The Thue-Morse-Pascal double sequence and similar structures

Mihai Prunescu

Abstract

If a recurrent two-dimensional sequence with initial conditions defined by linear substitution and a two-dimensional sequence that is generated by planar substitution are identical over a sufficiently large initial square, then they will coincide over all. After proving this general principle, we apply it to some concrete examples. One of them, the Thue-Morse-Pascal two-dimensional sequence, is defined by two copies of the Prouhet-Thue-Morse sequence as pair of initial conditions and by the Pascal Triangle Addition modulo 2 as rule of recurrence. As it follows, the Thue-Morse-Pascal two-dimensional sequence is the result of 15 substitution rules, each of them consisting of the substitution of some $4 \times 4$ matrix with an $8 \times 8$ matrix.

Key Words: recurrent two-dimensional sequence, periodic initial conditions, expansive system of context-free substitutions, automatic proof procedure, homomorphisms of finite abelian $p$-groups, Pascal’s Triangle, Prouhet-Thue-Morse Sequence, Doubling Period Sequence, Thue-Morse-Pascal Two-dimensional Sequence.

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1 Introduction

This note reveals a new touching point between recurrence and substitution. Both notions occur in a field of interdisciplinary investigations unifying very heterogenous motivations and techniques. The recurrence - although a very classical task - is more and more present in studies concerning cellular automata, see [26], [12], [7] or the monograph [27]. Substitutions occur in various contexts such as automatic sequences [11], [1], [2], aperiodic tilings [25], [21], [13], [6], [5], various fractal constructions [8], [9], [22] or mathematic quasicrystals [10], [4].

Definition 1.1 Let $A$ be a finite set and $f : A^3 \rightarrow A$ a fixed function. We call the set $A$ alphabet and the function $f$ recurrence. We will refer to the function $f$ as $f(x,y,z)$. We also fix two sequences $u,v : \mathbb{N} \rightarrow A$ with $u(0) = v(0)$, called initial conditions. We say that the tuple $(A,f,u,v)$ defines a recurrent two-dimensional sequence $a : \mathbb{N}^2 \rightarrow A$ if the following conditions are fulfilled:

1. $\forall k \in \mathbb{N} \ a(k,0) = u(k)$ and $a(0,k) = v(k)$.
2. $\forall i,j > 0 \ a(i,j) = f(a(i,j-1), a(i-1,j-1), a(i-1,j))$.

In the case that $u = v$ we mention just one of them in the tuple. If $u$ or $v$ are periodic, we just write down the period. Here we must stress the fact that the notion of recurrence defined here has a very different meaning than those used for one-dimensional sequences and their associated dynamical systems. In that case, recurrence means that each block occurring in the sequence, occurs infinitely often.
2 Definitions and main result

Although the notions of block decomposition, covering and system of substitutions will be used only in \( n = 1 \) or \( n = 2 \) dimensions, it seems to be more practical to define those notions for an arbitrary dimension \( n \).

**Definition 2.1** Let \( A \) be a finite set (alphabet). An \( n \)-dimensional sequence is a function \( a : \mathbb{N}^n \to A \). An \( n \)-dimensional cube is a function \( X : [0,d-1]^n \to A \). Such a set is called a \( d^n \)-cube. We say that \( X \) occurs in \( A \) at \( \vec{u} \in \mathbb{N}^n \) if \( a \mid \vec{u} + [0,d-1]^n = X \mid [0,d-1]^n \). We say for \( d \in \mathbb{N} \) that \( d \mid \vec{u} \) if there is a \( \vec{v} \in \mathbb{N}^n \), \( \vec{v} = d \vec{u} \). We say that \( X \) occurs in \( a \) if there is a \( \vec{u} \) such that \( X \) occurs in \( a \) at \( \vec{u} \).

**Definition 2.2** Let \( s \in \mathbb{N} \) be a natural number \( \geq 2 \) and let \( Y : [0,ds-1]^n \to A \) be some \((ds)^n\)-cube over \( A \). We define the \( d \)-block decomposition of \( Y \) to be set of all \( d^n \)-cubes occurring in \( Y \) in some \( d \)-position, and we denote it by \( B_d(Y) \).

**Definition 2.3** Let \( Y : [0,sd-1]^n \to A \) be a \((sd)^n\)-cube. Let \( d \geq 1 \) be a positive integer. We define the \( 2d \)-covering of \( Y \) to be set of all \((2d)\)-cubes occurring in \( Y \) in some \( d \)-position, and we denote it by \( C_{d}(Y) \).

We observe that copies of the elements of \( C_{d}(Y) \) cover \( Y \) with overlappings. This is a very important difference between \( C_{d}(Y) \) and \( B_d(Y) \).

**Definition 2.4** Let \( d \geq 1 \) and \( s \geq 2 \) two natural numbers. A system of substitutions of type \( d \to sd \) over the finite set \( A \) is a tuple of finite sets \((\mathcal{X}, \mathcal{Y}, X_1, \Sigma)\), as follows: \( \mathcal{X} \) is a set of \( d^n \)-cubes over \( A \), \( \mathcal{Y} \) is a set of \((ds)^n\)-cubes over \( A \) such that for every \( Y \in \mathcal{Y} \), \( B_d(Y) \subseteq \mathcal{X} \). \( X_1 \in \mathcal{X} \) is a special element called start-symbol. \( \Sigma : \mathcal{X} \to \mathcal{Y} \) is called the set of substitution rules, or simply the substitution. \( \Sigma \) has a natural extension defined on the set of cubes \( Z \) such that \( B_d(Z) \subseteq \mathcal{X} \). We remark that if \( B_d(Z) \subseteq \mathcal{X} \) then \( B_d(\Sigma(Z)) \subseteq \mathcal{X} \), so \( \Sigma \) can be applied again to \( \Sigma(Z) \). Last but not least, \( \Sigma \) must fulfill the following condition:

\[
\Sigma(X_1) \mid [0,d-1]^n = X_1
\]

We say that the substitution \( \Sigma \) is expansive. Also, we can call \( d \) primary granulation and \( s \) factor of expansion.
Definition 2.5 As one immediately can prove by induction, the expansivity of \( \Sigma \) means that for all \( m \in \mathbb{N} \) one has that \( \Sigma^m(X_1) \cap [0, ds^{m-1}] = \Sigma^{m-1}(X_1) \). So we can define the \( n \)-dimensional sequence \( b \):

\[
b := \lim_{m \to \infty} \Sigma^m(X_1)
\]

We say that the \( n \)-dimensional sequence \( a \) is defined by substitution.

Substitution in multi-dimensional sequences and many aspects of this tool can be also found in [23, 24, 14, 3] and in the references therein.

One-dimensional Examples: The Prouhet-Thue-Morse Sequence uses the alphabet \( A = \{0, 1\} \) and the rules of type \( 1 \to 2: 0 \to 01, 1 \to 10 \), with starting symbol 0. The Doubling Period Sequence is defined almost similarly, with the rules \( 0 \to 01, 1 \to 00 \). However, not all substitution sequences found in the literature match to our definition. For example, the Fibonacci Sequence produced by the rules \( 1 \to 10 \) and \( 0 \to 1 \) does not match with our definition, because the substitutes have different lengths. See [1] and [2] for examples.

Two-dimensional Examples: A lot of two-dimensional examples have been described by the author in [16, 17, 19] and [20]. In particular it is proven in [16] that the recurrent two-dimensional sequences \( a = (F_p, x + my + z, 1) \) are always generated by a system of substitutions of type \( 1 \to p \), given by the \( p \) many rules. Call the matrix \( F = a | [0, p - 1]^2 \) the fundamental block. Then the rules are:

\[
[x \to xF]_{x \in F_p}
\]

The most easy example is of course that of the Pascal Triangle modulo 2, defined as \((\mathbb{F}_2, x + z, 1)\), which can be generated with start symbol 1 by the rules:

\[
1 \to \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad 0 \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

Theorem 2.6 Let \( A \) be a finite set, \( f : A^d \to A \) a function and \( a : \mathbb{N}^2 \to A \) a recurrent two-dimensional sequence satisfying \( a(i, j) = f(a(i, j - 1), a(i - 1, j - 1), a(i - 1, j)) \) for all \( i, j \in \mathbb{N} \). Let \( (A, \mathcal{X}, \mathcal{Y}, X_1, \Sigma) \) be a substitution of type \( d \to sd \) generating a two-dimensional sequence \( b \). If the following conditions are satisfied:

1. \( \forall k \in \mathbb{N} \) \( b(k, 0) = a(k, 0) \) and \( b(0, k) = a(0, k) \).
2. There exists \( M \in \mathbb{N} \) such that \( a | [0, ds^M - 1] = \Sigma^M(Y_1) \) and \( C_d(\Sigma^M(Y_1)) = C_d(\Sigma^{M-1}(Y_1)) \).

Then \( a = b \).

Proof: In this proof we write for short \( \Sigma^m \) for the set \( \Sigma^m(Y_1) \). The conclusion is equivalent to \( a | [0, ds^m - 1] = \Sigma^m \) for all \( m \geq 1 \). This is true for all \( m \leq M \) by condition 2 and the fact that \( \Sigma \) is expansive. We prove this equality by induction for all \( m > M \).

Definition 2.7 Some finite or infinite matrix \((u(i, j))\) is called an \( f \)-matrix if \( \forall i, j \geq 1 \) one has:

\[
u(i, j) = f(u(i - 1, j), u(i - 1, j - 1), u(i, j - 1)).
\]

We observe that \( \Sigma^m = a | [0, ds^m - 1] \) if and only if (I) \( \Sigma^m \) fulfills the initial conditions of the recurrence and (II) \( \Sigma^m \) is an \( f \)-matrix. (I) is true by assumption. In order to prove (II), we observe that it is enough to prove that all the elements of \( C_d(\Sigma^m) \) are \( f \)-matrices, because those \( 2d \times 2d \)-matrices occurring in \( d \)-positions cover \( \Sigma^m \) with overplings. We denote by \( C_d \) the set \( C_d(\Sigma^m) = C_d(\Sigma^{M-1}) \). All the elements of \( C_d \) are \( f \)-matrices, because they all occur in \( a | [0, ds^m - 1] \). So if we prove that the sequence \((C_d(\Sigma^m))_{m \in \mathbb{N}}\) becomes stationary at \( m = M - 1 \) we are done.
Lemma 2.8  With the notations introduced above, if \( m \geq M - 1 \) then \( C_d(\Sigma^m) = C_d \).

Proof of the Lemma 2.8: The conclusion is true for \( m = M - 1 \) and \( m = M \) by definition. Consider that we have already proven that \( C_d(\Sigma^m) = C_d \) for some \( m \geq M \). Let \( U \) be a \( 2d \times 2d \) matrix occurring in \( d \)-position somewhere in \( \Sigma^{m+1} \). \( U \) can lie inside some \( \Sigma(X) \), or can lie on the border between some \( \Sigma(X) \) and its neighbor \( \Sigma(Y) \) or can even lie such that the four \( d \times d \)-blocks building together \( U \) are adjacent corners in four neighboring matrices got by substitution rules:

\[
\begin{pmatrix}
\Sigma(X) & \Sigma(Y) \\
\Sigma(Z) & \Sigma(T)
\end{pmatrix} = \Sigma \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}.
\]

where \( X, Y, Z, W \in \mathcal{X} \). In all cases, there exists a \( 2d \times 2d \)-matrix \( V \) occurring in \( \Sigma^m \) in \( d \)-position such that \( U \) is covered by \( \Sigma(V) \). We know that \( V \in C_d(\Sigma^m) = C_d \), the last equality being the hypothesis of induction. But as we know that \( C_d(\Sigma^{m-1}) = C_d \), we conclude that \( C_d(\Sigma^{m-1}) \), so that \( U \) occurs already in \( \Sigma^m \) in some \( d \)-position, as a sub-block in some occurrence of the block \( \Sigma(V) \). This means that \( C_d(\Sigma^{m+1}) = C_d(\Sigma^m) = C_d \). \( \square \)
3 Applications

Definition 3.1 Let $(t(k))_{k \geq 0}$ be the Prouhet-Thue-Morse Sequence, got applying the substitution rules 0 → 01 and 1 → 10 with start symbol 0 and $(t'(k))_{k \geq 0}$ be the complementary Prouhet-Thue-Morse Sequence, got by the same rules but with start symbol 1. Let $(d(k))_{k \geq 0}$ the Doubling-Period Sequence, got applying the substitution rules 0 → 01 and 1 → 00 with start symbol 0 and $(d'(k))_{k \geq 0}$ be the complementary Doubling-Period Sequence, got by the rules 1 → 10 and 0 → 11 with start symbol 1. Then we call the following two-dimensional sequences: $(\mathbb{Z}/2^k \mathbb{Z}, x + z, t, t)$ Thue-Morse-Pascal modulo $2^k$, $(\mathbb{Z}/2^k \mathbb{Z}, x + z, t', t')$ complementary Thue-Morse-Pascal modulo $2^k$, $(\mathbb{Z}/2^k \mathbb{Z}, x + z, d, d)$ Doubling-Period-Pascal modulo $2^k$, $(\mathbb{Z}/2^k \mathbb{Z}, x + z, d', d')$ complementary Doubling-Period-Pascal modulo $2^k$, $(\mathbb{Z}/2^k \mathbb{Z}, x + z, t, d)$ TMDP modulo $2^k$, and finally $(\mathbb{Z}/3 \mathbb{Z}, x + y + z, \{0, 0 \rightarrow 010, 1 \rightarrow 111\})$ Essential Heraldry. In the case that $k = 1$ we do not say “modulo 2” anymore, and speak only about the Thue-Morse-Pascal two-dimensional sequence, etc.
Corollary 3.2  The following results can be automatically proven using a computer and the principle expressed by Theorem 2.6:

1. Both Thue-Morse-Pascal and the complementary Thue-Morse-Pascal are produced by systems of substitution of type $4 \to 8$ with 15 rules.

2. Both Doubling-Period-Pascal and the complementary Doubling-Period-Pascal are produced by systems of substitution of type $8 \to 16$ with 70 rules.

3. TMDP is produced by a system of substitution of type $4 \to 8$ with 47 rules.

4. Thue-Morse-Pascal modulo 4 is produced by a system of substitutions of type $8 \to 16$ with 284 rules.

5. TMDP modulo 4 is produced by a system of substitutions of type $8 \to 16$ with 1712 rules.

6. Essential Heraldry is produced by a system of substitutions of type $3 \to 9$ with 171 rules.
Proof: We exemplarily develop only the case of the Thue-Morse-Pascal two-dimensional sequence, because with 15 rules of type $4 \rightarrow 8$ it can be completely described here. The system of substitutions $(\{0, 1\}, \mathcal{X}, \mathcal{Y}, X_1, \Sigma)$ consists of the following sets. The set $\mathcal{X} = \{X_1, \ldots, X_{15}\}$:

\[
X_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},
\]

\[
X_5 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X_8 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix},
\]

\[
X_9 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad X_{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
X_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad X_{14} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad X_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.
\]

The set $\mathcal{Y} = \{Y_1, \ldots, Y_{15}\}$:

\[
Y_1 = \begin{pmatrix} X_1 & X_2 \\ X_5 & X_6 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} X_3 & X_4 \\ X_7 & X_8 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} X_3 & X_4 \\ X_9 & X_6 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} X_1 & X_2 \\ X_{10} & X_8 \end{pmatrix},
\]

\[
Y_5 = \begin{pmatrix} X_3 & X_7 \\ X_9 & X_{14} \end{pmatrix}, \quad Y_6 = \begin{pmatrix} X_1 & X_8 \\ X_{11} & X_6 \end{pmatrix}, \quad Y_7 = \begin{pmatrix} X_3 & X_7 \\ X_7 & X_{11} \end{pmatrix}, \quad Y_8 = \begin{pmatrix} X_{12} & X_{13} \\ X_{12} & X_8 \end{pmatrix},
\]

\[
Y_9 = \begin{pmatrix} X_1 & X_{10} \\ X_5 & X_{14} \end{pmatrix}, \quad Y_{10} = \begin{pmatrix} X_1 & X_{10} \\ X_{10} & X_{11} \end{pmatrix}, \quad Y_{11} = \begin{pmatrix} X_{11} & X_{11} \\ X_{11} & X_{11} \end{pmatrix}, \quad Y_{12} = \begin{pmatrix} X_{12} & X_{12} \\ X_{12} & X_{11} \end{pmatrix},
\]

\[
Y_{13} = \begin{pmatrix} X_{11} & X_8 \\ X_{11} & X_8 \end{pmatrix}, \quad Y_{14} = \begin{pmatrix} X_{12} & X_{12} \\ X_{15} & X_{14} \end{pmatrix}, \quad Y_{15} = \begin{pmatrix} X_{11} & X_{11} \\ X_{14} & X_{14} \end{pmatrix}.
\]

The matrix $X_1$ is the start symbol, and $\forall i \Sigma(X_i) = Y_i$.

We can verify by hand the first condition of the Theorem 2.6. Only the substitution rules for $X_1$, $X_2$, $X_3$ and $X_4$ are needed. Let $M_1 = X_1 \cap \{y = 0\}$. The relevant part of the substitution rules reads $M_1 \rightarrow M_1 M_2$, $M_2 \rightarrow M_3 M_4$, $M_3 \rightarrow M_3 M_4$ and $M_4 \rightarrow M_1 M_2$, where $M_1 = M_4 = 0110$ and $M_2 = M_3 = 1001$. So the horizontal border is the sequence given by start word 0110 and by the rules $\alpha : 0110 \rightarrow 01101001$ and $\beta : 1001 \rightarrow 10010110$. Let $h : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be the homomorphism of monoids whose fix-point is the Prouhet-Thue-Morse sequence, i.e. this one generated by $h(0) = 01$ and $h(1) = 10$. The start word 0110 = $h^2(0)$ and every further complete substitution step using both rules $\alpha$ and $\beta$ has the same effect as once applying $h^2$. It follows by induction that the horizontal border is exactly the Prouhet-Thue-Morse sequence. The proof for the vertical border is similar.
The second condition of the Theorem 2.6 has been checked using a computer program. The
program generated a 8000 × 8000 initial square of the recurrent double sequence, checked that
this square was identical with the correspondent square produced by substitution, and checked the
fact that all 8²-squares occurring in 4-position have already occurred in some 4-positions in the
4000 × 4000 left-upper quarter of this initial square.

References


their applications (Singapore 1998), Springer Series Discrete Mathematics in Theoretical 


2005.

1996.


Advanced Study Institute on Long Range Aperiodic Order. Kluwer Academic Publishings, 
1997.


[14] N. Priebe Frank: Multi-dimensional constant-length substitution sequences. Topology and 
Applications, 152, 44 - 69, 2005.


[16] M. Prunescu: Self-similar carpets over finite fields. European Journal of Combinatorics, 


