable bijection, and satisfies the appropriate version of Fubini. Since the ultraproduct is \aleph_1 -saturated, it is (vacuously) σ -additive.

Pseudofinite dimension

Put $d(\varphi) = \prod_{\mathcal{U}} \log(|\varphi(\mathfrak{M}_n)|) \in \prod_{\mathcal{U}} \mathbb{R} = \mathbb{R}^*$.

We define *pseudofinite dimension* with respect to an infinite convex additive subgroup $\mathbb{Z} \leq C \leq \mathbb{R}^* =$ as

$$d_C(\varphi) = d(\varphi) + C \in \mathbb{R}^*/C.$$

If C is the convex hull of \mathbb{Z} , this is *fine* pseudofinite dimension d_{fine} . It satisfies $d_{\text{fine}}(\varphi) < d_{\text{fine}}(\psi)$ iff $\mu_\psi(\varphi) = 0$ iff $\mu_\varphi(\psi) = \infty$.

Often, we normalize with respect to an \emptyset -interpretable infinite set ψ , taking C to be the maximal convex additive subgroup avoiding $d(\psi)$. Alternatively, we can directly define *coarse pseudofinite dimension with respect to ψ* as

$$d_\psi(\varphi) = \text{std} \left(\frac{d(\varphi)}{d(\psi)} \right) \in \mathbb{R}_0^+ \cup \{\pm\infty\}.$$

We put $d_\psi(\emptyset) = -\infty$. Note that $d_\psi(\psi) = 1$.

Invariance and continuity

Pseudofinite dimension is invariant under definable correspondences.

Neither measure nor dimension need be invariant under automorphisms.

However, if we expand the language by *cardinality comparison quantifiers*, i.e. predicates $R_{\varphi,\psi}(a, b)$ for $|\varphi(x, a)| < |\psi(y, b)|$, they become not only automorphism-invariant, but also continuous, where $f : M \rightarrow \mathbb{R}$ is *continuous* if for all $\alpha < \beta$ there is an \emptyset -definable set D such that

$$\{a : f(a) \leq \alpha\} \subseteq D \subseteq \{a : f(a) < \beta\}.$$

Fine versus coarse dimension

- $d_{\text{fine}}(\varphi) \leq d_{\text{fine}}(\psi)$ iff there is $k \in \mathbb{N}$ such that $|\varphi(\mathfrak{M}_n)| \leq k |\psi(\mathfrak{M}_n)|$ for all n .
- $d_{\psi}(\varphi) \leq d_{\psi}(\psi)$ iff $|\varphi(\mathfrak{M}_n)| \leq |\psi(\mathfrak{M}_n)|^{1+\epsilon}$ for all $\epsilon > 0$ and almost all n .

For instance, a symmetric subset A containing 1 of a group is an *approximate subgroup* if there is a subset K with $d_{\text{fine}}(K) = 0$ and $A \cdot A \subseteq K \cdot A$. Finite approximate subgroups have been classified by Breuillard, Green and Tao, following work of Hrushovski.

The problem seems much harder if we weaken the condition $d_{\text{fine}}(K) = 0$ to $d_A(K) = 0$.

Larsen-Pink

Theorem (Larsen-Pink, Hrushovski-W.)

Let G be a simple group of finite Morley rank (or a finite power of such). Then for any Zariski-dense pseudofinite subgroup Γ and definable $X \subseteq G$ we have

$$d_C(\Gamma \cap X) \leq \frac{RM(X)}{RM(G)} d_C(\Gamma).$$

Corollary (Larsen-Pink, Hrushovski-W.)

If in addition $X = C_G(a)$ or $X = a^G$ for $a \in \Gamma$, then equality holds.

Theorem (Hrushovski-W.)

Let K be a field of finite Morley rank and characteristic $p > 0$, and X a definable subset of K^n . Then

$$d_C(X \cap \mathbb{F}_p^{alg}) \leq \frac{RM(X)}{RM(K)} d_C(\mathbb{F}_p^{alg}).$$

Amenability

A definable set X is

- *amenable* if there is a finitely additive mean μ normalized at X (a *probability measure*) on the family of all subsets of X ;
- *definably amenable* if there is such a mean (a *Keisler measure*) on the family of all definable subsets of X .

If X is a group, we ask that the mean be left-invariant.

No assumptions are made on automorphism-invariance or continuity of the mean.

A lattice \mathcal{D} of subsets of a group G is *G -invariant* if $gX \in \mathcal{D}$ for all $X \in \mathcal{D}$; we also call it a *G -lattice*. Often, we only need the mean to exist on some such \mathcal{D} , not on the family of all definable subsets of G . A mean μ on \mathcal{D} is *normalized at $X \in \mathcal{D}$* if $\mu(X) = 1$.

We do *not* assume that $G \in \mathcal{D}$. This definition can be extended in a natural way to hyperdefinable groups, and to group actions.

The Stabilizer Theorem I

Theorem (Sanders, Massicot-W, Hrushovski-Krupinski-Pillay)

Let $X \subseteq G$, and let \mathcal{D} be a G -lattice containing X and YX , where Y is any finite intersection of left translates of X . Suppose \mathcal{D} carries a mean normalized at X . Then there is an X -generic symmetric set $Z \subseteq G$ positively definable in $\langle G, \cdot, X \rangle$ with $Z^8 \subseteq X^4$.

Nothing is said about definability of X . It may be type-definable, or \forall -definable; positive definability of Z ensures that Z is as definable as X .

Z is X -generic if boundedly many left translates of Z cover X .

If the hypotheses are always inherited by Z , we can iterate and obtain (even for \forall -definable X) a type-definable generic subgroup of X^4 .

Applications

Theorem (Hrushovski)

Let A be a pseudofinite approximate subgroup. Then A has a Lie model, i.e. there is a type-definable generic subgroup $H \subseteq A^4$ normalized by A such that $\langle A \rangle / H$ is locally compact.

This is the first step of the classification of finite approximate groups by Breuillard, Green and Tao. $\langle A \rangle / H$ is nilpotent.

Theorem (Hrushovski-Krupinski-Pillay)

Let $H \leq G$ be type-definable, normalized by $G(\mathfrak{M})$, and $N = \langle H \rangle^G$. If $S_{G/H}(\mathfrak{M})$ or $S_{H \setminus G}(\mathfrak{M})$ carries a $G(\mathfrak{M})$ -invariant Borel probability measure, then $G_{\mathfrak{M}}^{00} \leq N G_{\mathfrak{M}}^{000}$.

Theorem (Hrushovski-Krupinski-Pillay)

Suppose T is amenable: Every $p \in S(\emptyset)$ extends to a global invariant Keisler measure. Then T is G -compact: Kim-Pillay strong type equals Lascar strong type.

S_1 -ideals

An invariant ideal \mathcal{I} of definable proper subsets of an \emptyset -invariant set X is S_1 if for any indiscernible sequence $(X_i : i < \omega)$ of definable subsets, if $X_0 \cap X_1 \in \mathcal{I}$ then $X_0 \in \mathcal{I}$.

An S_1 -ideal contains all forking formulas.

Membership in an S_1 -ideal is considered a smallness notion; a type is \mathcal{I} -wide (or just *wide* if \mathcal{I} is clear) if it does not imply a formula in \mathcal{I} . Note that wide types exist, since subsets in \mathcal{I} are proper, and do not fork.

If X is \bigvee -definable over \emptyset and μ is a non-trivial invariant mean on definable subsets of X , then

$$\mathcal{I} = \{Y \subseteq X \text{ definable} : \mu(Y) = 0\}$$

is an S_1 -ideal.

For this to work, the mean must be defined at least on all forking formulas.

The Stabilizer Theorem II

Theorem (Hrushovski)

Let X be \forall -definable over \emptyset and \mathcal{I} an S_1 -ideal on X . Let \mathfrak{M} be a model, and p a global wide type finitely satisfiable in \mathfrak{M} .

1. If $b \models p|_{\mathfrak{M}}$ and $a \models p|_{\mathfrak{M}b}$ then $\text{tp}(b/\mathfrak{M}a)$ is wide.
2. We can choose \mathfrak{M} and p such that for a, b as above, $\text{tp}(a/\mathfrak{M}b)$ is wide.

Theorem (Hrushovski)

Let \mathfrak{M} be a model, G a \forall -definable group, \mathcal{I} an \mathfrak{M} -invariant S_1 G -ideal on G , and p a wide type in G over \mathfrak{M} . Suppose there are $a, b \models p|_{\mathfrak{M}}$ such that neither $\text{tp}(a/\mathfrak{M}b)$ nor $\text{tp}(b/\mathfrak{M}a)$ fork over \mathfrak{M} . Then there is a wide normal subgroup H of G type-definable over \mathfrak{M} . In fact, $H = (p^{-1}p)^2$.

Applied to an approximate subgroup, this yields a Lie-model.

Regularity

For a finite graph (V, E) and $A, B \subseteq V$ put $E(A, B) = E \cap (A \times B)$.

For $\epsilon > 0$ we call (A, B) ϵ -regular if there is $\delta \in [0, 1]$ such that $|E(A', B') - \delta|A'||B'||| < \epsilon|A||B|$ for all $A' \subseteq A$ and $B' \subseteq B$.

Theorem (Szemerédi's Regularity Lemma)

For all ϵ there is $n \in \mathbb{N}$ such that any finite graph (V, E) can be partitioned $V = V_1 \cup \dots \cup V_n$ such that all but at most ϵn^2 pairs (V_i, V_j) are ϵ -regular.

We can in addition require that $||V_i| - |V_j|| \leq 1$ for all $i, j \leq n$ (equipartition).

The set $\Sigma = \{(i, j) : (V_i, V_j) \text{ is not } \epsilon\text{-regular}\}$ is called the *exceptional set*.

Under additional model-theoretic hypotheses, this can be considerably strengthened.

Regularity for dependent hypergraphs

Theorem (Chernikov-Starchenko)

Let $k \geq 2$, and V_1, \dots, V_k be definable in a dependent structure. For every definable k -ary relation E there is c such that for all $\epsilon > 0$ and any generically stable Keisler measures μ_i on V_i there are partitions \mathcal{P}_i of V_i and an exceptional set $\Sigma \subset \mathcal{P}_1 \times \dots \times \mathcal{P}_n$ such that

- $|\mathcal{P}_i| \leq (\frac{1}{\epsilon})^c$,
- $\sum_{\bar{X} \in \Sigma} \mu(X_1 \times \dots \times X_k) \leq \epsilon$, where $\mu = \mu_1 \otimes \dots \otimes \mu_k$,
- For all $\bar{X} \in \mathcal{P}_1 \times \dots \times \mathcal{P}_k \setminus \Sigma$ there is $\delta_{\bar{X}} \in \{0, 1\}$ with $|\mu(E(\bar{X})) - \delta_{\bar{X}} \mu(X_1 \times \dots \times X_k)| < \epsilon \mu(X_1 \times \dots \times X_k)$.

If the ambient structure is stable, $\Sigma = \emptyset$. If it is distal, then $E(\bar{X})$ is either total or empty, for all $\bar{X} \notin \Sigma$ (Erdős-Hajnal property).

In addition, all parts are defined by boolean combinations of fibres of E , independent of ϵ in the distal case.

Remarks

This generalizes earlier results of Malliaris-Shelah for the stable case, and of Chernikov-Starchenko and Simon for the distal case.

A measure is *generically stable* if it can be approximated by the counting measure on finitely many points. The pseudofinite counting measure is generically stable.

A theory is *distal* if every generically stable measure has a unique extension to every elementary superstructure, it is *stable* if no formula can order an infinite set, and *dependent* if no formula can uniformly cut out all subsets of arbitrarily large finite sets.

The main ingredient in the proof is a form of the Vapnik-Chervonenkis Theorem for generically stable measures in a dependent theory.

Sum-product phenomena

Theorem (Erdős-Szemerédi)

There is $c > 0$ such that for all finite $A \subset \mathbb{R}$

$$\max\{|A + A|, |A \cdot A|\} \geq |A|^{1+c}.$$

This is a special case of:

Theorem (Elekes-Szabó)

For any complex irreducible algebraic surface $V \subseteq \mathbb{C}^3$ projecting dominantly onto any two coordinates, either V is in coordinatewise algebraic correspondence with the graph of multiplication of a 1-dimensional algebraic group, or there is $\epsilon > 0$ such that $|V(A, B, C)| \leq O(n^{2-\epsilon})$ for any $A \times B \times C \subset \mathbb{C}^3$ with $|A| = |B| = |C| = n$.

Model-theoretic Elekes-Szabó

A relation $R \subset X_1 \times \cdots \times X_k$ is *fibre-algebraic* if there is d such that any projection to $k - 1$ coordinates has fibre size $\leq d$.

Let \mathcal{F}_i be a definable family of subsets of X_i , and $n \in \mathbb{N}$. A set $A_i \subset X_i$ is in (\mathcal{F}_i, n) -*general position* if $|A_i \cap F| \leq n$ for all $F \in \mathcal{F}_i$. A tuple \bar{A} is in $(\bar{\mathcal{F}}, n)$ -*general position* if each A_i is in (\mathcal{F}_i, n) -general position.

R admits *power saving* if there is $\epsilon > 0$ and some definable families $\bar{\mathcal{F}}$ on \bar{X} such that for all $n \in \mathbb{N}$ and any $A_i \subset X_i$ in $(\bar{\mathcal{F}}, n)$ -general position and $|A_1| = \cdots = |A_n|$ we have $|R(\bar{A})| = O_n(|A_1|^{k-1-\epsilon})$.

R is in *generic correspondence with multiplication* in a group G if over some parameters there are $k - 1$ independent generic elements g_1, \dots, g_{k-1} , and for all $i \leq k$ there is generic $a_i \in X_i$ interalgebraic with g_i (where $g_k^{-1} = g_1 g_2 \cdots g_{k-1}$), such that \bar{a} is generic in R .

Elekes-Szabó for stable and \mathcal{o} -minimal hypergraphs

Theorem (Bays-Breuillard, Chernikov-Peterzil-Starchenko)

Let R be a fibre-algebraic relation on $X_1 \times \cdots \times X_k$ definable in a stable theory with a distal expansion, or in an \mathcal{o} -minimal group. Then either R admits power saving, or R is in generic correspondence with multiplication in a type-definable abelian group.

In the \mathcal{o} -minimal case, the power saving bound is explicit.

The hypotheses imply that the X_i are in definable correspondence with one another.

Genericity in X_i and in R is *a priori* not well-defined; in the \mathcal{o} -minimal case it is with respect to topological (\mathcal{o} -minimal) dimension, in the stable case it is with respect to a pre-chosen set of interalgebraic types $p_i \vdash X_i$.

The proof uses a higher arity abelian group configuration theorem, as well as distal cell decomposition.

Szemerédi-Trotter

Theorem (Szemerédi-Trotter)

Let P be a finite set of points and L a finite set of lines in the real plane. Then

$$|\{(p, \ell) \in P \times L : p \in \ell\}| \leq 4|P|^{\frac{2}{3}}|L|^{\frac{2}{3}} + 4|P| + |L|.$$

Note that incidence avoids the complete bipartite graph $K_{2,2}$.

Theorem (Chernikov-Starchenko)

Let $R \subset X \times Y$ be $K_{d,s}$ -free, and admit a distal cell decomposition of exponent t . Then for all finite subsets $A \subset X$ and $B \subset Y$

$$|R(A, B)| = O_R(|A|^{\frac{(t-1)d}{td-1}} |B|^{\frac{t(d-1)}{td-1}} + |A| + |B|).$$

Theorem (Bays-Martin)

If K is a finitely generated extension of \mathbb{F}_p , $X = K^n$, $Y = K^m$ and R the zero set of a set of polynomials. If R is $K_{d,s}$ -free, then there is $\epsilon > 0$ such that for all finite subsets $A \subset X$ and $B \subset Y$

$$|R(A, B)| = O_R(|A|^{1-\epsilon} |B|^{\frac{(d-1)}{d}(1+\epsilon)} + |A| + |B|).$$

Thank you !