The Thue-Morse-Pascal double sequence and similar structures

Mihai Prunescu *

Abstract

If a recurrent two-dimensional sequence with initial conditions defined by linear substitution and a two-dimensional sequence that is generated by planar substitution are identical over a sufficiently large initial square, then they will coincide over all. After proving this general principle, we apply it to some concrete examples. One of them, the Thue-Morse-Pascal two-dimensional sequence, is defined by two copies of the Prouhet-Thue-Morse sequence as pair of initial conditions and by the Pascal Triangle Addition modulo 2 as rule of recurrence. As it follows, the Thue-Morse-Pascal two-dimensional sequence is the result of 15 substitution rules, each of them consisting of the substitution of some 4×4 matrix with an 8×8 matrix.

Key Words: recurrent two-dimensional sequence, periodic initial conditions, expansive system of context-free substitutions, automatic proof procedure, homomorphisms of finite abelian p-groups, Pascal's Triangle, Prouhet-Thue-Morse Sequence, Doubling Period Sequence, Thue-Morse-Pascal Two-dimensional Sequence.

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1 Introduction

This note reveals a new touching point between recurrence and substitution. Both notions occur in a field of interdisciplinary investigations unifying very heterogenous motivations and techniques. The recurrence - although a very classical task - is more and more present in studies concerning cellular automata, see [26], [12], [7] or the monograph [27]. Substitutions occur in various contexts such as automatic sequences [11], [1], [2], aperiodic tilings [25], [21], [13], [6], [5], various fractal constructions [8], [9], [22] or mathematic quasicrystals [10], [4].

Definition 1.1 *Let A be a finite set and $f : A^3 \to A$ a fixed function. We call the set A alphabet and the function f recurrence. We will refer to the function f as $f(x, y, z)$. We also fix two sequences $u, v : \mathbb{N} \to A$ with $u(0) = v(0)$, called initial conditions. We say that the tuple (A, f, u, v) defines a recurrent two-dimensional sequence $a : \mathbb{N}^2 \to A$ if the following conditions are fulfilled:*

1. $\forall\ k \in \mathbb{N}\ a(k, 0) = u(k)$ *and* $a(0, k) = v(k)$.
2. $\forall\ i, j > 0\ a(i, j) = f(a(i, j-1), a(i-1, j-1), a(i-1, j))$.

In the case that $u = v$ we mention just one of them in the tuple. If u or v are periodic, we just write down the period. Here we must stress the fact that the notion of recurrence defined here has a very different meaning than those used for one-dimensional sequences and their associated dynamical systems. In that case, recurrence means that each block occurring in the sequence, occurs infinitely often.

*Brain Products, Freiburg, Germany and Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit 5, P. O. Box 1-764, RO-014700 Bucharest, Romania. mihai.prunescu@math.uni-freiburg.de.

The author proved in [15] that recurrent two-dimensional sequences are Turing complete. After understanding the self-similar nature of a narrow class of recurrent two-dimensional sequences in [16], the author finally conjectured that all recurrent two-dimensional sequences given by homomorphisms of finite abelian p-groups and periodic borders (initial conditions) are produced by systems of substitution, see [16], [17], [18] , [19] and [20].

The factor of expansion in the case with periodic borders seemed to depend of the p-group homomorphism only, and not of the periodic borders. This suggested that this factor is an implicit property of the p-group homomorphism. The author supposed that two-dimensional substitution could also occur by applying iterated p-group homomorphisms on initial conditions (borders) which are not periodic, but quasi-periodic; such some produced by one-dimensional substitutions of appropriate factor of expansion. Here we present the first examples of this kind. However, we do not dare to formulate a general conjecture this time.

We observe also that every line $a(i,k) \,|\, i \in \mathbb{N}$ or column $a(k,j) \,|\, j \in \mathbb{N}$ in a recurrent two-dimensional sequence with periodic borders is ultimately periodic, and could never be an essentially non-periodic sequence - like the Prouhet-Thue-Morse sequence. This means that the cases studied here are definitely not covered by anything studied by the author before.

2 Definitions and main result

Although the notions of block decomposition, covering and system of substitutions will be used only in $n = 1$ or $n = 2$ dimensions, it seems to be more practical to define those notions for an arbitrary dimension n.

Definition 2.1 *Let A be a finite set (alphabet). An n-dimensional sequence is a function $a : \mathbb{N}^n \to A$. An n-dimensional cube is a function $X : [0, d-1]^n \to A$. Such a set is called a d^n-cube. We say that X occurs in A at $\vec{u} \in \mathbb{N}^n$ if $a \,|\, \vec{u} + [0, d-1]^n \equiv X \,|\, [0, d-1]^n$. We say for $d \in \mathbb{N}$ that $d \,|\, \vec{u}$ if there is a $\vec{v} \in \mathbb{N}^n$, $\vec{v} = d\vec{u}$. We say that X occurs in a if there is a $\vec{u}$ such that X occurs in a at $\vec{u}$. We say that X occurs at some d-position in a if moreover $d \,|\, \vec{u}$.*

Definition 2.2 *Let $s \in \mathbb{N}$ be a natural number ≥ 2 and let $Y : [0, ds-1]^n \to A$ be some $(ds)^n$-cube over A. We define the d-block decomposition of Y to be set of all d^n-cubes occurring in Y in some d-position, and we denote it by $B_d(Y)$.*

Definition 2.3 *Let $Y : [0, sd-1]^n \to A$ be a $(sd)^n$-cube. Let $d \geq 1$ be a positive integer. We define the $2d$-covering of Y to be set of all $(2d)^n$ cubes occurring in Y in some d-position, and we denote it by $C_d(Y)$.*

We observe that copies of the elements of $C_d(Y)$ cover Y with overlappings. This is a very important difference between $C_d(Y)$ and $B_d(Y)$.

Definition 2.4 *Let $d \geq 1$ and $s \geq 2$ two natural numbers. A system of substitutions of type $d \to sd$ over the finite set A is a tuple of finite sets $(\mathcal{X}, \mathcal{Y}, X_1, \Sigma)$, as follows: $\mathcal{X}$ is a set of d^n-cubes over A. $\mathcal{Y}$ is a set of $(ds)^n$-cubes over A such that for every $Y \in \mathcal{Y}$, $B_d(Y) \subset \mathcal{X}$. $X_1 \in \mathcal{X}$ is a special element called start-symbol. $\Sigma : \mathcal{X} \to \mathcal{Y}$ is called the set of substitution rules, or simply the substitution. Σ has a natural extension defined on the set of cubes Z such that $B_d(Z) \subseteq \mathcal{X}$. We remark that if $B_d(Z) \subseteq \mathcal{X}$ then $B_d(\Sigma(Z)) \subseteq \mathcal{X}$, so Σ can be applied again to $\Sigma(Z)$. Last but not least, Σ must fulfill the following condition:*

$$\Sigma(X_1) \,|\, [0, d-1]^n = X_1$$

We say that the substitution Σ is expansive. Also, we can call d primary granulation and s factor of expansion.

Definition 2.5 *As one immediately can prove by induction, the expansivity of* Σ *means that for all* $m \in \mathbb{N}$ *one has that* $\Sigma^m(X_1) \,|\, [0, ds^{m-1}]^n = \Sigma^{m-1}(X_1)$*. So we can define the* n*-dimensional sequence* b*:*

$$b := \lim_{m \to \infty} \Sigma^m(X_1)$$

We say that the n*-dimensional sequence* a *is defined by substitution.*

Substitution in multi-dimensional sequences and many aspects of this tool can be also found in [23], [24], [14], [3] and in the references therein.

One-dimensional Examples: The Prouhet-Thue-Morse Sequence uses the alphabet $A = \{0, 1\}$ and the rules of type $1 \to 2$: $0 \to 01$, $1 \to 10$, with starting symbol 0. The Doubling Period Sequence is defined almost similarly, with the rules $0 \to 01$, $1 \to 00$. However, not all substitution sequences found in the literature match to our definition. For example, the Fibonacci Sequence produced by the rules $1 \to 10$ and $0 \to 1$ does not match with our definition, because the substitutes have different lengths. See [1] and [2] for examples.

Two-dimensional Examples: A lot of two-dimensional examples have been described by the author in [16], [17], [19] and [20]. In particular it is proven in [16] that the recurrent two-dimensional sequences $a = (\mathbb{F}_p, x + my + z, 1)$ are always generated by a system of substitutions of type $1 \to p$, given by the p many rules. Call the matrix $F = a \,|\, [0, p-1]^2$ the fundamental block. Then the rules are:

$$[\, x \to xF \,]_{x \in \mathbb{F}_p}$$

The most easy example is of course that of the Pascal Triangle modulo 2, defined as $(\mathbb{F}_2, x + z, 1)$, which can be generated with start symbol 1 by the rules:

$$1 \to \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad 0 \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Theorem 2.6 *Let* A *be a finite set,* $f : A^3 \to A$ *a function and* $a : \mathbb{N}^2 \to A$ *a recurrent two-dimensional sequence satisfying* $a(i,j) = f(a(i, j-1), a(i-1, j-1), a(i-1, j))$ *for all* $i, j \in \mathbb{N}$*. Let* $(A, \mathcal{X}, \mathcal{Y}, X_1, \Sigma)$ *be a substitution of type* $d \to sd$ *generating a two-dimensional sequence* b*. If the following conditions are satisfied:*

1. $\forall\, k \in \mathbb{N}\;\; b(k, 0) = a(k, 0)$ *and* $b(0, k) = a(0, k)$*.*
2. *There exists* $M \in \mathbb{N}$ *such that* $a \,|\, [0, ds^M - 1] = \Sigma^M(Y_1)$ *and* $C_d(\Sigma^M(Y_1)) = C_d(\Sigma^{M-1}(Y_1))$*.*

Then $a = b$*.*

Proof: In this proof we write for short Σ^m for the set $\Sigma^m(Y_1)$. The conclusion is equivalent to $a \,|\, [0, ds^m - 1] = \Sigma^m$ for all $m \geq 1$. This is true for all $m \leq M$ by condition 2 and the fact that Σ is expansive. We prove this equality by induction for all $m > M$.

Definition 2.7 *Some finite or infinite matrix* $(u(i,j))$ *is called an* f*-matrix if* $\forall\, i, j \geq 1$ *one has:*

$$u(i,j) = f(u(i-1, j), u(i-1, j-1), u(i, j-1)).$$

We observe that $\Sigma^m = a \,|\, [0, ds^m - 1]$ if and only if (I) (Σ^m fulfills the initial conditions of the recurrence) and (II) (Σ^m is an f-matrix). (I) is true by assumption. In order to prove (II), we observe that it is enough to prove that all the elements of $C_d(\Sigma^m)$ are f-matrices, because those $2d \times 2d$-matrices occurring in d-positions cover Σ^m with overlappings. We denote by C_d the set $C_d(\Sigma^M) = C_d(\Sigma^{M-1})$. All the elements of C_d are f-matrices, because they all occur in $a \,|\, [0, ds^M - 1]$. So if we prove that the sequence $(C_d(\Sigma^m))_{m \in \mathbb{N}}$ becomes stationary at $m = M - 1$ we are done.

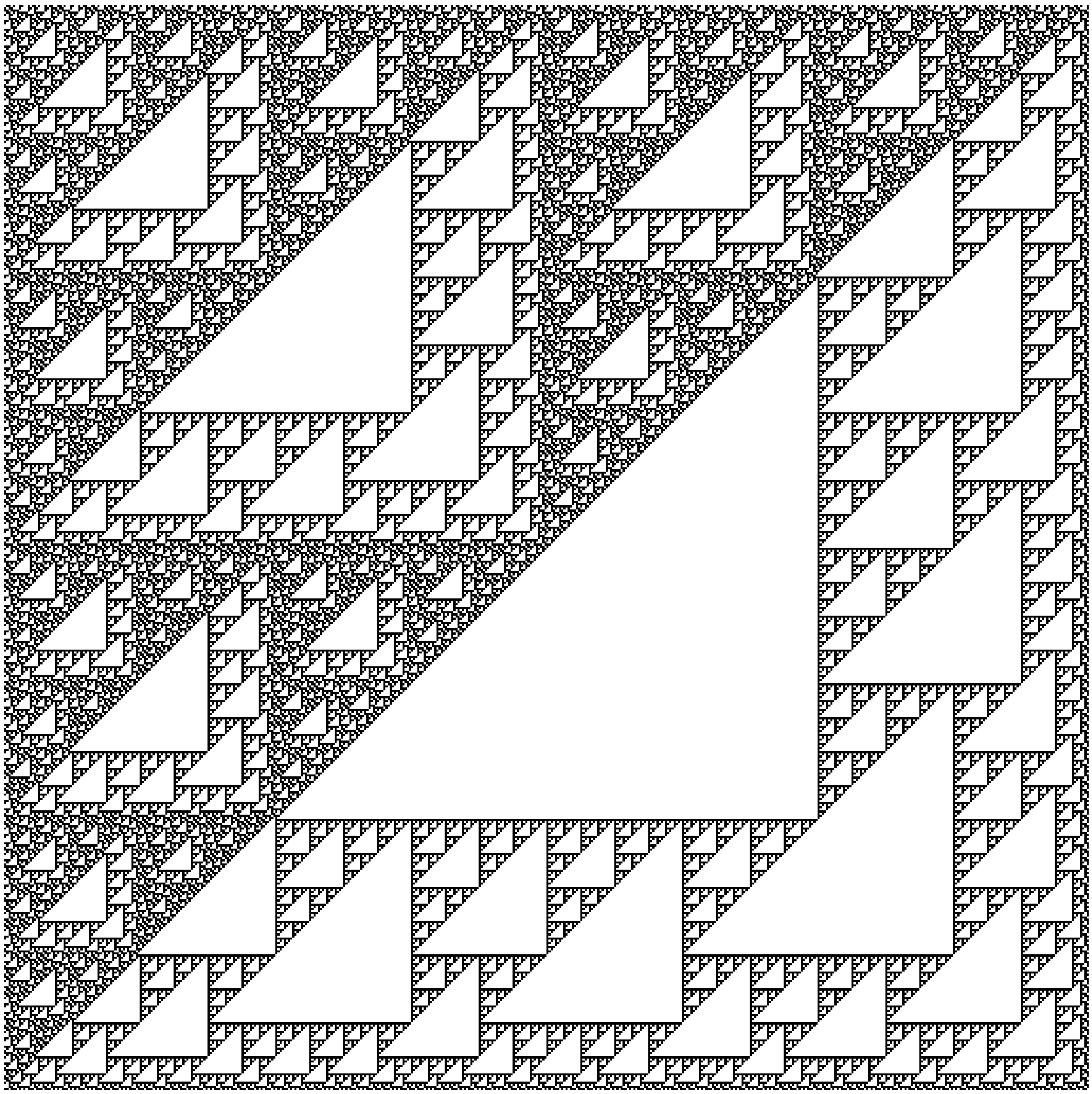

Fig. 1: Thue-Morse-Pascal, 512×512.

Lemma 2.8 *With the notations introduced above, if* $m \geq M - 1$ *then* $C_d(\Sigma^m) = C_d$.

Proof of the Lemma 2.8: The conclusion is true for $m = M - 1$ and $m = M$by definition. Consider that we have already proven that $C_d(\Sigma^m) = C_d$ for some $m \geq M$. Let U be a $2d \times 2d$ matrix occurring in d-position somewhere in Σ^{m+1}. U can lie inside some $\Sigma(X)$, or can lie on the border between some $\Sigma(X)$ and its neighbor $\Sigma(Y)$ or can even lie such that the four $d \times d$-blocks building together U are adiacent corners in four neighboring matrices got by substitution rules:

$$\begin{pmatrix} \Sigma(X) & \Sigma(Y) \\ \Sigma(Z) & \Sigma(T) \end{pmatrix} = \Sigma \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}.$$

where $X, Y, Z, W \in \mathcal{X}$. In all cases, there exists a $2d \times 2d$-matrix V occuring in Σ^m in d-position such that U is covered by $\Sigma(V)$. We know that $V \in C_d(\Sigma^m) = C_d$, the last equality being the hypothesis of induction. But as we know that $C_d(\Sigma^{m-1}) = C_d$, we conclude that $C_d(\Sigma^{m-1})$, so that U occurs already in Σ^m in some d-position, as a sub-block in some occurrence of the block $\Sigma(V)$. This means that $C_d(\Sigma^{m+1}) = C_d(\Sigma^m) = C_d$. □

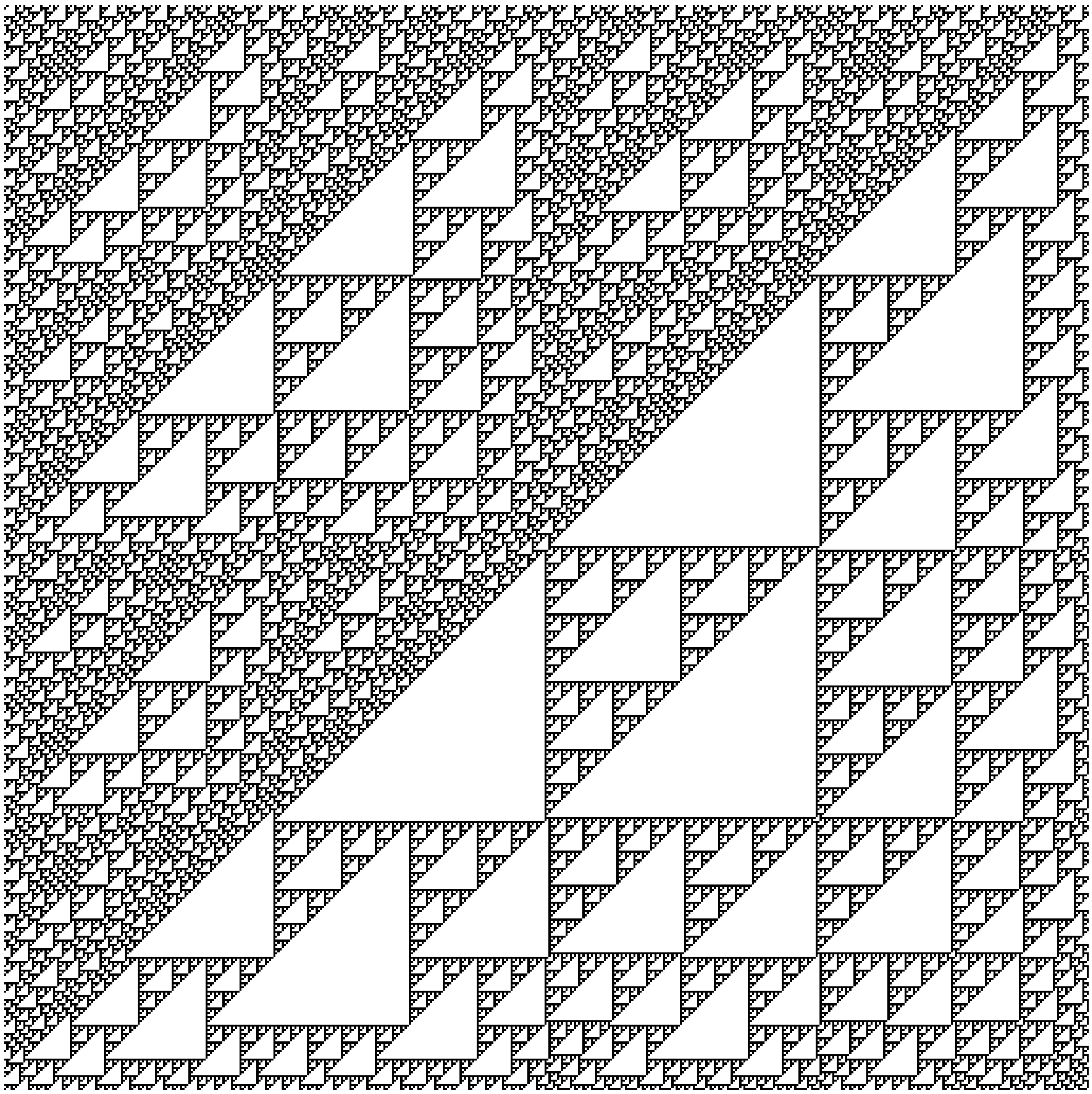

Fig. 2: Doubling-Period-Pascal, 512×512.

3 Applications

Definition 3.1 *Let $(t(k))_{k\geq 0}$ be the Prouhet-Thue-Morse Sequence, got applying the substitution rules $0 \to 01$ and $1 \to 10$ with start symbol 0 and $(t'(k))_{k\geq 0}$ be the complementary Prouhet-Thue-Morse Sequence, got by the same rules but with start symbol 1. Let $(d(k))_{k\geq 0}$ the Doubling-Period Sequence, got applying the substitution rules $0 \to 01$ and $1 \to 00$ with start symbol 0 and $(d'(k))_{k\geq 0}$ be the complementary Doubling-Period Sequence, got by the rules $1 \to 10$ and $0 \to 11$ with start symbol 1. Then we call the following two-dimensional sequences: $(\mathbb{Z}/2^k\mathbb{Z}, x+z, t, t)$ Thue-Morse-Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, t', t')$ complementary Thue-Morse-Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, d, d)$ Doubling-Period-Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, d', d')$ complementary Doubling-Period-Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, t, d)$ TMDP modulo 2^k, and finally $(\mathbb{Z}/3\mathbb{Z}, x+y+z, \{0, 0 \to 010, 1 \to 111\})$* Essential Heraldry. *In the case that $k = 1$ we do not say "modulo 2" anymore, and speak only about the Thue-Morse-Pascal two-dimensional sequence, etc.*

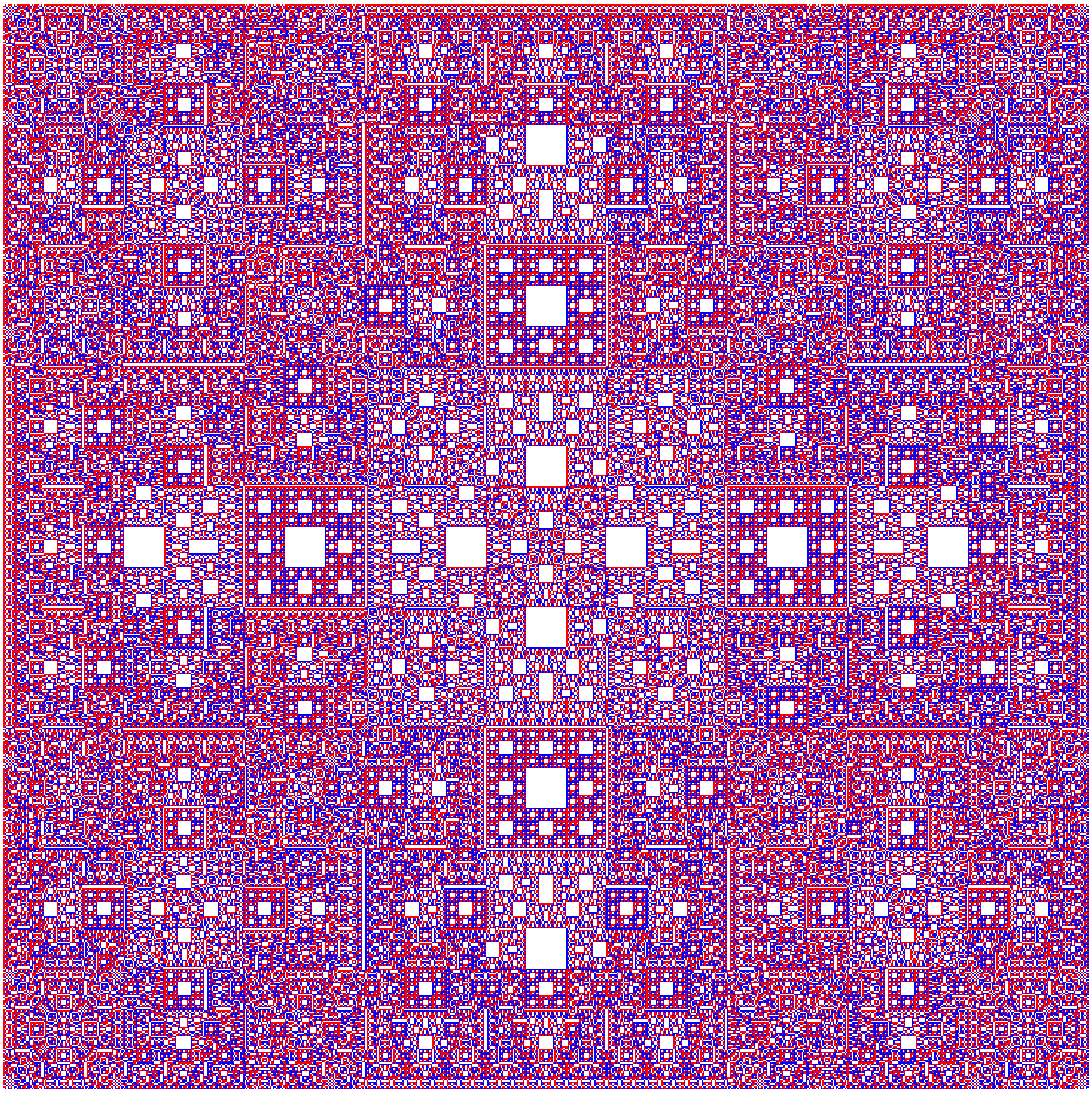

Fig. 3: Essential Heraldry, 729×729.

Corollary 3.2 *The following results can be automatically proven using a computer and the principle expressed by Theorem 2.6:*

1. *Both Thue-Morse-Pascal and the complementary Thue-Morse-Pascal are produced by systems of substitution of type* $4 \to 8$ *with* 15 *rules.*

2. *Both Doubling-Period-Pascal and the complemetary Doubling-Period-Pascal are produced by systems of substitution of type* $8 \to 16$ *with* 70 *rules.*

3. *TMDP is produced by a system of substitution of type* $4 \to 8$ *with* 47 *rules.*

4. *Thue-Morse-Pascal modulo* 4 *is produced by a system of substitutions of type* $8 \to 16$ *with* 284 *rules.*

5. *TMDP modulo* 4 *is produced by a system of substitutions of type* $8 \to 16$ *with* 1712 *rules.*

6. *Essential Heraldry is produced by a system of substitutions of type* $3 \to 9$ *with* 171 *rules.*

Proof: We exemplarily develop only the case of the Thue-Morse-Pascal two-dimensional sequence, because with 15 rules of type $4 \to 8$ it can be completely described here. The system of substitutions $(\{0,1\}, \mathcal{X}, \mathcal{Y}, X_1, \Sigma)$ consists of the following sets. The set $\mathcal{X} = \{X_1, \ldots, X_{15}\}$:

$$X_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \; X_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \; X_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \; X_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$X_5 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \; X_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \; X_7 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \; X_8 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$X_9 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \; X_{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; X_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; X_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \; X_{14} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \; X_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

The set $\mathcal{Y} = \{Y_1, \ldots, Y_{15}\}$:

$$Y_1 = \begin{pmatrix} X_1 & X_2 \\ X_5 & X_6 \end{pmatrix} \quad Y_2 = \begin{pmatrix} X_3 & X_4 \\ X_7 & X_8 \end{pmatrix} \quad Y_3 = \begin{pmatrix} X_3 & X_4 \\ X_9 & X_6 \end{pmatrix} \quad Y_4 = \begin{pmatrix} X_1 & X_2 \\ X_{10} & X_8 \end{pmatrix}$$

$$Y_5 = \begin{pmatrix} X_3 & X_7 \\ X_9 & X_{14} \end{pmatrix} \quad Y_6 = \begin{pmatrix} X_{11} & X_8 \\ X_{14} & X_6 \end{pmatrix} \quad Y_7 = \begin{pmatrix} X_3 & X_7 \\ X_7 & X_{11} \end{pmatrix} \quad Y_8 = \begin{pmatrix} X_{12} & X_{13} \\ X_{12} & X_8 \end{pmatrix}$$

$$Y_9 = \begin{pmatrix} X_1 & X_{10} \\ X_5 & X_{14} \end{pmatrix} \; Y_{10} = \begin{pmatrix} X_1 & X_{10} \\ X_{10} & X_{11} \end{pmatrix} \; Y_{11} = \begin{pmatrix} X_{11} & X_{11} \\ X_{11} & X_{11} \end{pmatrix} \; Y_{12} = \begin{pmatrix} X_{12} & X_{12} \\ X_{12} & X_{11} \end{pmatrix}$$

$$Y_{13} = \begin{pmatrix} X_{11} & X_8 \\ X_{11} & X_8 \end{pmatrix} \quad Y_{14} = \begin{pmatrix} X_{12} & X_{12} \\ X_{15} & X_{14} \end{pmatrix} \quad Y_{15} = \begin{pmatrix} X_{11} & X_{11} \\ X_{14} & X_{14} \end{pmatrix}$$

The matrix X_1 is the start symbol, and $\forall\, i \; \Sigma(X_i) = Y_i$.

We can verify by hand the first condition of the Theorem 2.6. Only the substitution rules for X_1, X_2, X_3 and X_4 touch the horizontal border. Let $M_i = X_i \cap (y = 0)$. The relevant part of the substitution rules reads $M_1 \to M_1 M_2$, $M_2 \to M_3 M_4$, $M_3 \to M_3 M_4$ and $M_4 = M_1 M_2$, where $M_1 = M_4 = 0110$ and $M_2 = M_3 = 1001$. So the horizontal border is the sequence given by start word 0110 and by the rules $\alpha : 0110 \to 01101001$ and $\beta : 1001 \to 10010110$. Let $h : \{0,1\}^* \to \{0,1\}^*$ be the homomorphism of monoids whose fix-point is the Prouhet-Thue-Morse sequence, i. e. this one generated by $h(0) = 01$ and $h(1) = 10$. The start word $0110 = h^2(0)$ and every further complete substitution step using both rules α and β has the same effect as once applying h^2. It follows by induction that the horizontal border is exactly the Prouhet-Thue-Morse sequence. The proof for the vertical border is similar.

The second condition of the Theorem 2.6 has been checked using a computer program. The program generated a 8000×8000 initial square of the recurrent double sequence, checked that this square was identic with the correspondent square produced by substitution, and checked the fact that all 8^2-squares occurring in 4-position have already occurred in some 4-positions in the 4000×4000 left-upper quarter of this initial square. □

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