Recurrent two-dimensional sequences generated by homomorphisms of finite abelian p-groups with periodic initial conditions

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Abstract

We prove that if a recurrent two-dimensional sequence with periodic initial conditions coincides in a sufficiently large starting square with a two-dimensional sequence produced by an expansive system of context-free substitutions, then they must coincide everywhere. We apply this result for some examples built up by homomorphisms of finite abelian p-groups, in particular for Pascal's Triangle modulo p^k, Pascal's Triangles modulo 2 with non-trivial periodic borders, and Sierpinski's Carpets with non-trivial periodic border. All these particular cases justify the conjecture that recurrent two-dimensional sequences generated by homomorphisms of finite abelian p-groups with periodic initial conditions can always be alternatively generated by expansive systems of context-free substitutions.

Key Words: recurrent two-dimensional sequence, periodic initial conditions, expansive system of context-free substitutions, automatic proof procedure, homomorphisms of finite abelian p-groups, Pascal's Triangle modulo p^k, Pascal's Triangle modulo 2 with periodic borders, Sierpinski's Carpet with periodic borders.

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1 Introduction

This article belongs to a series dedicated to the study of recurrent two-dimensional sequences over finite sets and their power of expression.

Definition 1.1 Let $(A, f, \lambda_0, \dots, \lambda_{n-1}, \mu_0, \dots, \mu_{m-1})$ be a finite structure with one ternary function f interpreted by a function $f : A^3 \to A$, and $n + m$ many constants interpreted by not necessarily distinct elements of A, such that both $m, n \geq 0$ and $\lambda_0 = \mu_0$. The recurrent two-dimensional sequence $a : \mathbb{N} \times \mathbb{N} \to A$ defined by this structure is the unique two-dimensional sequence satisfying the following conditions:

1. $\forall\ i \geq 0\ \ a(i, 0) = \lambda(i \mod n)$.
2. $\forall\ j \geq 0\ \ a(0, j) = \mu(j \mod m)$.
3. $\forall\ i \geq 1,\ j \geq 1\ \ a(i, j) = f(a(i-1, j), a(i-1, j-1), a(i, j-1))$.

In this definition we wrote $\lambda(u)$ and $\mu(v)$ for λ_u and μ_v, just for readability. The finite sequences $\vec{\lambda} = (\lambda_0, \dots, \lambda_{n-1})$ and $\vec{\mu} = (\mu_0, \dots, \mu_{m-1})$ are called the periods, and the recurrent two-dimensional sequence can be denoted by $(A, f, \vec{\lambda}, \vec{\mu})$. If $\vec{\lambda} = \vec{\mu} \in A^n$, the recurrent two-dimensional sequence is simply denoted by $(A, f, \vec{\lambda})$.

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Author's previous articles on this subject (see [1], [2], [3] and [4]) were dealing with the particular case $m = n = 1$. Consequently, the recurrent two-dimensional sequences considered there could have been (and was) denoted like (A, f, λ_0). We also observe that a recurrent two-dimensional sequence with periods $\vec{\lambda}$ of length n and $\vec{\mu}$ of length m can be equivalently represented using other periods $\vec{\lambda}'$ and $\vec{\mu}'$, both of equal length $\text{lcm}(n, m)$. The author used before the notion *double sequence*. However, *two-dimensional sequence* seems to be indeed a less misleading notion.

In [1] is proven that the subclass consisting of the recurrent two-dimensional sequences $(A, f, 1)$ with $f(x, y, z) = f(x, 1, z) = f(z, 1, x)$ is already Turing complete. This shows already why is not really interesting to study different generalizations in more than two dimensions: commutative operations depending of only two variables are already complicated enough.

In [2] it is proved that if $A = \mathbb{F}_q$ is the finite field with $q = p^k$ elements and $f(x, y, z) = x + my + z$, where $m \in \mathbb{F}_q$ is an arbitrary fixed element, then the recurrent two-dimensional sequence $(A, f, 1)$ is a self-similar pattern. This solves a conjectures published in [5]. The paper contains also some other results: a classification of the symmetries of the paterns in the case $q = p$ and a characterization of some regular zeros of the recurrent two-dimensional sequence for $m \in \{1, 0, -2, -1/2\}$.

In [3] this situation is generalized by introducing expansive systems of substitutions. For another notable application of the two-dimensional substitution, see Penrose's aperiodic tilings in [6].

Definition 1.2 Fix two natural numbers $x \geq 1$ and $s \geq 2$. Let $y = xs$. Let $\mathcal{X}$ be a finite set of $x \times x$ matrices over A and $\mathcal{Y}$ be a set of $y \times y$ matrices over A such that every $Y \in \mathcal{Y}$ has a $s \times s$ block-representation by connected blocks $(X(i, j))_{0 \leq i,j < s}$ and all blocks $X(i, j) \in \mathcal{X}$. We call system of substitutions of type $x \to y$ the tuple $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$, where $\Sigma : \mathcal{X} \to \mathcal{Y}$ is a fixed function and $X_1 \in \mathcal{X}$ is a fixed element of $\mathcal{X}$, called start-symbol. If a $ux \times vx$ matrix Z has the block-representation $Z = (X(i, j))_{0 \leq i < u,\ 0 \leq j < v}$ with all $X(i, j) \in \mathcal{X}$ we define $\Sigma(Z)$ to be the $usx \times vsx$ matrix with block-representation $(\Sigma(X(i, j)))$. We define the sequence of matrices $(S(n))$ by $S(1) = X_1$ and $S(n) = \Sigma^{n-1}(X_1)$. The number s is called scaling factor of the system of sustitutions.

Definition 1.3 We call the system of substitutions $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ expansive if the block representation of the matrix $\Sigma(X_1) = (X(i, j))$, using matrices in $\mathcal{X}$, fulfills $X(0, 0) = X_1$. With other words, X_1 is a left upper minor of $\Sigma(X_1)$.

Lemma 1.4 *Let $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ be an expansive system of substitutions. Then for all $n > 0$ the matrix $S(n)$ is $xs^{n-1} \times xs^{n-1}$ left upper minor of the matrix $S(n + 1)$.*

Definition 1.5 An expansive system of substitutions defines a two-dimensional sequence $(a(i, j))$ in the following way: one chooses an $n \in \mathbb{N}$ which is sufficiently large such that $0 \leq i, j < xs^{n-1}$. If $S(n) = (s(i, j))$, one defines $a(i, j) = s(i, j)$.

The Lemma 1.4, which is proved in [3], makes this definition correct.

One can see this whole construction also in the following way:

Definition 1.6 Let U a finite set, which shall be called an alphabet. Let $g : U \to M_{s \times s}(U)$ be a map, whose values are $s \times s$ matrices with entries in U. Suppose $u_0 \in U$ is the left-upper element of $g(u_0)$. Then $\Lambda = \lim\limits_{n \to \infty} g^n(u_0)$ is a two-dimensional sequence over U. We call Λ a *substitution sequence* with scaling factor s, i. e. of type $1 \to s$ over U. Let V be another alphabet and $\pi : U \to M_{x \times x}(V)$ be a bijective map, where $\pi(u)$ is always an $x \times x$ matrix with entries in V. Then $\Lambda' = \pi(\Lambda)$ is called a substitution sequence of type $x \to xs$ over V.

One observes immediately that for $V = A$ and $U = \mathcal{X}$ we get back the first definition, with $\Sigma = \pi g \pi^{-1}$ as an application from a set of $x \times x$ matrices in a set of $sx \times sx$ matrices, both over the alphabet A.

The main result of [3] was that if a recurrent two-dimensional sequence coincides with the two-dimensional sequence produced by an expansive system of substitutions on a sufficiently large left upper minor, then they must coincide everywhere. This fact can be used for implementing an automatic proof method, able to show if some recurrent two-dimensional sequence is or is not produced by an expansive system of substitutions of a given type $x \to y$. This procedure was used in [3] to show that some concrete recurrent two-dimensional sequences are produced by expansive systems of substitutions. The studied particular cases were mainly given by sums of Frobenius automorphisms over finite fields. The author observed later that general homomorphisms of finite abelian p-groups seem to always produce substitution patterns.

Finally, [4] contains a recurrent two-dimensional sequence $(A, f, 1)$ that cannot be generated by any expansive system of substitutions.

In the present paper we prove an analogon of the result of [3] for the recurrent two-dimensional sequences with periodic initial conditions (Section 2). We conjecture the fact that all the recurrent two-dimensional sequences generated by homomorphisms of finite abelian p-groups with periodic initial conditions are generated by expansive systems of substitutions (Section 3). In the rest of the paper we use the method introduced in Section 2 for verifying the Conjecture in some particular cases which might be also of own interest.

For widely related literature see monographs like the classical one of Mandelbrot [7] or the more specialized one by Stephen Wolfram [8], which focusses on cellular automata. Kari's survey [9] or Wilson's classical article [10] are also to recommend.

2 Main result

In this section we prove the immediate generalization of the main result of [3]. We recall that all matrices are indexed starting with 0.

Definition 2.1 We say that a $u \times v$ matrix $K = (k(\alpha, \beta))$ occurs in the $w \times z$ matrix $T = (t(a, b))$ if for some $0 \leq i < w$ and $0 \leq j < z$ one has $i + u \leq w$, $j + v \leq z$ and for all $0 \leq \alpha < u$ and $0 \leq \beta < v$ one has $t(i + \alpha, j + \beta) = k(\alpha, \beta)$. We say that K occurs in x-position in T if moreover $x \mid i$ and $x \mid j$.

Definition 2.2 Let x be a fixed natural number and T be a $wx \times zx$ matrix over A. We denote $\mathcal{N}_x(T)$ the set of all $2x \times 2x$ matrices occurring in x-position in T.

Definition 2.3 Let $(A, f, \vec{\lambda}, \vec{\mu})$ produce a recurrent two-dimensional sequence $R = (a(i, j))$ and let $x \geq 1$, $s \geq 2$ be two fixed natural numbers. For $k \geq 1$ let $R(n)$ be the $xs^{n-1} \times xs^{n-1}$ left upper minor of R.

Theorem 2.4 *Suppose that $(A, f, \vec{\lambda}, \vec{\mu})$ produces a recurrent two-dimensional sequence $R = (a(i, j))$ and that an expansive system of substitutions $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ of type $x \to y = sx$ produces the two-dimensional sequence $S = (b(i, j))$. For $n \in \mathbb{N}$ we define $R(n)$ according to the type $x \to y$ like in Definition 2.3. We assume that:*

1. $\forall\ i \geq 0 \ \ b(i, 0) = \lambda(i \mod n)$.
2. $\forall\ j \geq 0 \ \ b(0, j) = \mu(j \mod m)$.
3. *There exists $M \in \mathbb{N}$ such that $R(M) = S(M)$ and $\mathcal{N}_x(S(M)) = \mathcal{N}_x(S(M-1))$.*

In this case, $R = S$.

Proof: The conclusion is equivalent with $R(n) = S(n)$ for all $n \geq 1$. This is true for all $n \leq M$ by condition 3 together with the expansiveness of Σ. We prove this equality by induction for all $n \geq M$.

Definition 2.5 Recall the convention that all our matrices are indexed starting with 0. Some finite or infinite matrix $(u(i,j))$ is called an f-matrix if $\forall\ i, j \geq 1$ one has:

$$u(i,j) = f(u(i-1,j), u(i-1,j-1), u(i,j-1)).$$

We observe that $S(n) = R(n)$ if and only if [($S(n)$ fulfills the initial conditions of the recurrence) (I) & ($S(n)$ is an f-matrix) (II)]. (I) is true by assumption. In order to prove (II), we observe that it is enough to prove that all the elements of $\mathcal{N}_x(S(n))$ are f-matrices, because those $2x \times 2x$-matrices starting in x-positions cover $S(n)$ with overlappings. We denote by $\mathcal{N}_x$ the set $\mathcal{N}_x(S(M)) = \mathcal{N}_x(S(M-1))$. All the elements of $\mathcal{N}_x$ are f-matrices, because they all occur in $R(M) = S(M)$. So if we prove that the sequence $(\mathcal{N}_x(S(n)))_{n\in\mathbb{N}}$ becomes stationary at $n = M$ we are done.

Lemma 2.6 *With the notations introduced above, if* $n \geq M$ *then* $\mathcal{N}_x(S(n)) = \mathcal{N}_x$.

Proof of the Lemma 2.6: The conclusion is true for $n = M$ by definition. Consider that we have already proven that $\mathcal{N}_x(S(n)) = \mathcal{N}_x$ for some $n \geq M$. Let U be a $2x \times 2x$ matrix occurring in x-position somewhere in $S(n+1)$. U can lie inside some $\Sigma(X)$, or can lie on the border between some $\Sigma(X)$ and its neighbor $\Sigma(Y)$ or can even lie such that the four $x \times x$-minors that build U are adiacent corners in four neighboring matrices got by substitution rules:

$$\begin{pmatrix} \Sigma(X) & \Sigma(Y) \\ \Sigma(Z) & \Sigma(T) \end{pmatrix} = \Sigma \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}.$$

where $X, Y, Z, W \in \mathcal{X}$. In all cases, there exists a $2x \times 2x$-matrix V occuring in $S(n)$ in x-position such that U is covered by $\Sigma(V)$. So $V \in \mathcal{N}_x(S(n))$. But as we know that $\mathcal{N}_x(S(n)) = \mathcal{N}_x$ and $\mathcal{N}_x(S(n-1)) = \mathcal{N}_x$, we conclude that $V \in \mathcal{N}_x(S(n-1))$, so that U occurs already in $S(n)$ in some x-position, in some occurrence of $\Sigma(V)$. This means that $\mathcal{N}_x(S(n+1)) = \mathcal{N}_x(S(n)) = \mathcal{N}_x$. □

Observation: Theorem 2.4 is very useful for implementing computer methods able to automatically prove that recurrent two-dimensional sequence can be generated by expansive systems of substitutions of given types. The input consists of the recurrence rule together with the pressuposed type and a value for M. The system of substitutions Σ is constructed by inspecting the recurrent two-dimensional sequence $R(M)$. Instances that fulfill the theorem are normally very large. A possible way to implement the method and to prevent out of memory errors works as follows: one initialize in the memory two linear buffers of length kx and ksx for a big value of k, like $k = 1000$ or 10000, and two 2-dimensional buffers of $x \times x$ and $y \times y$ respectively. In a given moment of the computation the first linear buffer contains values $(a(i,j))$ with $i = lx$ and $0 \leq j < kx$, the second linear buffer contains values $(a(i,j))$ with $i = ly$ and $0 \leq j < ky$, the first 2-dimensional buffer contains the $x \times x$ minor of R starting in (lx, tx) and the second 2-dimensional buffer contains the $y \times y$ minor of R starting at (ly, ty).

Although the principal subject of this article are recurrent two-dimensional sequences generated by homomorphisms of abelian p-groups, we point out that Theorem 2.4 can be applied for any kind of function, as in the following example:

Example 2.7 *The recurrent two-dimensional sequence Hedera (see Fig. 1) given by* $(\mathbb{F}_5, x^3z^3 + x^4y + yz^4 + 2xyz + 4, 1)$ *can be realized with an expansive system of substitutions of type* $256 \to 512$ *with* 1802 *rules.*

We recall here that over finite fields any function can be expresed by a polynomial in the corresponding number of variables. We do not know if this example of substitution pattern can be also realized by an appropriated homomorphism o finite abelian p-groups.

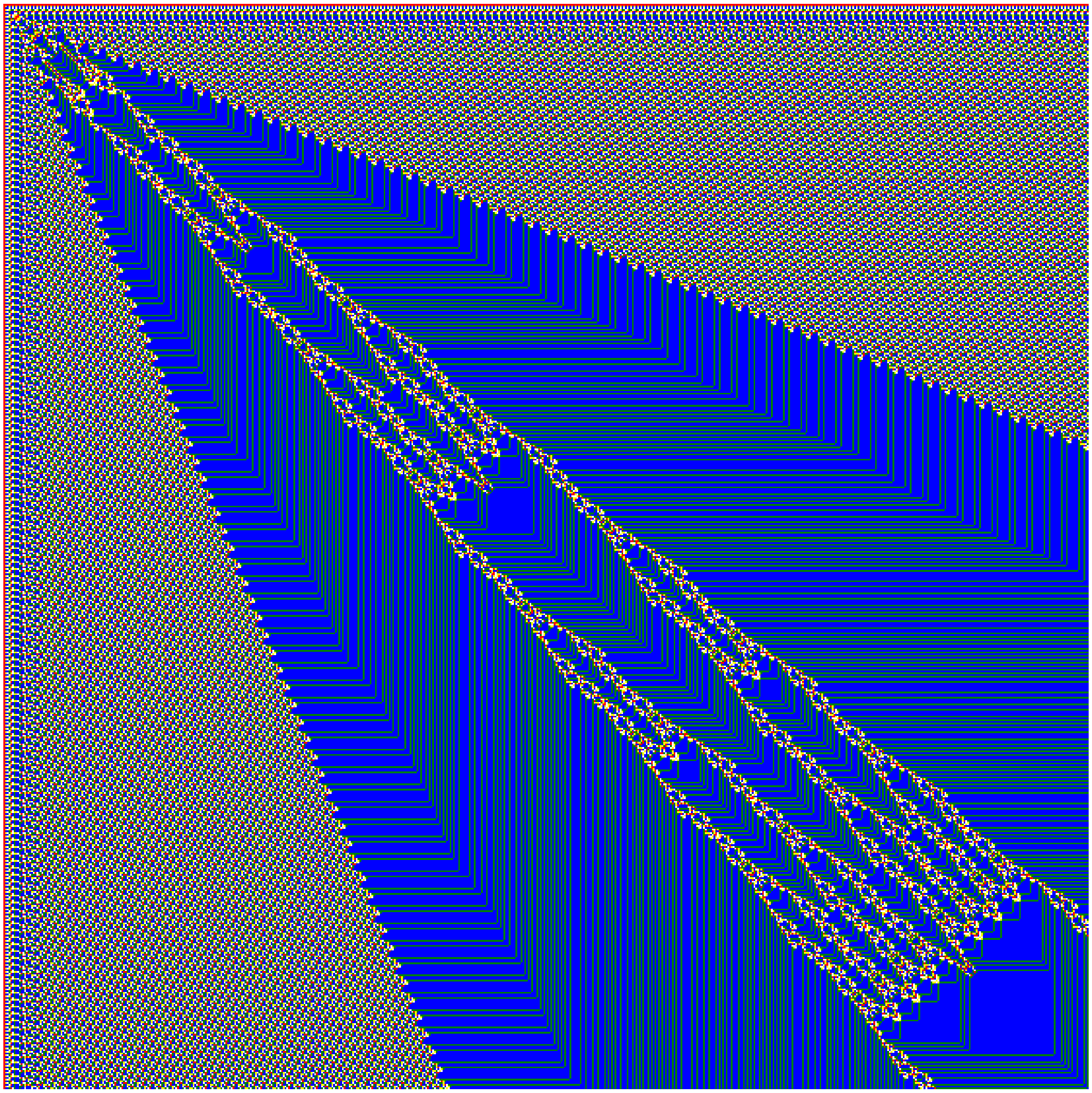

Fig. 1: Hedera, 525×525.

3 Homomorphisms of finite abelian p-groups

In order to state the main conjecture, we recall some folklore on finite abelian groups. A finite abelian group $(G, +, 0)$ is always isomorphic with a product of finite abelian p-groups $H_{p_1} \times H_{p_2} \times \dots \times H_{p_k}$ with $p_1 < p_2 < \dots < p_k$ different primes. Every finite abelian p-group H_p is a product of not necessarily different finite cyclic p-groups:

$$H_p = \mathbb{Z}/p^{e_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{e_u}\mathbb{Z},$$

with $e_1 \le e_2 \le \dots \le e_u$. Between two finite abelian p-groups corresponding to different primes there are no non-trivial homomorphisms. This fact simplifies very much the characterization of all homomorhisms: if $h : H_{p_1} \times H_{p_2} \times \dots \times H_{p_k} \to L_{q_1} \times L_{q_2} \times \dots \times L_{q_s}$ is a homomorphisms of groups, then $h = (h_1, h_2, \dots, h_s)$, where all $h_i : H_{p_1} \times H_{p_2} \times \dots \times H_{p_k} \to L_{q_i}$ is a homomorphism of groups. Further, if $q_i \notin \{p_1, \dots, p_k\}$, then $h_i = 0$. If $q_i = p_j$ for some j then there exists a homomorphisms of finite abelian p_j-groups $g : H_{p_j} \to L_{p_j}$ such that $\forall\ x_1 \in H_{p_1},\ \dots,\ x_k \in H_{p_k}\ h_i(x_1, \dots, x_k) = g(x_j)$. This way, it is enough to characterize the homomorphisms $h : H_p \to L_p$ between finite abelian p-groups corresponding to the same prime. Now let H_p be the group whose isomorphism type has been fixed above, and let L_p be:

$$L_p = \mathbb{Z}/p^{f_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{f_v}\mathbb{Z},$$

with $f_1 \le f_2 \le \cdots \le f_v$.

In this case h can be completely described by a $u \times v$ matrix of the form:

$$A = \begin{pmatrix} a_{1,1}p^{[f_1-e_1]} & a_{1,2}p^{[f_1-e_2]} & \dots & a_{1,u}p^{[f_1-e_u]} \\ a_{2,1}p^{[f_2-e_1]} & a_{2,2}p^{[f_2-e_2]} & \dots & a_{2,u}p^{[f_2-e_u]} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v,1}p^{[f_v-e_1]} & a_{v,2}p^{[f_v-e_2]} & \dots & a_{v,u}p^{[f_v-e_u]} \end{pmatrix},$$

where $[a-b] = a-b$ if $a \ge b$ and $[a-b] = 0$ if $a < b$, and $a_{i,j} \in \mathbb{Z}$. If an element of $x \in H_p$ has the representation $(x_1, \dots, x_u)$ where every $x_i \in \mathbb{Z}$ is a representative of the corresponding element of $\mathbb{Z}/p^{e_i}\mathbb{Z}$, then the vector $A\vec{x} = (y_1, \dots, y_v)$ has in every component $y_j \in \mathbb{Z}$ a representative of the projection of $h(x)$ in $\mathbb{Z}/p^{f_j}\mathbb{Z}$.

Example 3.1 *: The recurrent two-dimensional sequence called Twin Peaks, which has been studied in [3], is defined by* $(\mathbb{F}_4, f, 1)$ *with* $\mathbb{F}_4 = \mathbb{F}_2[\epsilon]$, $\epsilon^2 + \epsilon + 1 = 0$ *and* $f(x, y, z) = y + \epsilon(x+z) + \epsilon^2(x^2+y^2+z^2)$. *As an abelian group,* $\mathbb{F}_4 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *and* $f : (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *is a homomorphism of abelian groups.*

Indeed, f has the representation:

$$f(\vec{x}, \vec{y}, \vec{z}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vec{z}.$$

This is the case with all linear combinations of powers of Frobenius, and also with all linear polynomials over cyclic rings. Consequently, the conjecture presented here covers and extends all cases conjectured in [3].

Conjecture 3.2 *Let* G *be a finite abelian* p*-group,* $g : G^3 \to G$ *some homomorphism of finite abelian* p*-groups, and* $\vec{\lambda} \in G^n$, $\vec{\mu} \in G^m$ *two periods. In this situation, there always exists an expansive system of substitutions producing the same two-dimensional sequence like the recurrence* $(G, g, \vec{\lambda}, \vec{\mu})$.

In the rest of the paper we will apply Theorem 2.4 to check the conjecture in some particular examples. **All particular results given as examples in the following sections has been got by computer computations. All these computations was done with a program implementing the Theorem 2.4.**

4 Pascal's Triangle modulo p^k

The first class of examples deals again with the trivial period 1. The reason is that by writing [3] the author was so much obsessed to suggest how *beautiful* the substitution principle can be, that he really forgot to point out how *useful* it can be. If we extend the definition 1.1 to infinite structures, it is clear that the recurrent two-dimensional sequence given by $(\mathbb{Z}, x+z, 1)$ is Pascal's Triangle: $a(i,j) = \binom{i}{i+j}$. By substituting the group $\mathbb{Z}$ with groups $\mathbb{Z}/p^k\mathbb{Z}$ we are facing Pascal's Triangles modulo p^k - a subject that has been focused with different methods in the last decades.

As proven in [11] all Pascal's Triangles modulo p^k are automatic sequences, i. e. are produced by expansive systems of substitution. We display here some examples. In the following statements "produced by n rules of type $x \to y$" is an abbreviation for "produced by an expansive system of substitution of type $x \to y$ with n rules".

Example 4.1 *For $p = 2$ and $k \in \{1, 2, 3, 4, 5\}$ one has:*

- $(\mathbb{Z}/2\mathbb{Z}, x+z, 1)$ *can be produced by* 2 *rules of type* $1 \to 2$.
- $(\mathbb{Z}/4\mathbb{Z}, x+z, 1)$ *can be produced by* 8 *rules of type* $2 \to 4$.
- $(\mathbb{Z}/8\mathbb{Z}, x+z, 1)$ *can be produced by* 46 *rules of type* $2 \to 4$.
- $(\mathbb{Z}/16\mathbb{Z}, x+z, 1)$ *can be produced by* 278 *rules of type* $2 \to 4$.
- $(\mathbb{Z}/32\mathbb{Z}, x+z, 1)$ *can be produced by* 1706 *rules of type* $2 \to 4$.

Indeed, $(\mathbb{Z}/2\mathbb{Z}, x+z, 1)$ can be generated by the substitution rules $1 \to \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $0 \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ with start-symbol 1, as it was already proven in [2] by algebraic methods. This fact is already equivalent with Lucas' Theorem concerning the parity of the binomial coefficients. In [2] is also proven that for all primes p, Pascal's triangle modulo p is generated by p many substitution rules of type $1 \to p$. The Pascal's Triangles modulo 8, 16 and 32 are generated by expansive systems of substitution that are too complicated to be displayed here; but the system of substitutions generating Pascal's Triangle modulo 4 is small enough to be displayed.

Let $\mathcal{X}$ be the set consisting of the following matrices $X_1, \dots, X_8$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix}$$

Let $\mathcal{Y}$ consist of the following matrices $Y_1, \dots, Y_8$:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_5 & X_6 \end{pmatrix} \begin{pmatrix} X_1 & X_7 \\ X_3 & X_6 \end{pmatrix} \begin{pmatrix} X_4 & X_4 \\ X_4 & X_6 \end{pmatrix} \begin{pmatrix} X_8 & X_2 \\ X_5 & X_6 \end{pmatrix} \begin{pmatrix} X_6 & X_6 \\ X_6 & X_6 \end{pmatrix} \begin{pmatrix} X_8 & X_7 \\ X_3 & X_6 \end{pmatrix} \begin{pmatrix} X_8 & X_7 \\ X_5 & X_4 \end{pmatrix}$$

Let $\Sigma : \mathcal{X} \to \mathcal{Y}$ consist of all rules $X_i \rightsquigarrow Y_i$. See Fig. 2.

Example 4.2 *For $p = 3$ and $k \in \{1, 2, 3\}$ one has:*

- $(\mathbb{Z}/3\mathbb{Z}, x+z, 1)$ *can be produced by* 3 *rules of type* $1 \to 3$.
- $(\mathbb{Z}/9\mathbb{Z}, x+z, 1)$ *can be produced by* 39 *rules of type* $3 \to 9$.
- $(\mathbb{Z}/27\mathbb{Z}, x+z, 1)$ *can be produced by* 705 *rules of type* $3 \to 9$.

Example 4.3 *For $p = 5$ and $k \in \{1, 2\}$ one has:*

- $(\mathbb{Z}/5\mathbb{Z}, x+z, 1)$ *can be produced by* 5 *rules of type* $1 \to 5$.
- $(\mathbb{Z}/25\mathbb{Z}, x+z, 1)$ *can be produced by* 305 *rules of type* $5 \to 25$.

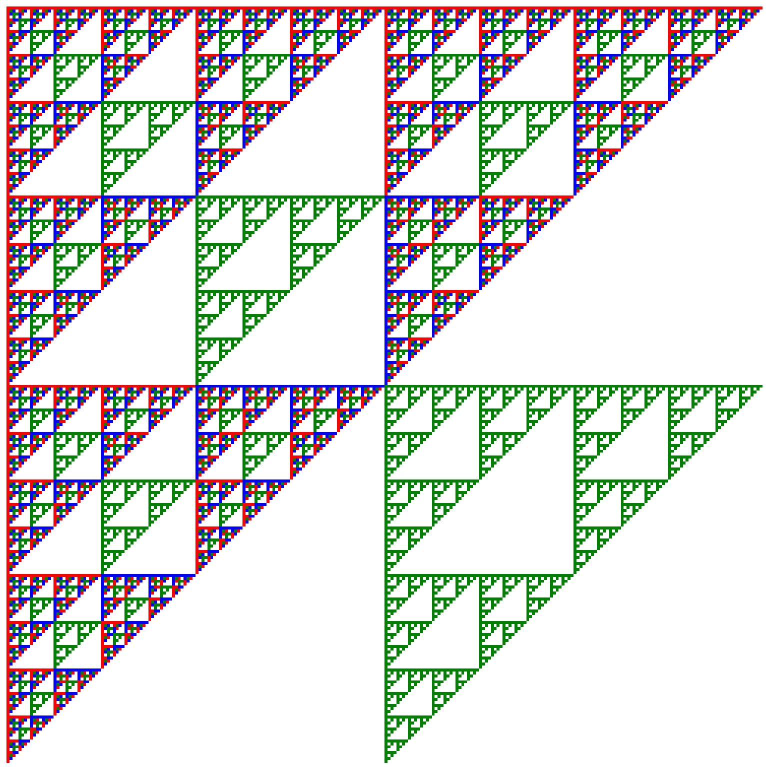

Fig. 2: Pascal's Triangle modulo 4, 256×256.

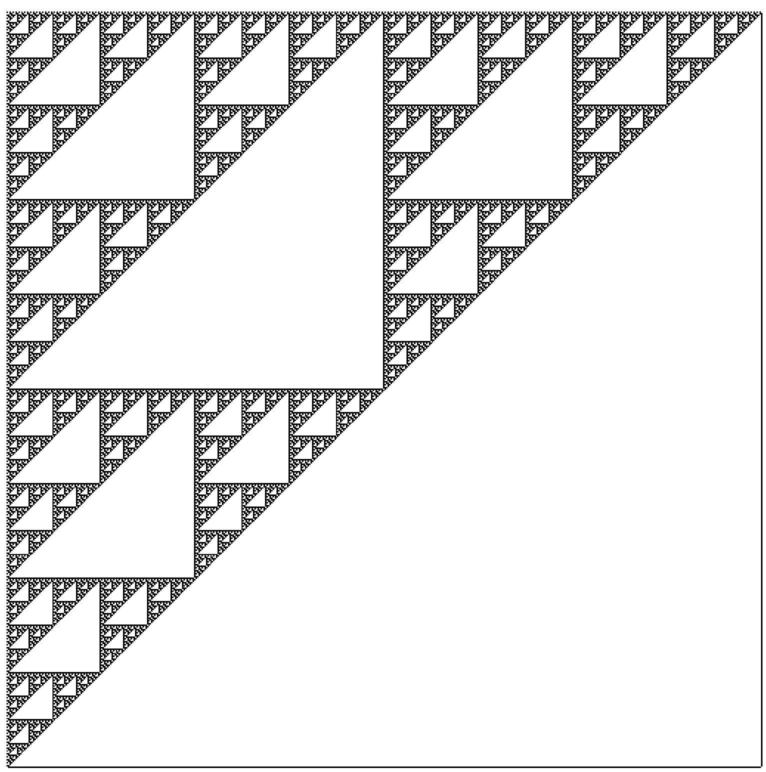

Fig. 3: Pascal's Triangle modulo 2 with period $\vec{\lambda} = 10$, 512×512.

5 Some new tissues on Pascal's Triangle modulo 2

In this section we keep the homomorphism $x + z : (\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z}$ fixed, only the periods $\vec{\lambda}$ will vary.

Example 5.1 $(\mathbb{Z}/2\mathbb{Z}, x + z, \vec{\lambda} = 10)$ *can be produced by* 8 *rules of type* $2 \to 4$*. See Fig. 3.*

The expansive system of substitutions occurring in Example 5.1 is again small enough to be displayed here.

Let $\mathcal{X}$ be the set consisting of the following matrices $X_1, \ldots, X_8$:

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$$

Let $\mathcal{Y}$ consist of the following substitution rules $Y_1, \ldots, Y_8$:

$$\begin{pmatrix}X_1 & X_2\\ X_3 & X_4\end{pmatrix} \begin{pmatrix}X_1 & X_2\\ X_5 & X_6\end{pmatrix} \begin{pmatrix}X_1 & X_5\\ X_3 & X_8\end{pmatrix} \begin{pmatrix}X_7 & X_6\\ X_8 & X_4\end{pmatrix} \begin{pmatrix}X_1 & X_5\\ X_5 & X_7\end{pmatrix} \begin{pmatrix}X_7 & X_6\\ X_7 & X_6\end{pmatrix} \begin{pmatrix}X_7 & X_7\\ X_7 & X_7\end{pmatrix} \begin{pmatrix}X_7 & X_7\\ X_8 & X_8\end{pmatrix}$$

Let $\Sigma : \mathcal{X} \to \mathcal{Y}$ consist of all rules $X_i \rightsquigarrow Y_i$. See Fig. 3.

Example 5.2 $(\mathbb{Z}/2\mathbb{Z}, x + z, \vec{\lambda} = 1000)$ *can be produced by* 8 *rules of type* $4 \to 8$*. See Fig. 4.*

Definition 5.3 We say that the two-dimensional sequences have the same structure of substitution if according to the Definition 1.6, they are generated by the same $1 \to s$ expansive substitution $g : U \to M_{s\times s}(U)$ with different bijective maps $\pi_1 : U \to M_{x_1\times x_1}(V_1)$ and $\pi_2 : U \to M_{x_2\times x_2}(V_2)$.

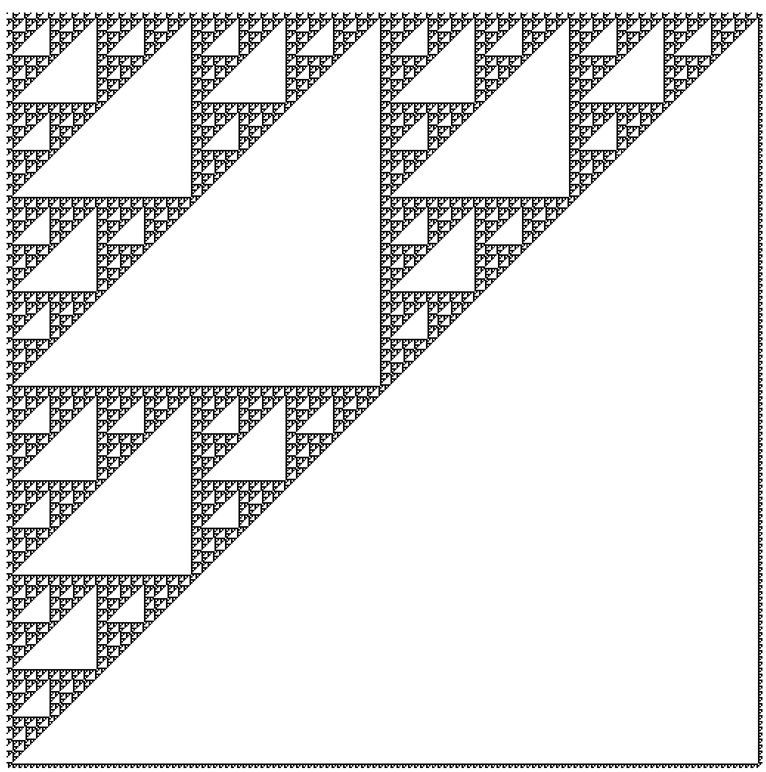

Fig. 4: Pascal's Triangle modulo 2 with period $\vec{\lambda} = 1000$, 512×512.

Interesting enough, the following example, Pascal's Triangle with period $\vec{\lambda} = 1000$, has the same substitution structure like Pascal's Triangle with period $\vec{\lambda} = 10$, but the expansive system of substitutions is that time of type $4 \to 8$. Although the elementary minors $X_1, \ldots, X_8$ cannot be computed from the corresponding minors of Pascal's Triangle with period $\vec{\lambda} = 10$, the rules of substitution $X_i \to Y_i$ are the same as there. That's why it shall be sufficient to describe the matrices X_i.

Here are the matrices $X_1, \ldots, X_4$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

And here are the matrices $X_5, \ldots, X_8$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Corollary 5.4 $(\mathbb{Z}/2\mathbb{Z}, x+z, \vec{\lambda} = 10)$ *and* $(\mathbb{Z}/2\mathbb{Z}, x+z, \vec{\lambda} = 1000)$ *have the same structure of substitutions without being of the same type.*

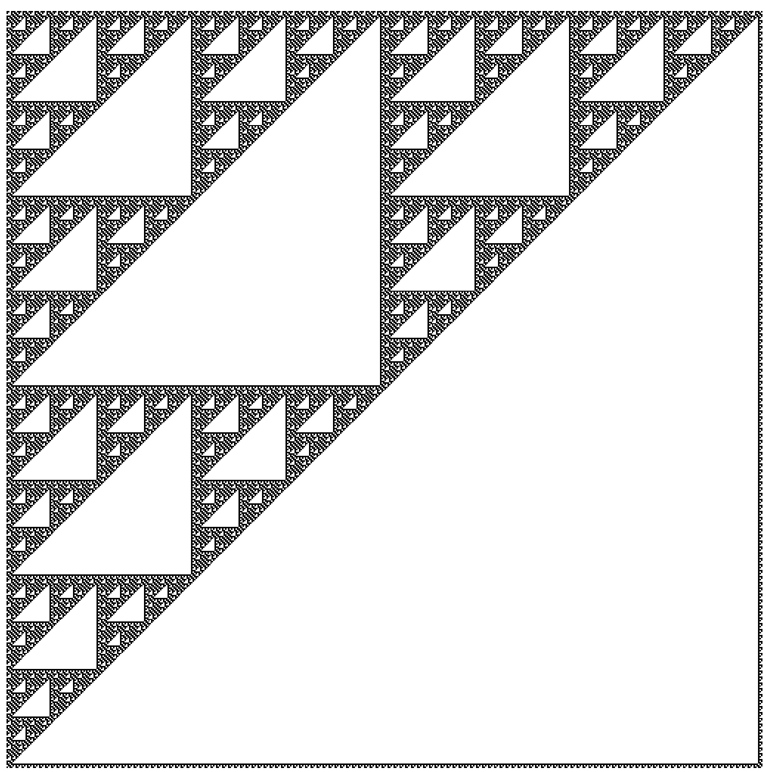

Fig. 5: Pascal's Triangle modulo 2 with period $\vec{\lambda} = 1101$, 512×512.

Example 5.5 $(\mathbb{Z}/2\mathbb{Z}, x+z, \vec{\lambda} = 1101)$ *can be produced by* 8 *rules of type* $4 \to 8$. *See Fig. 5.*

Pascal's Triangle with period 1101 and 1011 have the same type and the same structure of substitution like Pascal's Triangle with period 1000, but with different basic minors X_i. Their similarity in difference implicitly poses the problem to find a good definition able to mathematically express this similarity. The author is not sure that the following definition shall prove at the end to be the correct one for this kind of phenomenon. However, first tries must be done.

Definition 5.6 Let R be a recurrent two-dimensional sequence over a finite p-group G, and $x \geq 1$, $s \geq 1$ two natural numbers. For $n \geq 1$ let $R(n)$ be the $xs^{n-1} \times xs^{n-1}$ left upper minor of R. The scalling set $I_n \subset [0,1] \times [0,1]$ is defined as follows: The square $[0,1] \times [0,1]$ is divided in $xs^{n-1} \times xs^{n-1}$ many equal squares indexed $s(i,j)$. One excludes the interior of $s(i,j)$ from the unit square if and only if $a(i,j) = 0$.

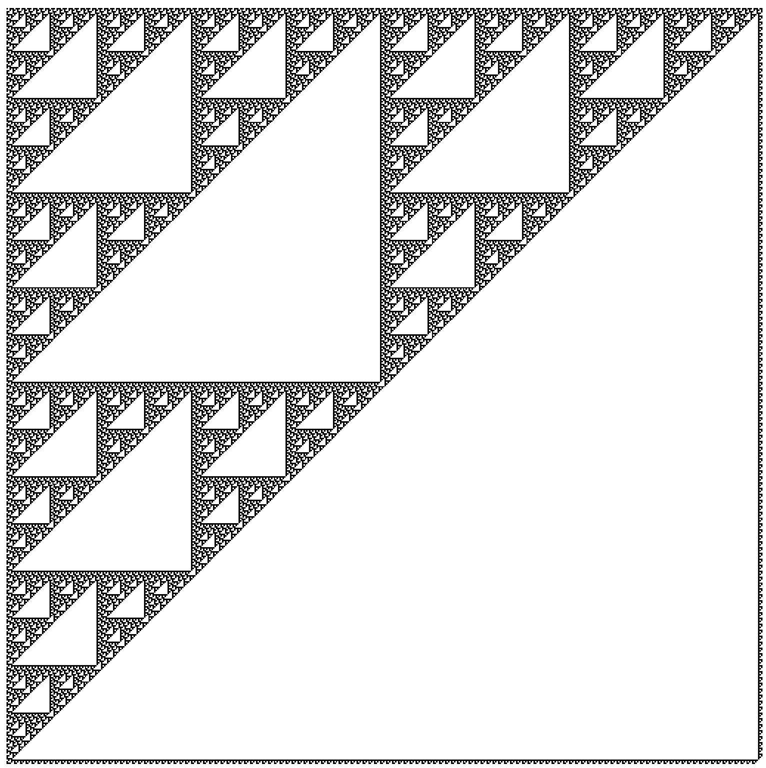

Fig. 6: Pascal's Triangle modulo 2 with period $\vec{\lambda} = 1011$, 512×512.

Example 5.7 $(\mathbb{Z}/2\mathbb{Z}, x + z, \vec{\lambda} = 1011)$ *can be produced by* 8 *rules of type* $4 \to 8$. *See Fig. 6.*

Definition 5.8 Let G be a finite abelian p-group and let $f : G^3 \to G$ be a homomorphism of p-groups. Let $k \geq 1$ a natural number. We say that f allows stable shapes for periods of length k if and only if: There exists $x, s \in \mathbb{N}$ such that for all periods $|\vec{\lambda}| = k$, $\vec{\lambda} \neq (0, \dots, 0)$, then the sequence of scaling sets (I_n) associated with the recurrent two-dimensional sequence $(G, f, \vec{\lambda})$ converges according to the Hausdorff Topology of the compact subsets of $\mathbb{R}^2$ to one and the same limit set I that does not depend of the chosen $\vec{\lambda} \neq (0, \dots 0)$.

Corollary 5.9 *Recall that* $\mathbb{F}_4$ *denotes the unique finite field with* 4 *elements and denote by* $\epsilon \in \mathbb{F}_4 \setminus \mathbb{F}_2$ *a fixed solution of the equation* $\epsilon^2 + \epsilon + 1 = 0$. *The homomorphism* $f(x, y, z) = y + \epsilon(x + z) + \epsilon^2(x^2 + y^2 + z^2)$, $f : (\mathbb{F}_4)^3 \to \mathbb{F}_4$, *that produces the two-dimensional sequence Twin Peaks studied in [3], allows stable shapes for periods of length* 1.

Proof: By direct computation. It has been also already mentioned in [3] that this function produces the same Hausdorff limit for border values 1, ϵ and ϵ^2.

Corollary 5.10 *The homomorphism* $f(x, y, z) = x + z : (\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z}$ *allows stable shapes for periods of length* 4.

Proof: By direct computation.

Corollary 5.11 *The homomorphism* $f : (\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z}$ *given by* $f(x, y, z) = x + z$ *does not allow stable shapes for periods of length* 3.

Proof: See the examples.

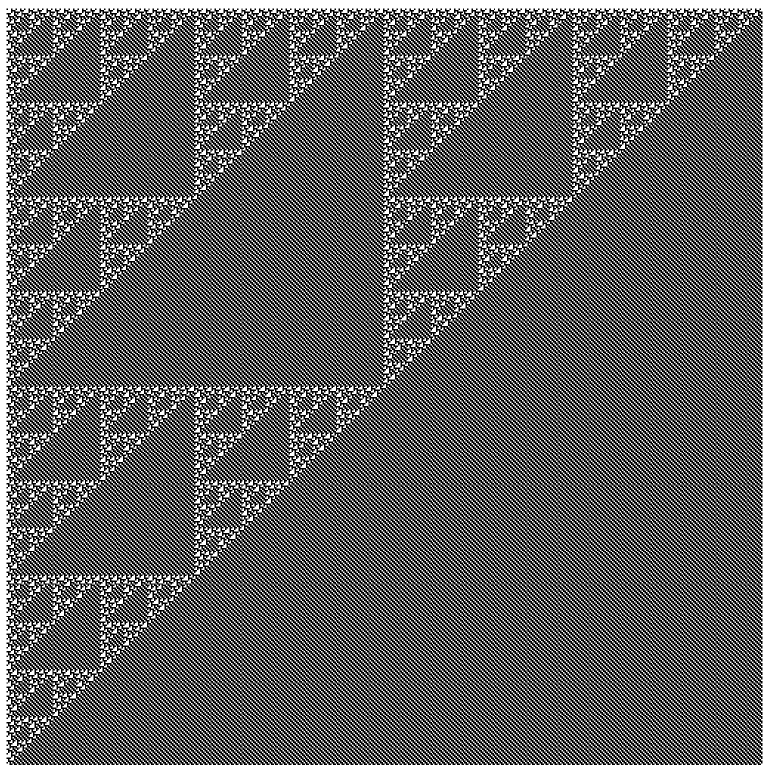

Fig. 7: Pascal's Triangle modulo 2 with period $\vec{\lambda} = 100$, 512×512.

Example 5.12 $(\mathbb{Z}/2\mathbb{Z}, x + z, \vec{\lambda} = 100)$ *can be produced by* 22 *rules of type* $3 \to 6$. *See Fig. 7.*

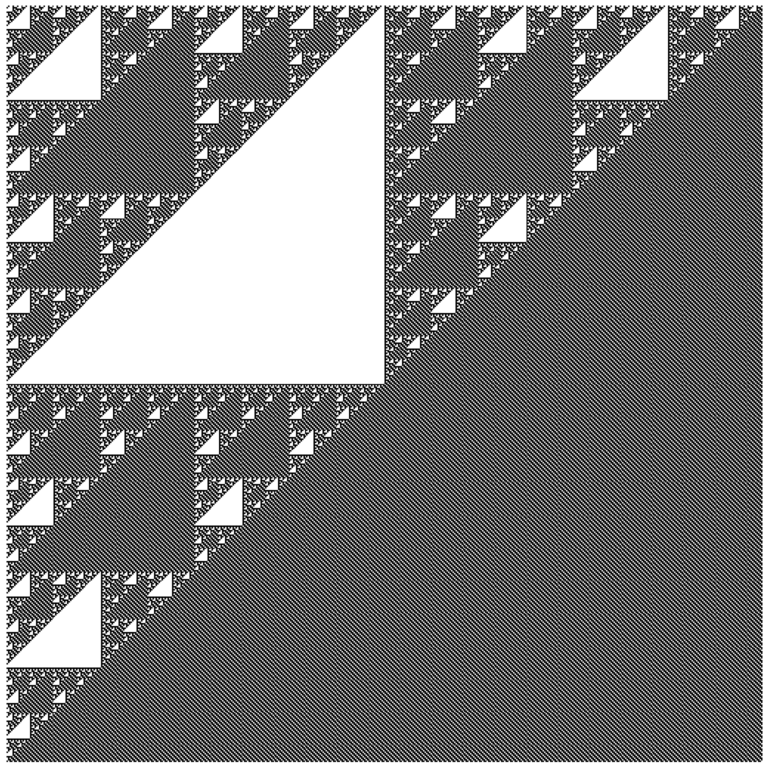

Fig. 8: Pascal's Triangle modulo 2 with period $\vec{\lambda} = 010$, 512×512.

Example 5.13 $(\mathbb{Z}/2\mathbb{Z}, x+z, \vec{\lambda} = 010)$ *can be produced by* 88 *rules of type* $12 \to 24$. *See Fig. 8.*

6 Some new tissues on Sierpinski's Carpet

If this article is seen as a kind of manifesto for the Conjecture 3.2, it couldn't be convincing without containing also examples that really depend of three variables. The first candidate to suffer periodic perturbations in the initial conditions is of course Sierpinski's Carpet, that has been recovered in [2] as the recurrent two-dimensional sequence generated by the sequence $(\mathbb{Z}/3\mathbb{Z}, x+y+z, 1)$. It has been proven there that this two-dimensional sequence is also generated by an expansive system of substitutions of type $1 \to 3$ with three rules, and that all the rules had the form $a \to aF$ for $a \in \mathbb{Z}/3\mathbb{Z}$, where F is the so called fundamental block:

$$F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Here aF means multiplication of a matrix with a number and the start-symbol is 1.

We present four examples of periodic initial conditions for this homomorphism.

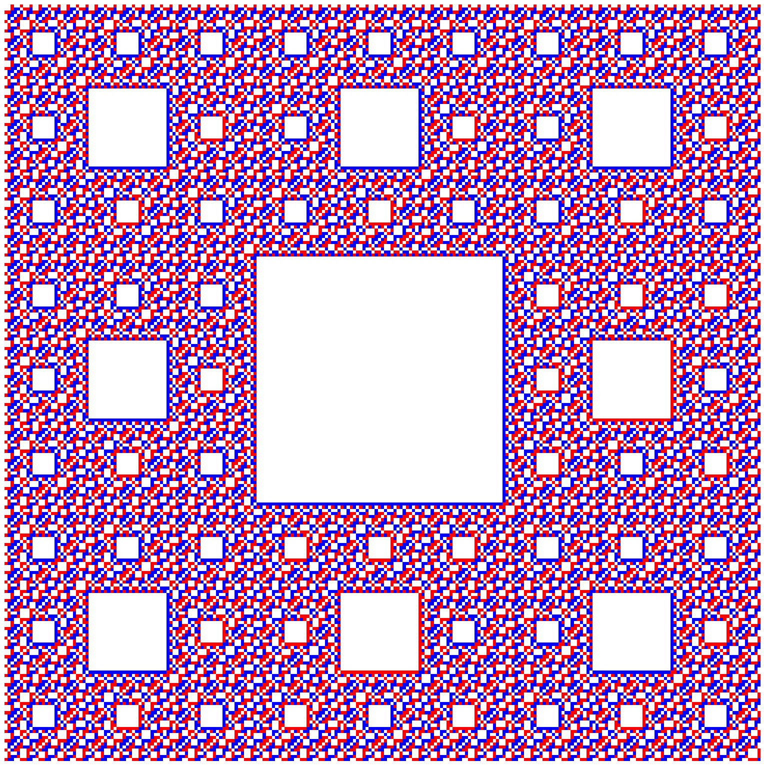

Fig. 9: Sierpinski's Carpet with period $\vec{\lambda} = 001$, 243×243.

Example 6.1 $(\mathbb{Z}/3\mathbb{Z}, x + y + z, \vec{\lambda} = 001)$ *can be produced by* 23 *rules of type* $3 \to 9$. *See Fig. 9.*

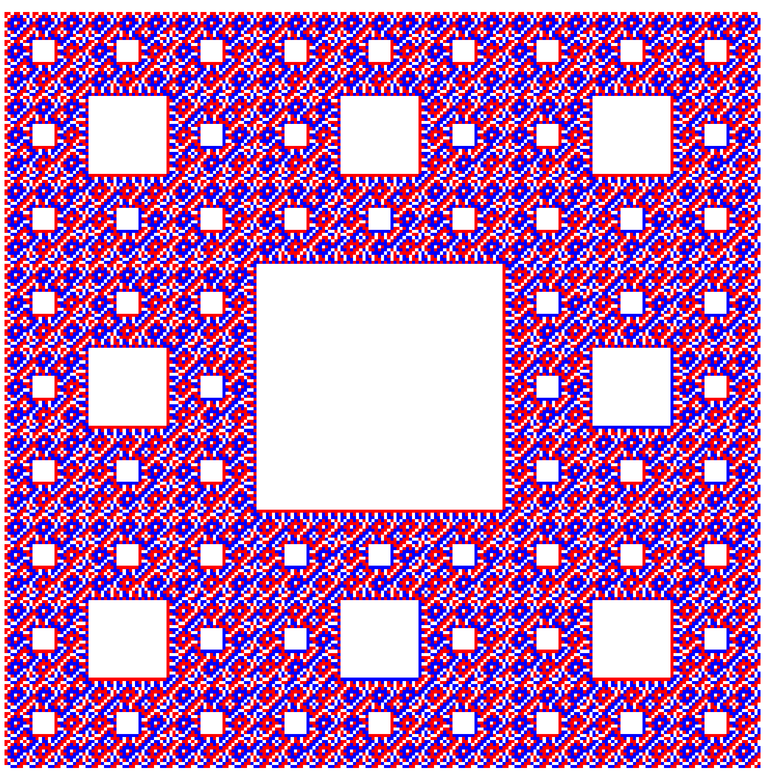

Fig. 10: Sierpinski's Carpet with period $\vec{\lambda} = 110$, 243×243.

Example 6.2 *($\mathbb{Z}/3\mathbb{Z}, x+y+z, \vec{\lambda}=110$) can be produced by 23 rules of type $3 \to 9$. See Fig. 10.*

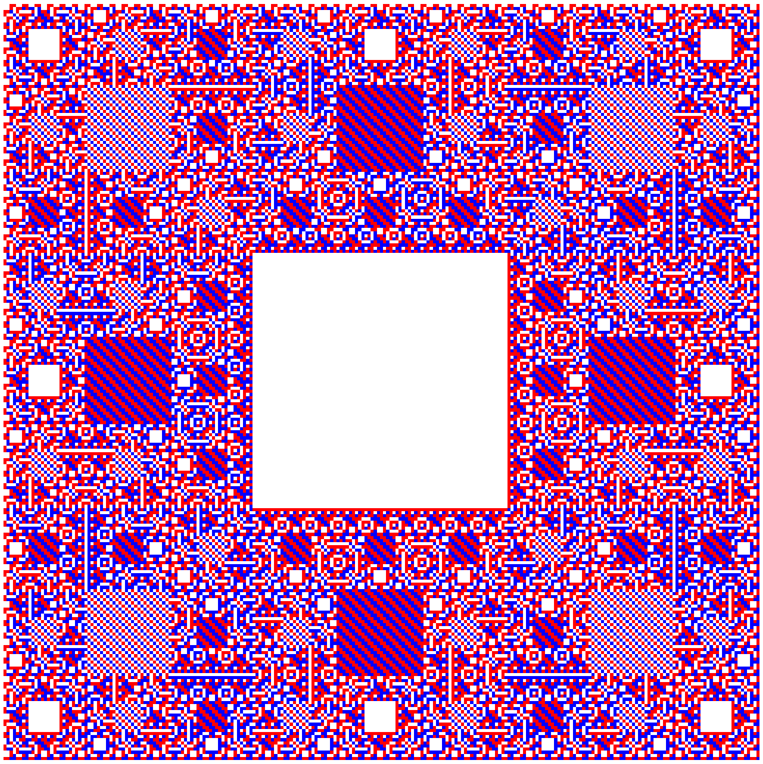

Fig. 11: Sierpinski's Carpet with period $\vec{\lambda} = 0011$, 243×243.

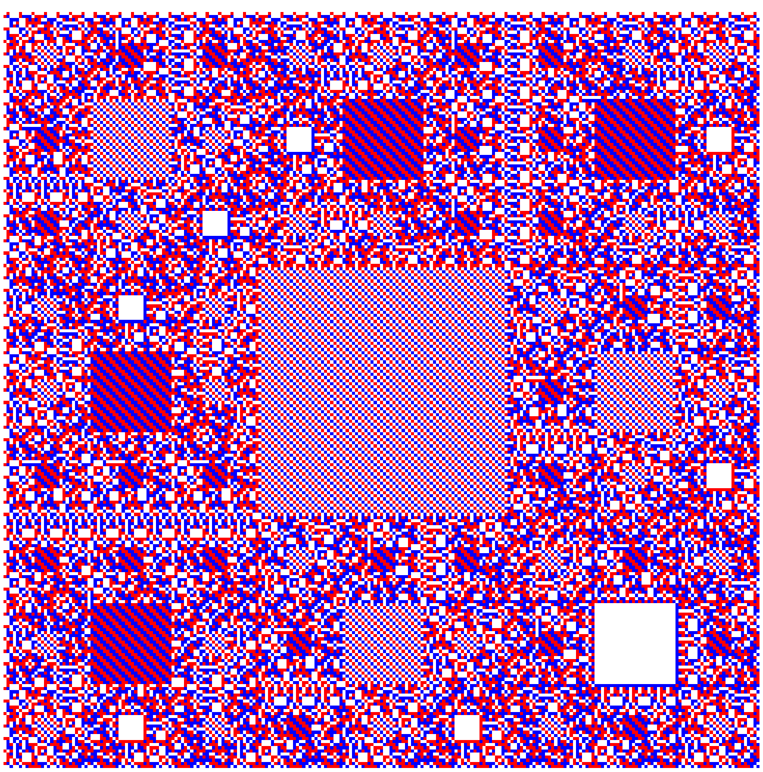

Fig. 12: Sierpinski's Carpet with period $\vec{\lambda} = 0100$, 243×243.

Example 6.3 $(\mathbb{Z}/3\mathbb{Z}, x+y+z, \vec{\lambda}=0011)$ *can be produced by* 1647 *rules of type* $12 \to 36$*. See Fig. 11.*

Example 6.4 $(\mathbb{Z}/3\mathbb{Z}, x+y+z, \vec{\lambda}=0100)$ *can be produced by* 3591 *rules of type* $12 \to 36$*. See Fig. 12.*

Corollary 6.5 *The homomorphism* $f : (\mathbb{Z}/3\mathbb{Z})^3 \to \mathbb{Z}/3\mathbb{Z}$ *given by* $f(x,y,z) = x+y+z$ *allows stable shapes for periods of length* 3*.*

Proof: By direct computation.

Corollary 6.6 *The homomorphism* $f : (\mathbb{Z}/3\mathbb{Z})^3 \to \mathbb{Z}/3\mathbb{Z}$ *given by* $f(x,y,z) = x+y+z$ *does not allow stable shapes for periods of length* 4*.*

Proof: See the last examples.

7 Triangular lava flow

We have kept for the end some examples chosen at random. The first example, called Patchwork, is given by $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, f, \vec{\lambda})$, where:

$$f(\vec{x},\vec{y},\vec{z}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{z}$$

and $\vec{\lambda} = (0,0)(0,0)(0,2)(0,1)(0,1)(0,0)$. The 243×243 left upper minor is represented in Fig 13.

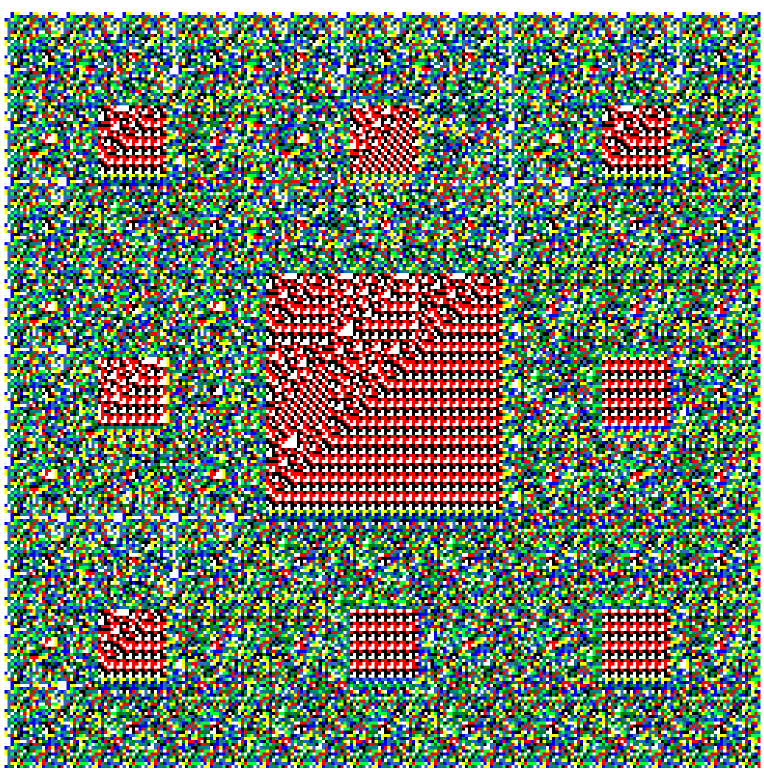

Fig. 13: Patchwork, 243×243.

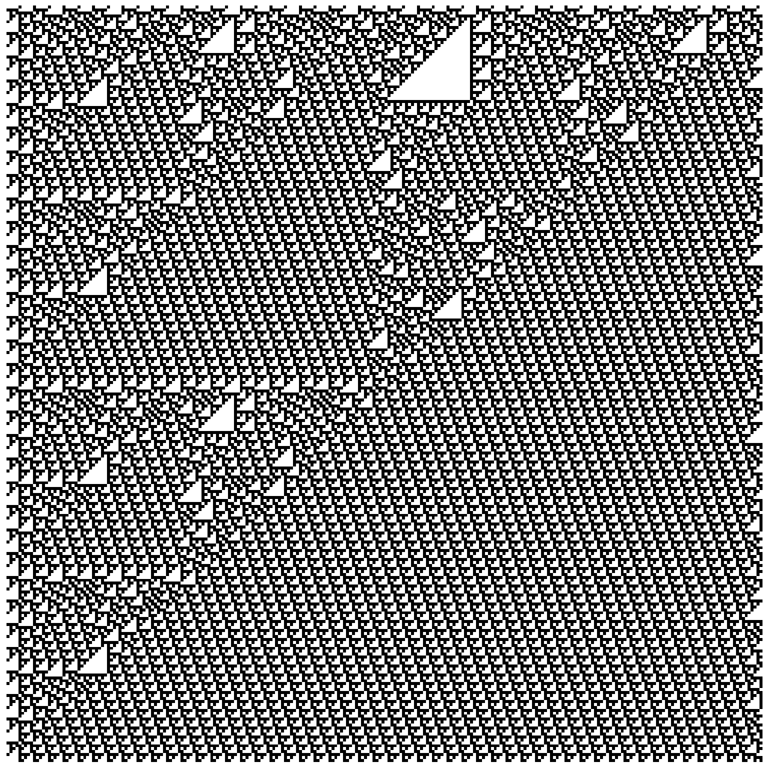

Fig. 14: Pascal's Triangle modulo 2 with periods $\vec{\lambda} = 00001$, $\vec{\mu} = 0100$, 256×256.

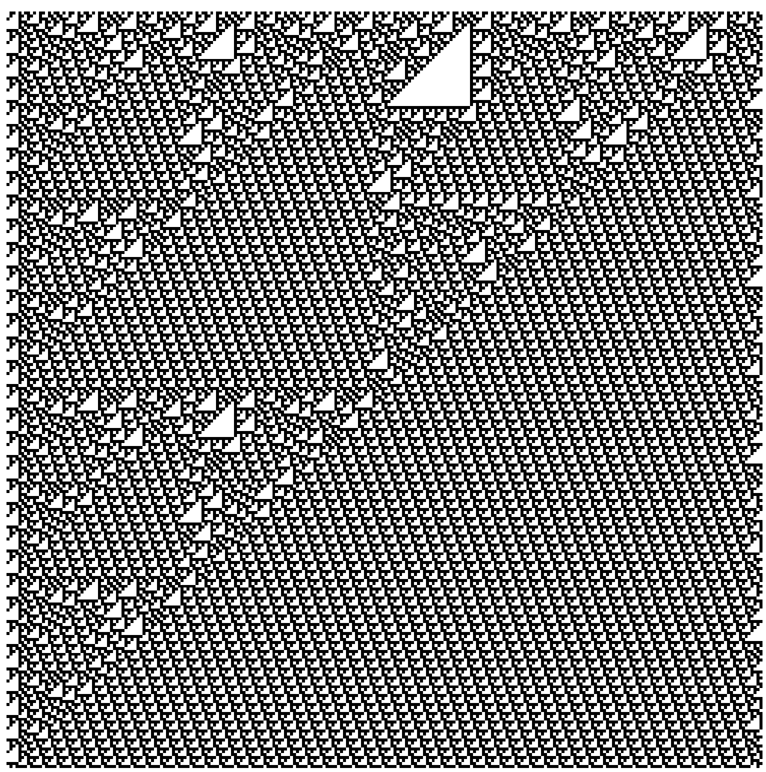

Fig. 15: Pascal's Triangle modulo 2 with periods $\vec{\lambda} = 01101$, $\vec{\mu} = 0010$, 256×256.

Example 7.1 *The sequence Patchwork defined above and shown in Fig. 13 can be produced by*

1080 *rules of type* $6 \to 18$.

In fact the choice was not completely random. Two other nice examples have been rejected because they had more than $100,000$ rules of substitution, although they were of almost the same type: both over $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, but generated with periods of length 4. This makes clear that we need to know more on the underlying mathematical theory, as soon as possible. The computational method used here just has arrived at its limits.

Another kind of examples belong to the case with different periods $\vec{\lambda} \neq \vec{\mu}$. Both examples chosen are Pascal's Triangles modulo 2. We displayed here one given by $(\mathbb{Z}/2\mathbb{Z}, x+z, \vec{\lambda} = 00001, \vec{\mu} = 0100)$ and another given by $(\mathbb{Z}/2\mathbb{Z}, x+z, \vec{\lambda} = 01101, \vec{\mu} = 0010)$.

Example 7.2 *Pascal's Triangles modulo* 2 *with different periods* $\vec{\lambda} \neq \vec{\mu}$ *defined above and shown in Fig. 14 and Fig. 15 can be produced by expansive systems of substitutions, both of type* $4 \to 8$ *with* 64 *rules.*

Often one gets recurrent two-dimensional sequences which are periodic. The most easy way to be periodic is if they consist of a square which is infinitely repeated in both directions. In Fig. 16, called Chess Table, we see the recurrent two-dimensional sequence $(\mathbb{Z}/2\mathbb{Z}, x+y+z, 10001010000)$.

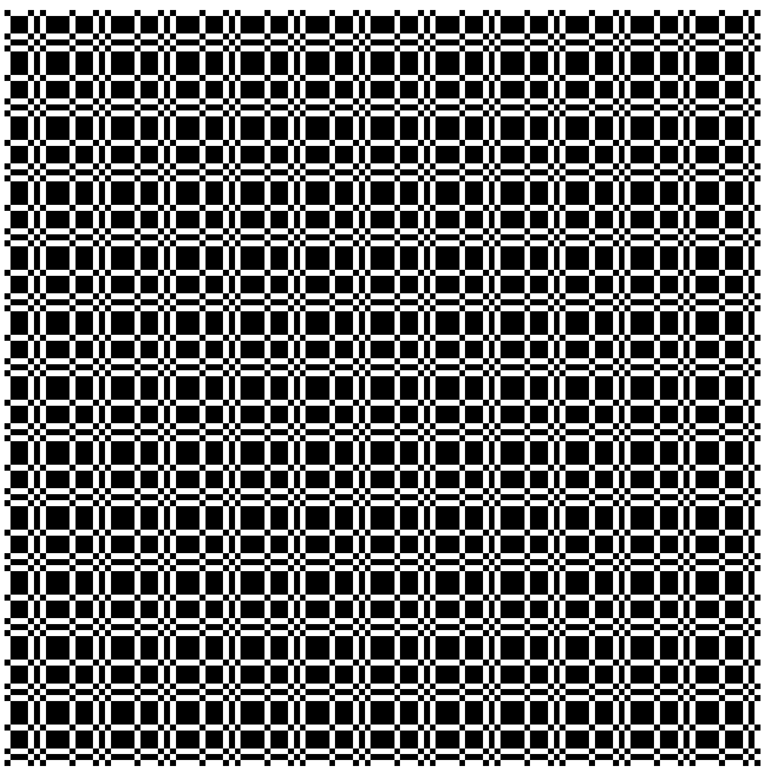

Fig. 16: Chess Table, 128×128.

Such situations are not at all a counterexample to the principle of substitution, because they can be always constructed by an expansive system of substitutions with the unique rule of substitution:

$$X_1 \rightsquigarrow \begin{pmatrix} X_1 & X_1 \\ X_1 & X_1 \end{pmatrix}$$

Other kinds of periodicity are also particular cases of substitution.

Finally, the author hopes that he was able to provide an insight into this new world, and to convince the reader of both theoretical and artistic interests of these objects.

References

[1] **Mihai Prunescu**: *Undecidable properties of recurrent double sequences.* Notre Dame Journal of Formal Logic, 49, 2, 143 - 151, 2008.

[2] **Mihai Prunescu**: *Self-similar carpets over finite fields.* European Journal of Combinatorics, 30, 4, 866 - 878, 2009.

[3] **Mihai Prunescu**: *Recurrent double sequences that can be generated by context-free substitutions.* Fractals, 18, 1, 65 - 73, 2010.

[4] **Mihai Prunescu**: *Some new counterexamples to context-free substitution in recurrent double sequences over finite sets.* Preprint 2010, submitted.

[5] **Dann E. Passoja, Akhlesh Lakhtakia**: *Carpets and rugs: an exercise in numbers.* Leonardo, 25, 1, 1992, 69 - 71.

[6] **Roger Penrose**: *Pentaplexity.* Mathematical Intelligencer 2 (1), 32 - 37, 1979.

[7] **Benoit B. Mandelbrot**: *The fractal geometry of nature.* W. H. Freeman and Company, San Francisco, 1977, 1982.

[8] **Stephen Wolfram**: *A new kind of science.* Wolfram Media, 2002. ISBN 1-57955-008-8

[9] **Jarkko Kari**: *Theory of cellular automata: A survey.* Theoretical Computer Science 334, 3 - 33, 2005.

[10] **Stephen J. Willson**: *Cellular automata can generate fractals.* Discrete Applied Mathematics, 8, 1984, 91 - 99.

[11] **J.-P. Allouche, F. von Haeseler, H.-O. Peitgen, G. Skordev**: *Linear cellular automata, finite automata, and Pascal's triangle.* Discrete Applied Mathematics 66, 1 - 22, 1996.