Recurrent double sequences that can be produced by context-free substitutions

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Abstract

We prove that if a recurrent double sequence coincide in a sufficiently large starting square with a double sequence produced by context-free substitutions,then they must coincide everywhere. We apply this result for some examples.

Key Words: recurrent double sequence, context-free substitution, Frobenius' automorphism, Hausdorff metric.

A.M.S.-Classification: 05B45, 28A80, 03D03.

1 Introduction

This article belongs to a series dedicated to the study of recurrent double sequences over finite sets and their power of expression.

Definition 1.1 Let $(A, f, 1)$ be a finite structure with one ternary function f, and one constant 1. The recurrent double sequence $a : \mathbb{N} \times \mathbb{N} \to A$ starts with the values $a(i, 0) = a(0, j) = 1$ and satisfies the recurrence $a(i, j) = f(a(i-1, j), a(i-1, j-1), a(i, j-1))$.

In [1] it is proved that if the $A = \mathbb{F}_q$ is the finite field with q elements and $f(x, y, z) = x + my + z$, where $m \in \mathbb{F}_q$ is an arbitrary fixed element, one gets a self-similar pattern. In the case when $q = p$ is prime and so $\mathbb{F}_q = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ as ring of classes of remainders modulo p, the pattern can be done by substitutions of type $x \to xB$, where B is a $p \times p$ matrix called fundamental block. This fact is not explicitly said in [1], but is very easy to see it applying the Kronecker product representation theorem in the case $\mathbb{F}_q = \mathbb{F}_p$, where the only one automorphism of Frobenius is the identity. This result generalizes and solves conjectures from [2]. In [3] the general recurrent double sequences with binary commutative recurrence f are considered. It is proved there that this class of recurrent double sequences is Turing complete. If follows that the ternary recurrences over finite sets are also Turing complete, even if we consider only those f which are symmetric in x and z.

The present paper is dedicated to a class of objects lying between those extrema. On a hand the substitution $x \to xB$ arrising in [1] is similar with those used to define the regular grammars in the sense of Chomsky. On the other hand the general recurrent double sequences are Turing complete. Now we meet recurrent double sequences which are somehow analogous with the context-free grammars. More exactly, the recurrent double sequences presented here can be produced using a set of substitutions of type $X \to Y$, where X denotes different $x \times x$ matrices, Y different $y \times y$ matrices, $x \,|\, y$, every substitution is uniquely determined by the left matrix X, and every right matrix Y consists of blocks that are left matrices X, such that the substitution process can be continued.

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We explain how to use the computer in order to prove that a concrete recurrent double seuquence can be produced by such a substitution scheme. Every automatic proof succeeds by exploring only a finite part of the double sequence. One example is intensively analyzed, other examples are only shortly discussed. At the end of the paper we state some conjectures concerning the general phenomenon.

For related literature see the monograph [4], surveys like [5] and papers like [6].

2 Definitions and main result

For this section we fix a finite set A.

Definition 2.1 Let $(A, 1, f)$ be a finite structure with ternary function. Denote by R the recurrent double sequence already defined in Definition 1.1. Suppose that two natural numbers $x \geq 1$ and $s \geq 2$ have been fixed. For $n \geq 1$ denote by $R(n)$ the matrix $(a(i,j))$ with $0 \leq i, j \leq xs^{n-1} - 1$. Call f-matrix every $u \times v$ matrix $(b(i,j))$ with elements in A such that for all $1 \leq i < u$ and $1 \leq j < v$ one has $b(i,j) = f(b(i-1,j), b(i-1,j-1), b(i,j-1))$, where the indexes start with 0. A $xs^{n-1} \times xs^{n-1}$ matrix $(b(i,j))$ is $R(n)$ if and only if is a f-matrix and fulfills $b(i,0) = b(0,j) = 1$.

Definition 2.2 The fixed natural numbers $x \geq 1$ and $s \geq 2$ used in this definition can be considered to be the same as x and s used in the Definition 2.1. Let $y = xs$. Let $\mathcal{X}$ be a finite set of $x \times x$ matrices over A and $\mathcal{Y}$ be a set of $y \times y$ matrices over A such that every $Y \in \mathcal{Y}$ has a $s \times s$ block matrix representation $(X(i,j))_{0 \leq i,j < s}$ and all blocks $X(i,j) \in \mathcal{X}$. We call system of substitutions of type $x \to y$ the tuple $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$, where $\Sigma : \mathcal{X} \to \mathcal{Y}$ is a fixed function and $X_1 \in \mathcal{X}$ is a fixed element of $\mathcal{X}$, called start-symbol. If a $u \times v$ matrix Z consists only of neigbouring blocks $X(i,j) \in \mathcal{X}$, $Z = (X(i,j))_{0 \leq i < u,\ 0 \leq j < v}$, we define $\Sigma(Z)$ to be the $su \times sv$ matrix with block reprezentation $(\Sigma(X(i,j)))$. We define the sequence of matrices $(S(n))$ by $S(1) = X_1$ and $S(n) = \Sigma^{n-1}(X_1)$. The number s is called scaling factor of the system of sustitutions.

Definition 2.3 We call the system of substitutions $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ expansive if the block representation of the matrix $\Sigma(X_1) = (X(i,j))$ using matrices in $\mathcal{X}$ fulfills $X(0,0) = X_1$.

Lemma 2.4 *Let $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ be an expansive system of substitutions. Then for all $n > 0$ the matrix $S(n)$ is $xs^{n-1} \times xs^{n-1}$ left upper minor of the matrix $S(n+1)$.*

Proof: By induction. For $n = 1$ the statement is exactly the definition of an expansive system of substitutions. Let n be > 1 and suppose that $S(n-1)$ is the left upper minor of $S(n)$. This means that

$$S(n) = \begin{pmatrix} S(n-1) & U \\ V & W \end{pmatrix},$$

where U, V and W are matrix blocks with appropriate dimensions. It follows that:

$$S(n+1) = \Sigma(S(n)) = \begin{pmatrix} \Sigma(S(n-1)) & \Sigma(U) \\ \Sigma(V) & \Sigma(W) \end{pmatrix} = \begin{pmatrix} S(n) & \Sigma(U) \\ \Sigma(V) & \Sigma(W) \end{pmatrix}.$$

$\square$

Definition 2.5 We say that a $u \times v$ matrix $K = (k(\alpha, \beta))$ occurs in the $w \times z$ matrix $T = (t(a,b))$ if for some $0 \leq i < w$ and $0 \leq j < z$ one has $i + u \leq w$, $j + v \leq z$ and for all $0 \leq \alpha < u$ and $0 \leq \beta < v$ one has $t(i + \alpha, j + \beta) = k(\alpha, \beta)$. We say that K occurs in x-position in T if moreover $x \mid i$ and $x \mid j$.

Definition 2.6 Let x be a fixed natural number and T be a $wx \times zx$ matrix over A. We denote $\mathcal{N}_x(T)$ the set of all $2x \times 2x$ matrices occurring in x-position in T.

Definition 2.7 Let x be a fixed natural number and T be a $wx \times zx$ matrix over A. We denote by $\mathcal{J}_x(T)$ the set of all $x \times x$ matrices occurring in T in the positions $\{(0, kx) \,|\, k \in \mathbb{N}\}$. Analogously, we denote by $\mathcal{I}_x(T)$ the set of all $x \times x$ matrices occurring in T in the positions $\{(kx, 0) \,|\, k \in \mathbb{N}\}$.

Theorem 2.8 *Let $(A, f, 1)$ be a finite structure with ternary function f and let $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ be an expansive system of substitutions of type $x \to y$ over A. We define the matrices $(R(n))_{n \geq 1}$ according to x and $s = y/x$ given by the system of substitutions. Suppose that for some $m > 1$ following conditions hold:*

- $R(m) = S(m)$.
- $\mathcal{N}_x(S(m-1)) = \mathcal{N}_x(S(m))$.
- $\mathcal{J}_x(S(m-1)) = \mathcal{J}_x(S(m))$ *and* $\mathcal{I}_x(S(m-1)) = \mathcal{I}_x(S(m))$.

Then for all $n \geq 1$ one has $R(n) = S(n)$.

Proof: The system of sustitutions being expansive, the matrices $S(1)$, $S(2)$, $\dots$, $S(m-1)$ are successively left upper minors of $S(m)$, and so they are identic with $R(1)$, $R(2)$, $\dots$, $R(m-1)$ respectively. The condition $\mathcal{N}_x(S(m-1)) = \mathcal{N}_x(S(m))$ implies that the set $\mathcal{N} := \mathcal{N}_x(S(m))$ is Σ-closed in the following sense: for all $N \in \mathcal{N}$ one has $\mathcal{N}_x(\Sigma(N)) \subseteq \mathcal{N}$. Of course, $N \in \mathcal{N}_x(S(m-1))$, so $\Sigma(N)$ is a $2sx \times 2sx = 2y \times 2y$ matrix occurring in $S(m)$ in sx-position. All $2x \times 2x$ matrices occurring in $\Sigma(N)$ in x-position are also $2x \times 2x$ matrices occurring in $S(m)$ in x-position, so they belong to $\mathcal{N}$.

We observe that all elements of $\mathcal{N}$ are f-matrices, because they occur in $S(m) = R(m)$, which is an f-matrix. From the facts that the elements of $\mathcal{N}$ covers $R(m)$ with overlappings and that $\mathcal{N}$ is Σ-closed follows that for all $n \geq m$, $\mathcal{N}_x(S(n)) = \mathcal{N}$ and so all $S(n)$ are covered with overlappings by f-matrices. So for all $n \geq m$, $S(n)$ are f-matrices. In order to prove the statement it remains to show that for all $n \geq m$, the matrices $R(n)$ satisfies the conditions $a(i, 0) = a(0, j) = 1$.

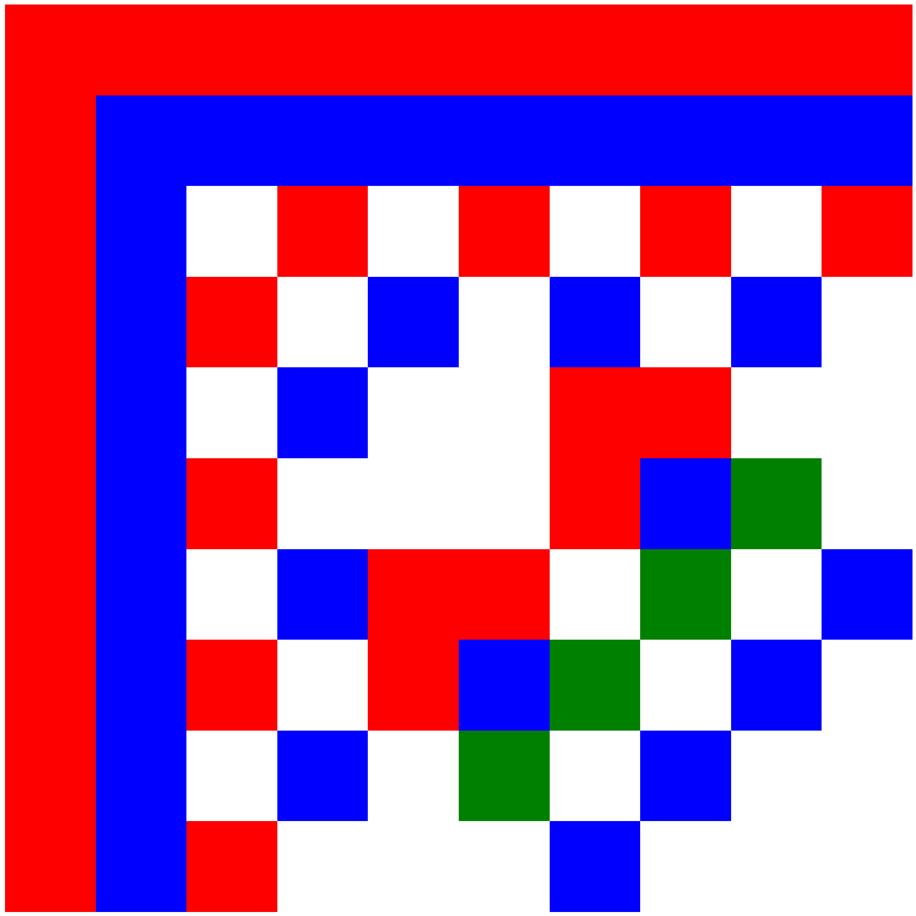
The condition $\mathcal{J}_x(S(m-1)) = \mathcal{J}_x(S(m)) := \mathcal{J}$ means that the set of $\mathcal{X}$-matrices occurring at the first row stabilized by $S(m)$, so all elements of $\mathcal{J}$ have a first row constant $1 \in A$ because $R(m) = S(m)$. All $S(n)$, $n \geq m$, have a first row constant 1. Using $\mathcal{I}$ instead of $\mathcal{J}$ we get that the first column is always constant 1.

We have shown that all $S(n)$ are f-matrices and satisfies $a(i, 0) = a(0, j) = 1$. So for every n, $S(n) = R(n)$. □

The Theorem 2.8 suggests an automatic method to check if a particular recurrent double sequence can be generated by an expansive system of substitutions. Let $x \to y$ be a given type. One computes the corresponding matrix $R(k)$ for a relatively big value of k, and reads all x-matrices occurring in x-position. Every time when such a matrix X starting at position (ax, bx) has been read, one reads the corresponding y-matrix Y starting at position (ay, by). If X is read for the first time, one saves the pair (X, Y). If (X, Y) has been already read before, one continues. If X has been read before, but has been save in pair with some y-matrix $Z \neq Y$, the recurrent double sequence cannot be generated by substitutions of the given type. In this situation we return **No** and stop. If not such contradiction occured, one verifies if the second and the third condition of theorem 2.8 are satisfied. If yes, we return **Yes** and stop. If no, we return **Maybe**. This means, that the explored $R(k)$ can be generated by substitutions of the given type, but k was chosen too small, and so we are not able to state that the whole recurrent double sequence can be generated by the system of substitutions computed so far.

3 Twin Peaks

This section contain the first example studied by the author, presented with details. The goal of the section is to show how one can use the substitution structure in order to better understand



The 10×10 left upper minor of Twin Peaks.

the geometric properties of the resulting pattern.

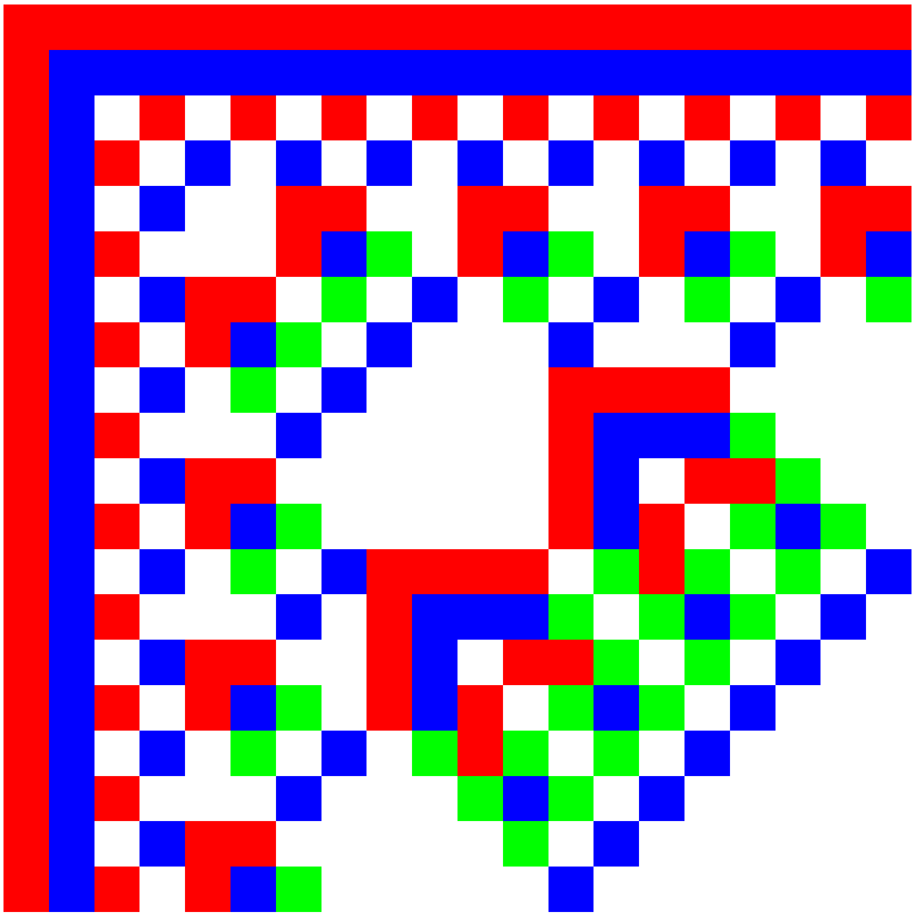
The recurrent double sequence, called Twin Peaks, has been first discovered as a recurrent double sequence of elements of the finite field $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$, more exactly given by $(\mathbb{F}_5, g, 1)$ with $g(x, y, z) = y + 3y^3 + 3x^2z^2$. The author observed that only four colours does really occur in the pattern and supposed that there must be polynomials over the finite field $\mathbb{F}_4 = \mathbb{F}_2[\epsilon]$, where $\epsilon^2 + \epsilon + 1 = 0$, producing the same pattern. After solving an under-determined interpolation problem, the author has chosen the solution $(\mathbb{F}_4, f, 1)$ with $f(x, y, z) = y + \epsilon(x + z) + \epsilon^2(x^2 + y^2 + z^2)$.

Definition 3.1 Let $T = (a(i, j))_{i,j \in \mathbb{N}}$ be the recurrent double sequence produced by the structure $(\mathbb{F}_4, f, 1)$, $f(x, y, z) = y + \epsilon(x + z) + \epsilon^2(x^2 + y^2 + z^2)$.

Definition 3.2 Let $TP(n) \subset \mathbb{R}^2$ a set obtained in the following way. If the unit square is the union of $2^n \times 2^n$ closed equal squares $s_{i,j}$, then $TP(n) = \bigcup\limits_{a(i,j) \neq 0} s_{i,j}$.

We prove that the sequence $TP(n)$ converges according to the Hausdorff topology of compact subsets of $\mathbb{R}^2$, fact which is also suggested by the displayed images. In order to simplify the notation we shall write 2 for ϵ and 3 for ϵ^2. In the figures following colours have been used: white for 0, red for 1, blue for 2 and green for 3.

Although the proof for structure and convergence has been done for the sequence of sets $TP(n)$, the images display left upper minors with edge $5 * 2^k$. This decision has been taken only in order to better point out the biggest pair of twin houses.



The 20×20 left upper minor of Twin Peaks.

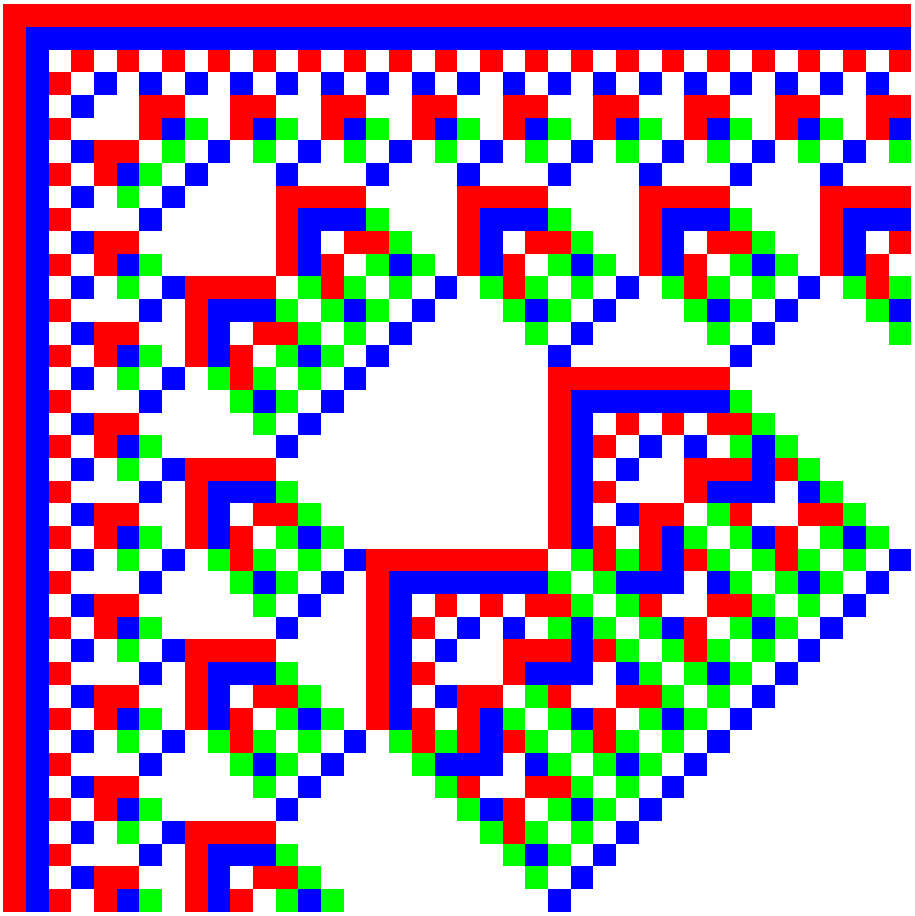
Definition 3.3 Let $\mathcal{X}$ be the set of the following 2×2 matrices denoted X_1, ..., X_{15}, and let $\mathcal{Y}$ be the set of 4×4 matrices defined together with the substitution $\Sigma : \mathcal{X} \to \mathcal{Y}$ as follows:

$$X_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_2 & X_2 \\ X_5 & X_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_3 & X_6 \\ X_3 & X_6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_7 & X_1 \\ X_1 & X_9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$



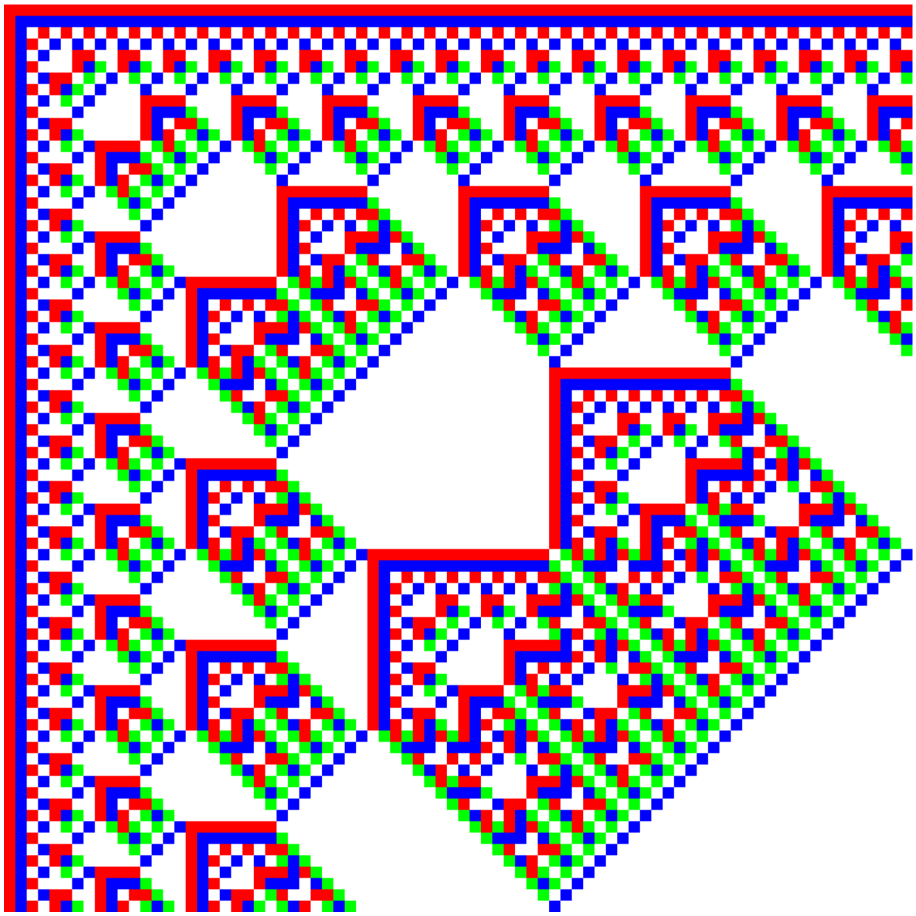
The 40×40 left upper minor of Twin Peaks.

$$X_5 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_8 & X_1 \\ X_{10} & X_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 \\ 0 & 2 & 0 & 3 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

$$X_6 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_{11} & X_{10} \\ X_1 & X_8 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

$$X_7 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_7 & X_7 \\ X_7 & X_7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_8 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_8 & X_7 \\ X_{12} & X_8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 \end{pmatrix}$$



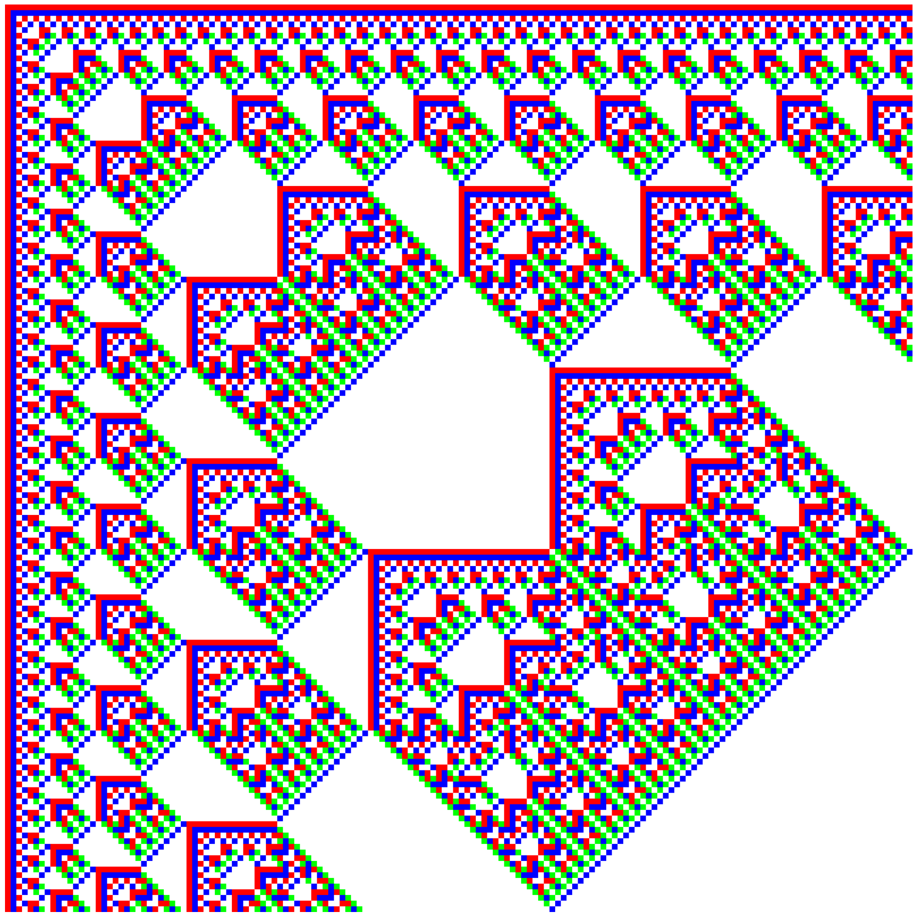
The 80×80 left upper minor of Twin Peaks.

$$X_9 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_9 & X_{12} \\ X_{12} & X_9 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 & 3 \\ 3 & 0 & 3 & 2 \\ 1 & 3 & 0 & 3 \\ 3 & 2 & 3 & 0 \end{pmatrix}$$

$$X_{10} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_9 & X_{10} \\ X_{10} & X_7 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{11} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_{11} & X_{12} \\ X_7 & X_{11} \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{12} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_{13} & X_{14} \\ X_{15} & X_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$



The 160×160 left upper minor of Twin Peaks.

$$X_{13} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_{13} & X_6 \\ X_5 & X_{10} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

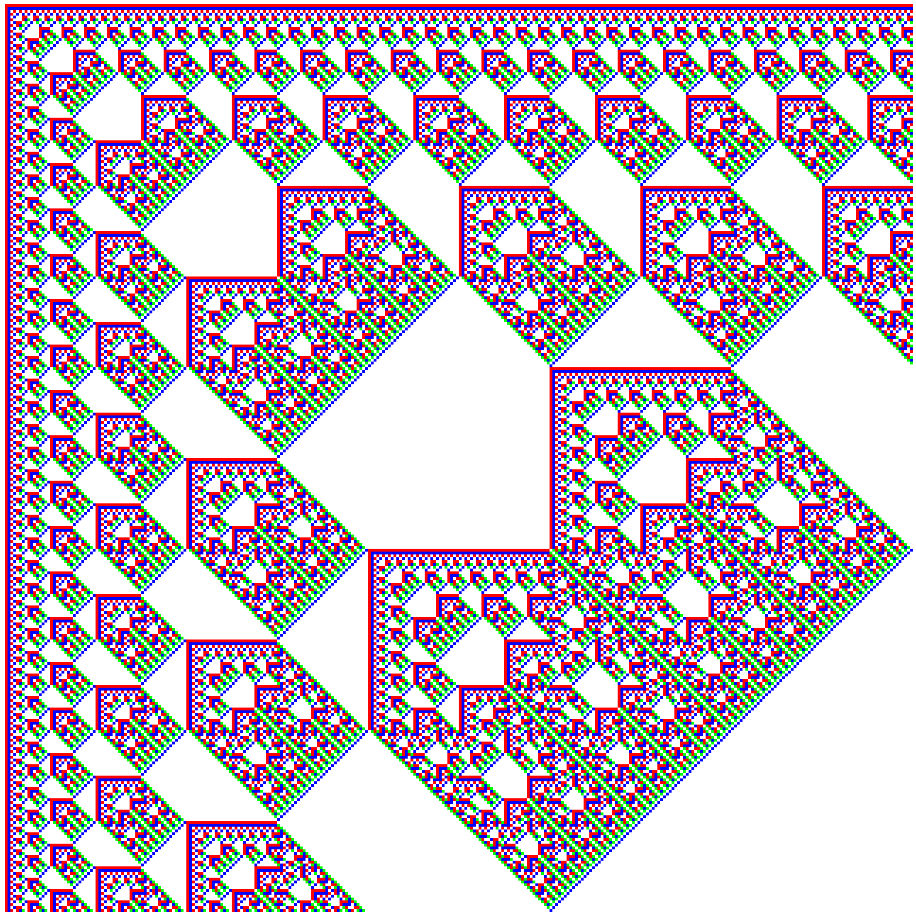
$$X_{14} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_3 & X_{14} \\ X_{11} & X_5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 0 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$X_{15} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} X_2 & X_8 \\ X_{15} & X_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 \\ 1 & 0 & 0 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

The pairs $\{2, 3\}$, $\{5, 6\}$, $\{8, 11\}$ and $\{14, 15\}$ consist of transposed matrices, and the same is true for their images through Σ. All the other blocks X_i and respective images $\Sigma(X_i)$ are symmetric.

Corollary 3.4 *The recurrent double sequence T is produced by the expansive system of substitutions $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ defined above.*

Proof: By automatic verification of the conditions of Theorem 2.8 over $T(9)$. The Σ-closed set $\mathcal{N}$ has 124 many elements. $\square$



The 320×320 left upper minor of Twin Peaks.

Corollary 3.5 *The sequence of compact plane sets* $(TP(n))$ *is convergent according to the Hausdorff topology.*

Proof: The situation is a little bit more difficult as those studied in [1], so we sketch a proof. Let $d(\cdot,\cdot)$ be the Hausdorff distance for compact plane sets. It follows from the Corollary 3.4 by considering the particular form of every individual rule of substitution, that $d(TP(n),TP(n+1)) \leq 1/2^n$ for all $n \geq 1$. Applying the inequality of triangle, one gets for $k \geq 1$:

$$d(TP(n),TP(n+k)) \leq d(TP(n),TP(n+1)) + d(TP(n+1),TP(n+2)) + \dots$$

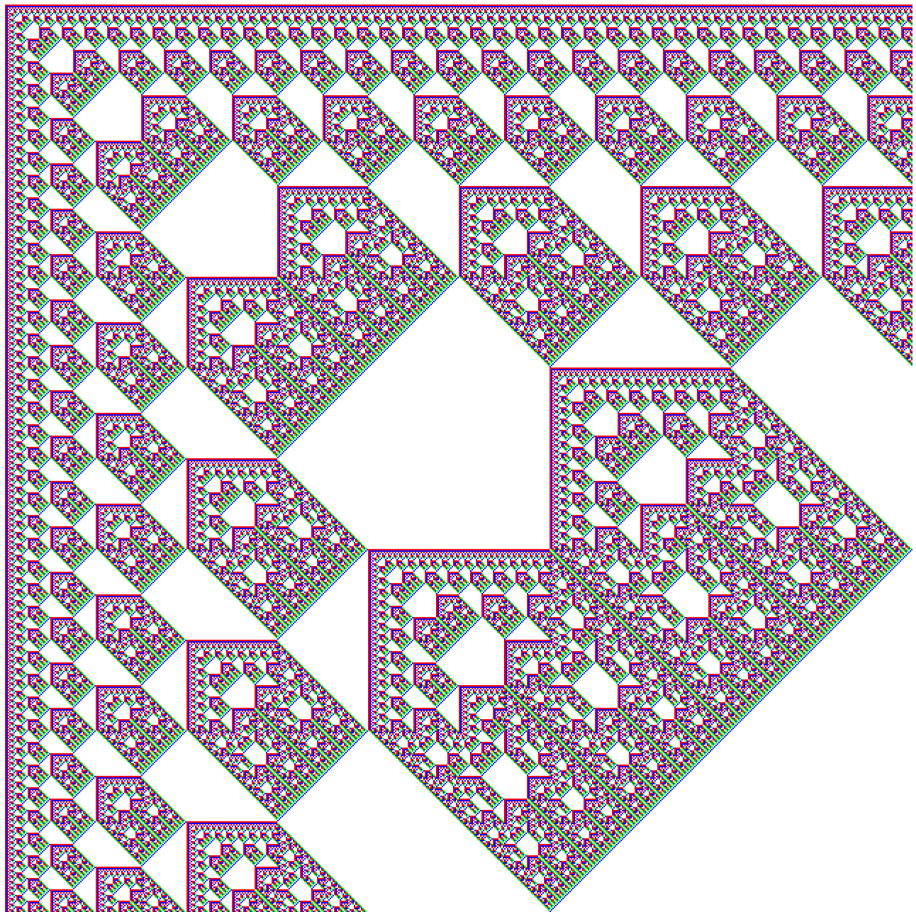
$$+d(TP(n+k-1),TP(n+k)) \leq \frac{1}{2^n}(1+\frac{1}{2}+\dots+\frac{1}{2^k}) < \frac{1}{2^{n-1}},$$

so $(TP(n))$ is a Cauchy sequence, and the metric space of all compact plane sets is complete. $\square$

Definition 3.6 $TP := \lim_{n\to\infty} TP(n)$.

Just some lines about the pattern itself. The most interesting question is where does the start symbol X_1 occur again in the interior of the infinite matrix T. It is clear that every such occurrence starts in T an even 2^k-minor which is isomorphic with the corresponding left upper 2^k-minor of T, for some $k \geq 2$. This has as result the existence of subsets in the limit TP which are homothetically homeomorphic with the whole set TP.

The occurrences of X_1 in T which are next to the first line or to the first column start with the cells $(4, 6+4m)$ and $(6+4m, 4)$, $m \in \mathbb{N}$.

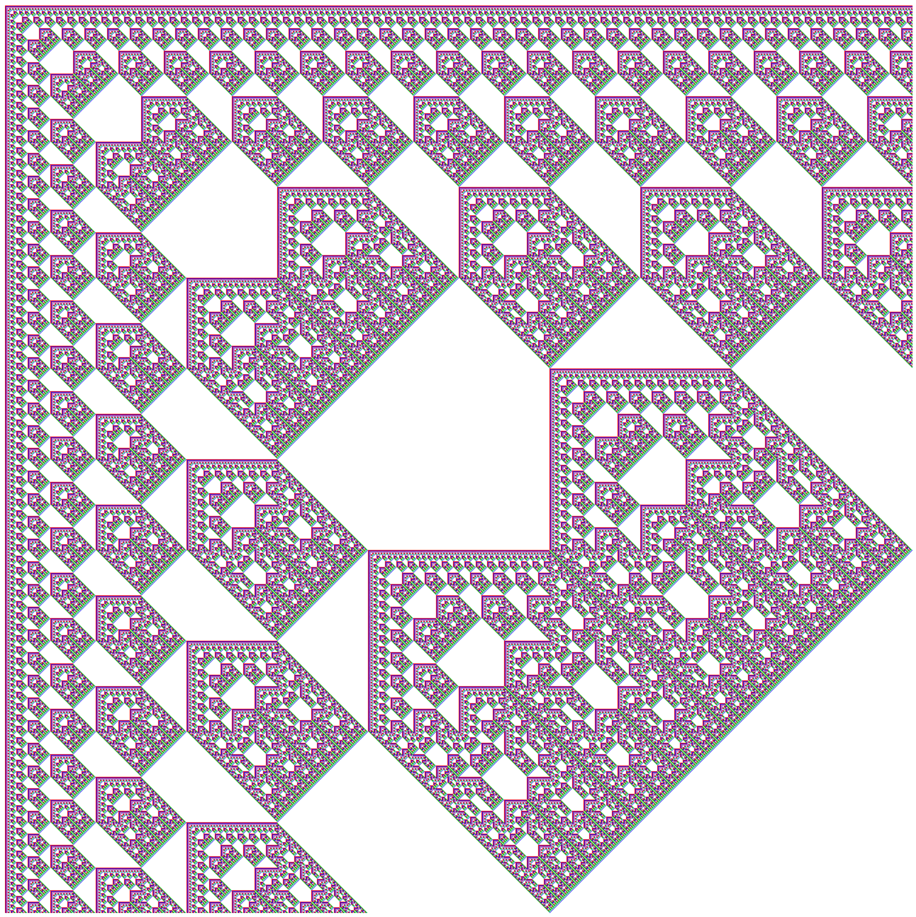


The 640×640 left upper minor of Twin Peaks.

Definition 3.7 Call the 4-minor H starting with $a(10,4)$ the primitive house, or house of order 0. Observe that $H = \binom{X_1 X_8}{X_{11} X_{10}}$. Call the even $2^{n+2} \times 2^{n+2}$-matrix $\Sigma^n(H)$ the house of order n. Call the left upper quarter of a house of arbitrary order its roof. Call the oposite quarter the botom of the house, and the remaining quarters the sides of the house. Finally let $HP := \lim_{n\to\infty} \Sigma^n(H)$. Call the 4×4-matrix J starting at $a(6,4)$ a primitive twin house, or twin house of order 0. $J = \binom{X_1 X_9}{X_{11} X_{10}}$. Twin house of order n, roof, simple side, common side, bottom of a twin house, and the limit set JP are defined analogously.

We observe that the roof of a (twin) house of order n is identic with the left upper 2^{n+1}-minor of T. We observe also that the (twin) house of order 3 contains a primitive twin house pair under (in) its roof, the (twin) house of order 4 contains primitive twin house pairs in its sides, and the (twin) house of order 5 contains a primitive twin house pair in its bottom. The same things happen also for the transposed of H or J.

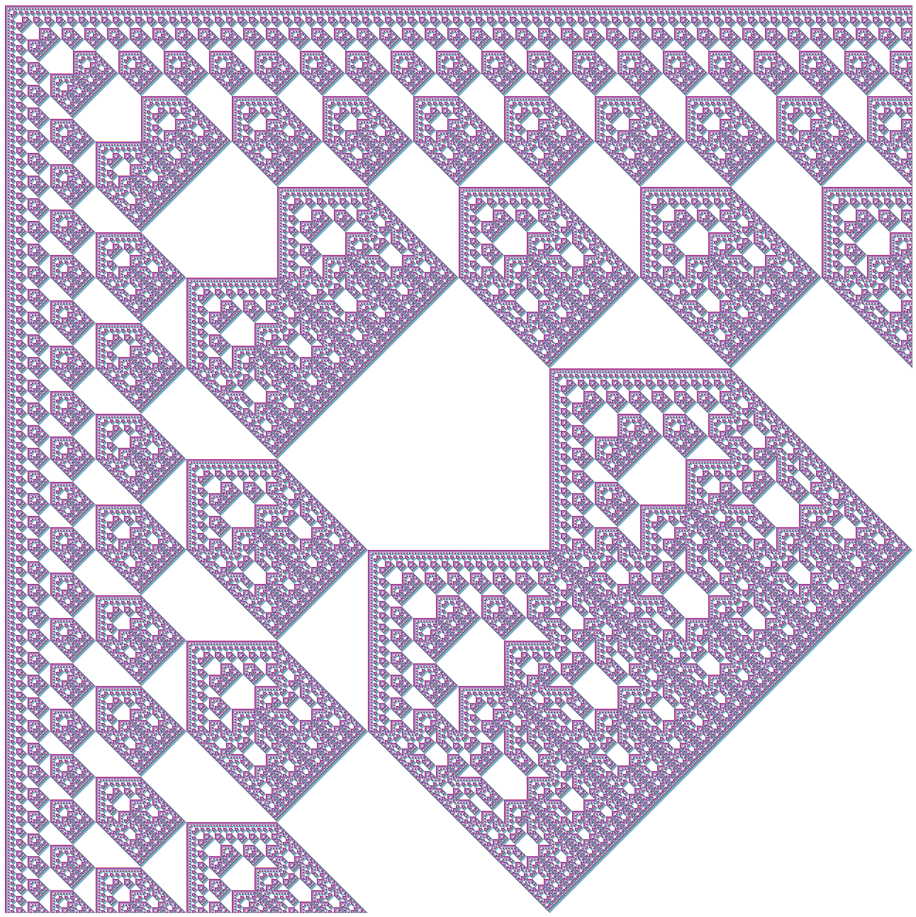
Consequently the limit TP contains a central pair of twin houses. If one navigates from this central pair to the upper left corner, one intersects a stratification of infinitely many geometrically decreasing (with ratio 1/2) rows of houses. All houses in TP are translated homothetic images of TH. A house in TP of edge of the roof equal d contains in its roof maximal houses of edge $d/8$, in its sides maximal houses of edge $d/16$ and in its bottom maximal houses of edge $d/32$. Every house contains an infinity of houses of different edges. All houses in TP have roofs which are homothetically similar with the whole set TP and geometrically congruent with a left upper square subset of TP.



The 1280×1280 left upper minor of Twin Peaks.

The related structures $(\mathbb{F}_4, f, x)$ produce similar patterns for $x \neq 0, 1$. More exactly, $(\mathbb{F}_4, f, 2)$ produces the infinite submatrix $(a(i+1, j+1))_{i,j\geq 0}$. This cannot be produced by an expansive system of substitutions of type $2 \to 4$, but by one of type $4 \to 8$ with 51 rules, and it converges to the same limit TP. The case $(\mathbb{F}_4, f, 3)$ seems to be more difficult. This structure produces a different pattern, in which houses are not bilaterally symmetric anymore. The pattern can be also produced by an expansive system of substitutions of type $4 \to 8$ with 51 rules. I conjecture that this pattern $(\mathbb{F}_4, f, 3)$ converges to the same limit TP.

We observe that T allows also a function of contraction, like the Penrose Tiling, see [7]. Indeed, let $\Gamma : \mathfrak{M} \to \mathbb{F}_4$ such that: $\Gamma(X_7, X_8, X_9, X_{11}) = 0$, $\Gamma(X_1, X_2, X_3, X_{13}) = 1$, $\Gamma(X_4, X_5, X_6, X_{10}) = 2$, $\Gamma(X_{12}, X_{14}, X_{15}) = 3$. Then $\Gamma(T(n+1)) = T(n)$. Of course the pattern is very different from the Penrose Tiling: as pointed out by a referee, an important difference is the role played by the scaling factor in both situations.

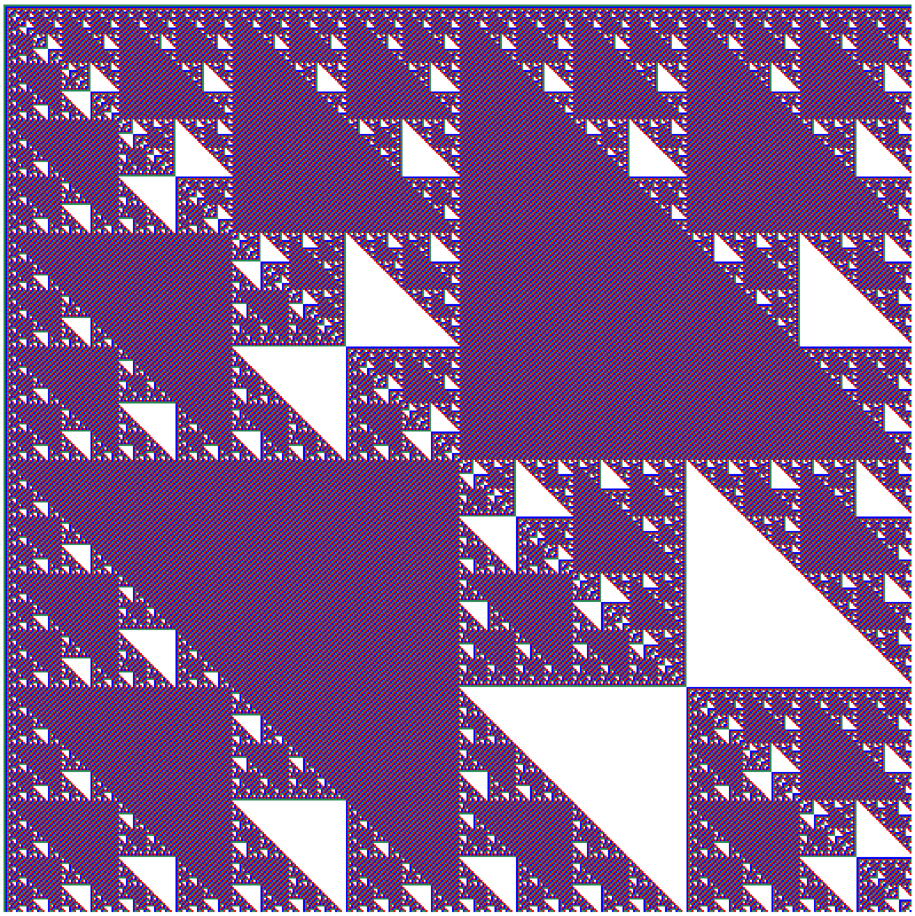


The 2560×2560 left upper minor of Twin Peaks.

4 Other examples

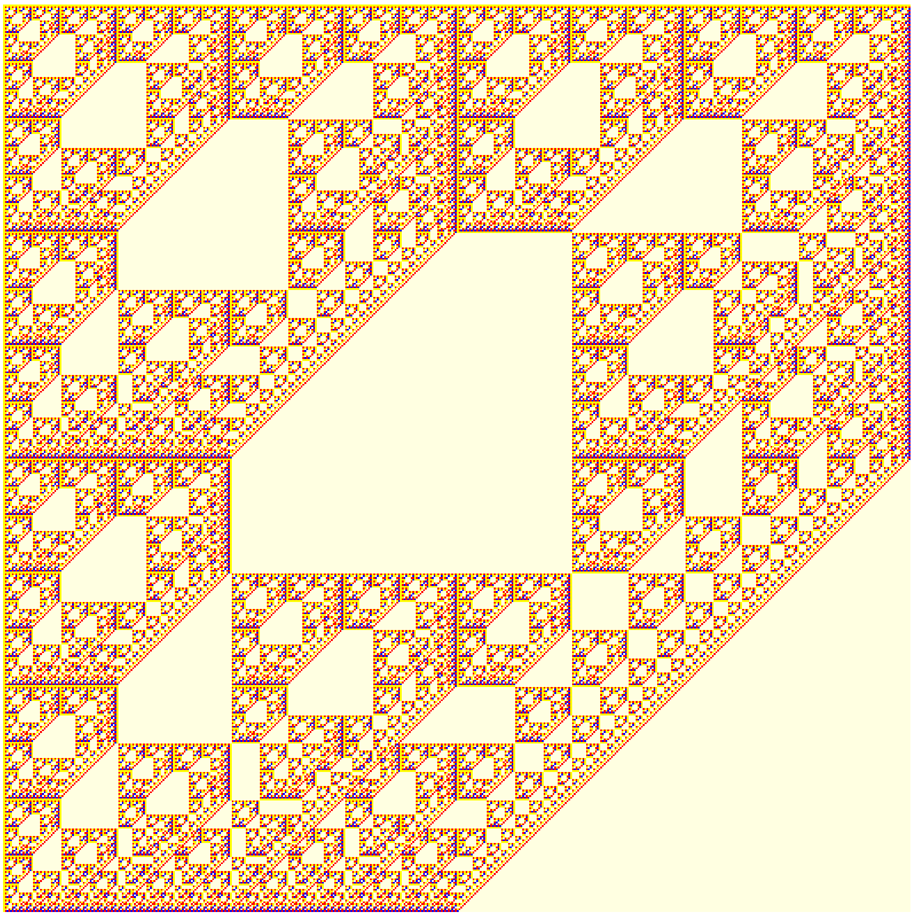
In the time between the first submission and the first referee reports, the author found around one hundred situations where the Theorem 2.8 applies. The number of examples is now too big to discuss or present any of them with details. Maybe the best thing to do is only to show some of them, and to formulate some conjectures and open problems. The examples displayed here arose over the finite field $\mathbb{F}_4$ and over the rings of remainder classes $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z}$. For the field $\mathbb{F}_4$ we keep the presentation chosen for the main example Twin Peaks, with ϵ denoted with 2 and $\epsilon^2 = \epsilon + 1$ denoted by 3. The rings of classes always have a canonical presentation.

Recall that for a finite field $\mathbb{F}_q$, with $q = p^k$, the automorphism of Frobenius $x \rightsquigarrow x^p$ generates the cyclic Galois group of the algebraic extension $\mathbb{F}_q/\mathbb{F}_p$. The other automorphisms are given by the polynomials $x^p, x^{p^2}, \dots, x^{p^{k-1}}, x^{p^k} = x = \mathrm{id}$. Let us call Frobenius polynomials the linear combinations of Frobeni with coefficients in $\mathbb{F}_q$. All those polynomials are additive homomorphisms of $\mathbb{F}_q$ in itself. I conjecture that all functions $f(x, y, z) = e(x)+g(y)+h(z)$, where $e(x)$, $g(y)$ and $h(z)$ are Frobenius polynomials, produces recurrent double sequences which are substitution patterns. However, they seem not to do this for all starting values in the field. Another mistery is the fact that patterns produced with such a polynomial by different starting values look macroscopically similar, in spite of the fact that they are microscopically different, have different substitution types and number of rules, or even a substitution pattern looks macroscopically similar to a pattern obtained by applying the same polynomial to another starting value, but that seem not to be a substitution pattern.



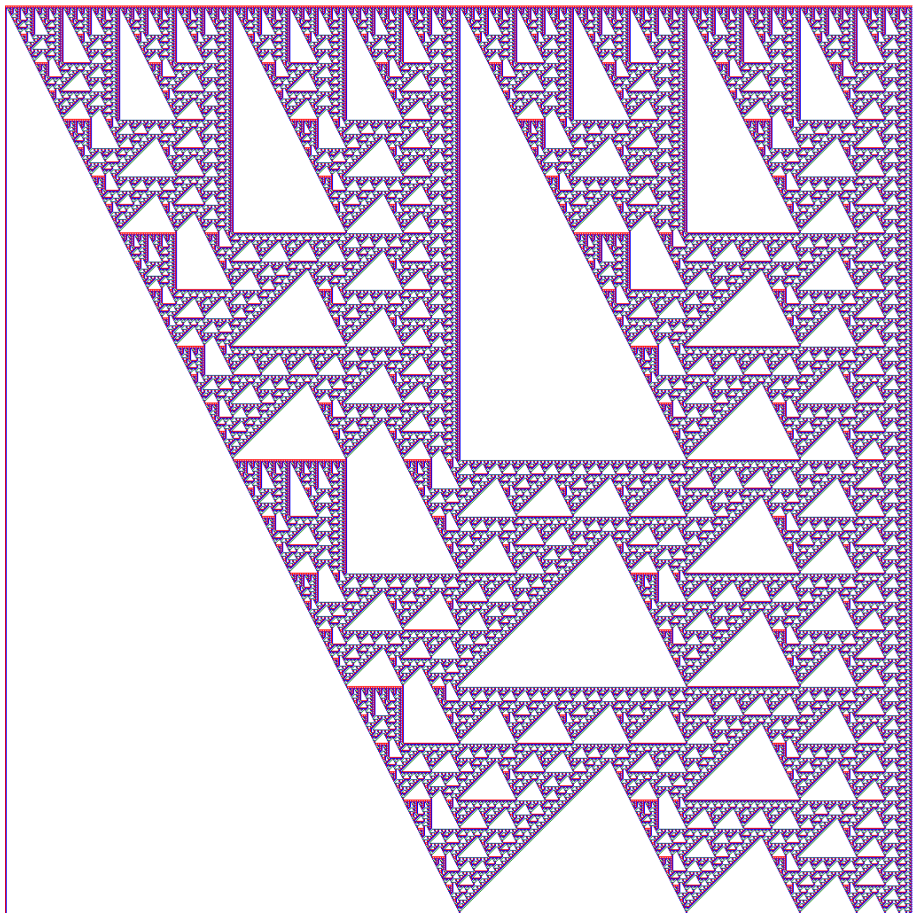
Butterflies, 512×512. $(\mathbb{F}_4, 2x + x^2 + 3y + 2z + z^2, 1)$.

Corollary 4.1 *The recurrent double sequence Butterflies generated by the structure of recurrence* $(\mathbb{F}_4, 2x+x^2+3y+2z+z^2, 1)$ *can be also generated by an expansive system of substitutions of type* $2 \to 4$ *with 34 rules.*



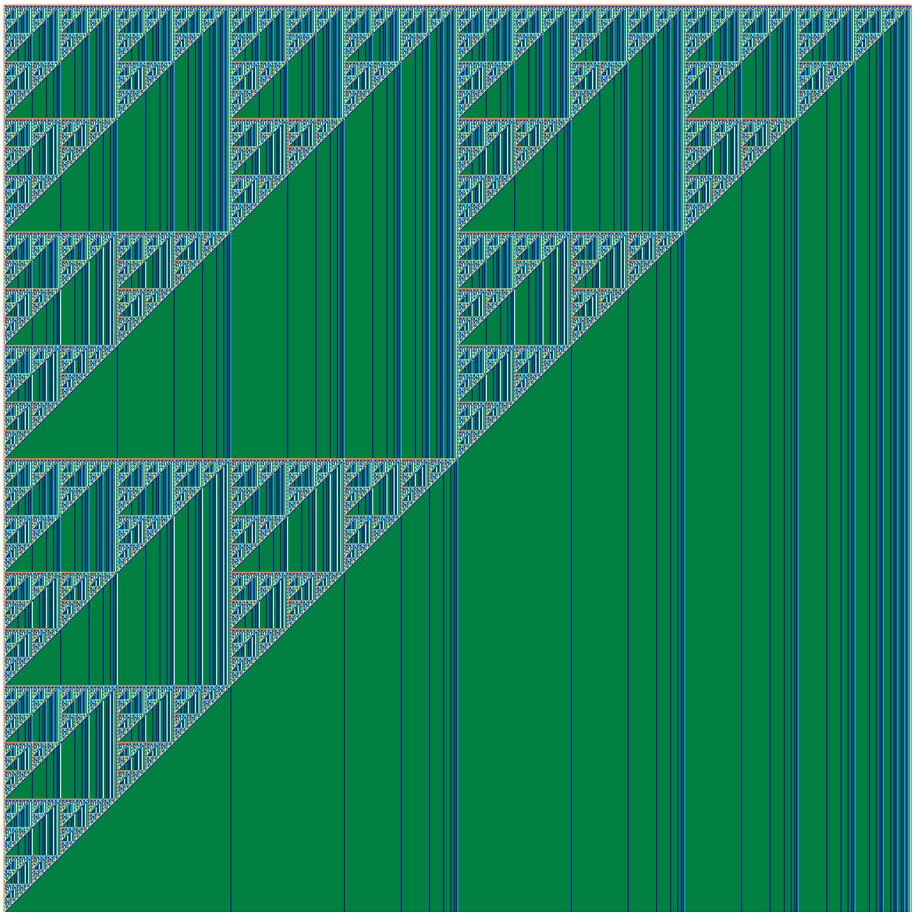
Diamond, 512×512. $(\mathbb{F}_4, 2x^2 + y + 3y^2 + 2z^2, 1)$.

Corollary 4.2 *The recurrent double sequence Diamond generated by the structure of recurrence* $(\mathbb{F}_4, 2x^2 + y + 3y^2 + 2z^2, 1)$ *can be also generated by an expansive system of substitutions of type* $4 \to 8$ *with* 42 *rules.*



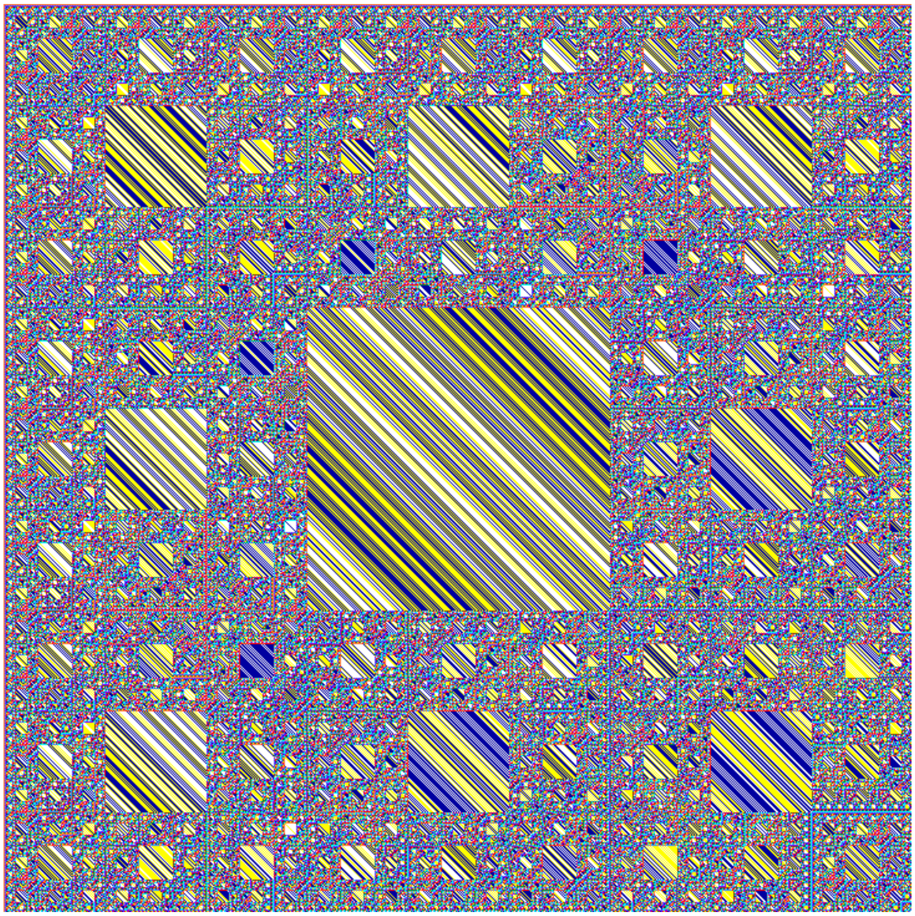
Pythagora vs. Pascal, 1024×1024. $(\mathbb{F}_4, 3x^2 + y + 2y^2 + z + z^2 + 3, 1)$.

Corollary 4.3 *The recurrent double sequence Pythagora vs. Pascal generated by the structure of recurrence* $(\mathbb{F}_4, 3x^2 + y + 2y^2 + z + z^2 + 3, 1)$ *can be also generated by an expansive system of substitutions of type* $4 \to 8$ *with* 14 *rules.*



Pascal Perturbed, 1024×1024. $(\mathbb{Z}/8\mathbb{Z}, x^2 + 2x^4 + 6y + 2z^2 + z, 1)$.

Corollary 4.4 *The recurrent double sequence Pascal Perturbed generated by the structure of recurrence* $(\mathbb{Z}/8\mathbb{Z}, x^2 + 2x^4 + 6y + 2z^2 + z, 1)$ *can be also generated by an expansive system of substitutions of type* $4 \to 8$ *with* 47 *rules.*



Sierpinski Perturbed, 729×729. $(\mathbb{Z}/9\mathbb{Z}, 6x^6 + 2x^3 + y + 2z^3 + 6z^6, 1)$.

Corollary 4.5 *The recurrent double sequence Sierpinski Perturbed generated by the structure of recurrence* $(\mathbb{Z}/9\mathbb{Z}, 6x^6 + 2x^3 + y + 2z^3 + 6z^6, 1)$ *can be also generated by an expansive system of substitutions of type* $9 \to 27$ *with* 4951 *rules.*

The case of rings of classes of remainders $\mathbb{Z}/p^k\mathbb{Z}$ the situation is more complicated. The cases $(\mathbb{Z}/p^k\mathbb{Z}, x + y + z, 1)$ with $k > 1$, presented by Passoja and Lakhtakia in [2] as non-self-similar examples (differently from the case $k = 1$) seem to be substitution patterns. For example:

Corollary 4.6 *The recurrent double sequences* $(\mathbb{Z}/9\mathbb{Z}, x+y+z, 1)$ *can be produced by an expansive system of substitutions of type* $3 \to 9$ *with* 57 *rules.*

I strongly conjecture that the cases $(\mathbb{Z}/p^k\mathbb{Z}, x + y + z, 1)$ are all substitution patterns, but for an automatic proof they seem to need the consideration of quite big starting squares, that cost a lot of time of computation. A general algebraic proof is needed here. I also conjecture that additive polynomials like $ax + by + cz$ always produce substitution patterns in these rings. The general form of polynomials producing substitution patterns over these rings is still unclear.

Finally, we are quite far away from a classification theory for the various geometric phenomena occuring in such objects.

References

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