$P \neq NP$ for the Reals with various Analytic Functions

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Abstract

We show that non-deterministic machines in the sense of [BSS] defined over wide classes of real analytic structures are more powerful than the corresponding deterministic machines.

Key Words: BSS-model, $P \neq NP$, non-determinism, analytic, semialgebraic.

1 Introduction and basic notions

[BSS] introduced a theory of computation and complexity over arbitrary rings, in particular for the ring $\mathbb{R}$ of real numbers. [Meer] considered machines of BSS-type which could carry out polynomial evaluations and the transcendental evaluation $x \rightarrow \sin(x)$, performing also tests like "$x \geq 0$?". According to the algebraic size of inputs given by size $(y_1, \ldots, y_n) = n$ he defined the classes $P_{\sin}$ of sets which can be deterministically decided in polynomial time by real sin-machines and $NP_{\sin}$ of sets which can be recognized in polynomial time using non-deterministic real sin-machines, and he proved that $P_{\sin} \neq NP_{\sin}$.

In this note we generalize the result of Meer for other examples of analytic structures over the reals. The principal instrument and most of the mathematical substance of this note is our Lemma. It was proved by [van den Dries] in order to show that several expansions of the ring of reals with total analytic functions lead to structures whose theories do not admit elimination of quantifiers. We was able to reuse his construction in a such direct way because it consists in **existential** quantifiers which cannot be removed. For the deeper connections between the elimination of quantifiers and the P versus NP problem consult [Poizat].

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K. Meer used global properties of the function sinus (set of all zeros) in order to construct a subset of $\mathbb{R}$ which is non-deterministically recognizable in a constant time by a sin-machine but is not decidable by any deterministic sin-machine with or without bound of time. We will use only local properties of real-analytic functions which are not semialgebraic. We will display an example to show that our situation is not compulsory so dramatic as by Meer: our sets can be even decidable, but not within uniform time-bounds.
Definitions and notations: Let $U \subseteq \mathbb{R}^n$ be not empty, open and connected. A real-analytic function $f : U \to \mathbb{R}$ will be called **tame** if it has not analytical singularities on the boundary of U. We empfasize two cases and introduce some notations. If $U = \mathbb{R}^n$ then f is surely tame. Such a tame function will be called **total**. A total function f will be in some context denoted also by $\tilde{f}$ or $\bar{f}$. On the other hand, if $U \neq \mathbb{R}^n$ and f is tame, then there are by analytic continuation open connected subsets $V, W \subseteq \mathbb{R}^n$ and an analytic function $\tilde{f} : W \to \mathbb{R}$ so that the topological closures $\overline{U} \subset V$, $\overline{V} \subset W$ and $f = \tilde{f}|_U$. In this case we call f **partial** and we denote the middle continuation $\tilde{f}|_V$ by $\overline{f}$. For a function f as before we call the domain dom $(f) := U$. In the same way dom $(\overline{f}) := V$, dom $(\tilde{f}) := W$.
The functions x^{-1} and $\log : (0, +\infty) \to \mathbb{R}$ are real-analytic but not tame. Their restrictions to $(1, +\infty)$ are tame. In the following we will understand by $\mathcal{F} = \{f_i \,|\, f_i : U_i \subset \mathbb{R}^{n_i} \to \mathbb{R}\}$ a family of tame functions. A family of analytic functions which does not contain only tame functions will be called **wild**. The families will be tame if the contrary will be not explicitly stated. The models of computations we are interested about are

$$\mathbb{R}_{\mathcal{F}} = (\mathbb{R}; 0, 1, +, -, \cdot, \mathcal{F}; \leq).$$

$\mathcal{F}$ may contain all possible constant functions, so our result applies also to models of computation which allow the use of arbitrary real parameters.
An $\mathcal{F}$-**machine** over $\mathbb{R}$ is an $\mathbb{R}_{\mathcal{F}}$-machine in the sense of [BSS]. We will work only with inputs of constant size, so it will be convenient to describe $\mathcal{F}$-machines using **while**-programs with a constant number of registers and allowing computation instructions for finitely many functions of $\mathcal{F}$. For the non-deterministic machines we use a supplementary instruction **guess**. The instruction **neverstop** has its evident meaning.
We call an $\mathcal{F}$-machine M **incorrect** if there is an input (or a combination of input and sequence of guesses, in the case that M is not deterministic) so that there is an instruction

$$y := f(x_1, \ldots, x_n)$$

which cannot be carried out because the instantaneous contents of the registers $(x_1, \ldots, x_n)$ does not belong to dom (f). We exclude incorrect machines.

Defining notions like **recognizable**, **computable** or **decidable** in our context suppose the existence of a correct program in the classical definition.
As last, we recall that a set $S \subseteq \mathbb{R}^n$ is called **semialgebraic** if there is a formal definition of S over $\mathbb{R}$. A formal definition is a well formed formula $\varphi(\vec{x}, y)$ in the language of ordered rings such that

$$\forall \vec{x} \in \mathbb{R}^n \; [\; \vec{x} \in S \;\Leftrightarrow\; \mathbb{R} \models \varphi(\vec{x}, y) \;].$$

If the formal definition contains also some function symbols interpretting real analytic functions in one or many variables but does not contain quantifiers, we say that the set $S \subseteq \mathbb{R}^n$ is **semianalytic**. If it contains quantifiers also, we call S a **subanalytic** set. The next paragraph contains the construction of a subanalytic set which is not semianalytic. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ will be called **semialgebraic** or **semianalytic** iff its graph $\{(\vec{x}, y) \,|\, f(\vec{x}) = y\} \subset \mathbb{R}^{n+1}$ is respectively such a set.

2 A non-semianalytic set

Now we are ready to give the principal construction necessary for the Lemma of van den Dries. Let $F : \mathbb{R}^m \to \mathbb{R}$ a total real-analytic function. We define $\Phi : \mathbb{R}^m \times (0, +\infty) \to \mathbb{R}$ as

$$\Phi(x_1, \dots, x_m, x_{m+1}) := x_{m+1} F(\frac{x_1}{x_{m+1}}, \dots, \frac{x_m}{x_{m+1}}).$$

Let Σ_F be the graph of Φ, subset of $\mathbb{R}^{m+2}$:

$$\Sigma_F = \{(x_1, \dots, x_{m+2}) \mid x_{m+1} > 0 \;\wedge\; x_{m+2} = x_{m+1} F(\frac{x_1}{x_{m+1}}, \dots, \frac{x_m}{x_{m+1}})\}.$$

The Lemma says shortly that if F is **not semialgebraic** then Σ_F is **not semianalytic**.
Lemma (van den Dries): *If the function F is not semialgebraic, there is no open ball $\mathcal{O} \subset \mathbb{R}^{m+2}$ around 0 such that $\Sigma_F \cap \mathcal{O}$ belongs to the boolean algebra of subsets of $\mathcal{O}$ generated by finitely many basic semianalytic sets $\{\vec{x} \in \mathcal{O} \mid F_i(\vec{x}) > 0\}$, respectively $\{\vec{x} \in \mathcal{O} \mid F_i(\vec{x}) = 0\}$ given by some real-analytic functions $F_1, \dots, F_k : \mathcal{O} \to \mathbb{R}$.*
Proof: We make a proof by contradiction. If the situation presented in the Lemma is true, we suppose k to be the minimal integer which makes such a description possible. Consequently, no F_i vanishes identically on $\mathcal{O}$. But there must be a point $c \in \Sigma_F \cap \mathcal{O}$ and an open neighborhood $\mathcal{N}$ around c such that some F_i, say F_1, is identically 0 on $\Sigma_F \cap \mathcal{O} \cap \mathcal{N}$; otherwise Σ_F would

contain a whole neighborhood in $\mathbb{R}^{m+2}$. We define $\psi : \mathbb{R}^m \times (0, +\infty) \to \mathbb{R}^{m+2}$ given by $\psi(\vec{x}) := (\vec{x}, \Phi(\vec{x}))$. $\mathcal{U} := \psi^{-1}(\mathcal{O}) \subset \mathbb{R}^{m+1}$ is open, connected and not empty, and the composed application $F_1 |_{\Sigma_F \cap \mathcal{O}} \circ \psi |_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}$ is a real-analytic function which is identically 0 on the open set $\psi^{-1}(\mathcal{N})$, so it must vanish identically on the whole connected $\mathcal{U}$. This implies that F_1 vanishes on $\Sigma_F \cap \mathcal{O}$.

It is easy to see that if $\mathcal{O}$ has the supposed property, all smaller balls around 0 have it also. It is not the case that some open ball around 0 does not intersect Σ_F because the function Φ is homogenous: if $\lambda > 0$ then $\Phi(\lambda\vec{x}) = \lambda\Phi(\vec{x})$.

We choose $\mathcal{O}$ small enough so that the Taylor series of F_1 in 0 coverges over $\mathcal{O}$. It has the form:

$$F_1(x_1, \dots, x_{m+2}) = \sum_{i=0}^{\infty} P_i(x_1, \dots, x_{m+2}),$$

where every P_i is a homogenous polynomial of degree i. We fix a point $(a_1, \dots, a_{m+2}) \in \Sigma_F \cap \mathcal{O}$. Since Φ is a homogenous function, for all $\lambda \in (0, 1)$ is $\lambda\vec{a} \in \Sigma_F \cap \mathcal{O}$. This means that the real-analytic function of one variable λ,

$$F_1(\lambda\vec{a}) = \sum_{i=0}^{\infty} \lambda^i P_i(a_1, \dots, a_{m+2}),$$

vanishes identically on $(0, 1)$, so all its coefficients must be 0. If we take a $d \in \mathbb{N}$ such that the corresponding P_d is not 0 as polynomial, we get for $P = P_d$ that:

$$\forall (x_1, \dots, x_{m+1}) \in \psi^{-1}(\mathcal{O}) \quad P(x_1, \dots, x_{m+1}, \Phi(x_1, \dots, x_{m+1})) = 0.$$

Using the homogenity of P and Φ and the fact that $\mathcal{O}$ is a ball we see that this identity holds in fact for all $\vec{x} \in \mathbb{R}^{m+1}$. Taking further $x_{m+1} = 1$ we get that for all $\vec{x} \in \mathbb{R}^m$ holds $P(\vec{x}, 1, F(\vec{x})) = 0$, or shortly:

$$\forall \vec{x} \in \mathbb{R}^m \quad Q(x_1, \dots, x_m, F(x_1, \dots, x_m)) = 0.$$

Q is not identically 0 as polynomial, because the homogenous polynomial P was also not 0. Following [Lojasiewicz] there is a decomposition $\mathbb{R}^m = A_0 \cup A_1 \cup \dots \cup A_s$ in disjoint connected semialgebraic sets so that $Q(\vec{x}, Y) \equiv 0$ for all $\vec{x} \in A_0$ and for $j \in \{1, \dots, s\}$ the real roots of $Q(\vec{x}, Y) = 0$ for $\vec{x} \in Aj$ are given by continouos functions $\rho_{j1}(\vec{x}) < \dots < \rho_{jk(j)}(\vec{x})$, all of them semialgebraic. Because of the continuity of F and of the connectedness of every A_j there will be for every $j \in \{1, \dots, s\}$ an l such that $F |_{A_j} = \rho_{jl}$, so $F |_{A_1 \cup \dots \cup A_s}$ is semialgebraic. But $\overline{A_1 \cup \dots \cup A_s} = \mathbb{R}^m$ and F is continuous, so F must be globally semialgebraic. This is a contradiction. $\square$

3 Main result

We will not consider a partial tame function to be not semialgebraic only because it comes from a semialgebraic function restricted to a non-semialgebraic open domain. That is why we state another more definition before stating our result:
Definition: We say that a tame function f is **essentially not semialgebraic** if there is no open set U such that $f\,|_U$ is a nonempty semialgebraic function. For a total analytic function it means the same as to be not semialgebraic.
Proposition: *Let $\mathcal{F}$ be a family of tame real-analytic functions containing a function f which is essentially not semialgebraic. Then there is a natural $t \geq 3$ and a subset $\Sigma \subset \mathbb{R}^t$ such that:*

- *Σ is recognized by a non-deterministic $\mathcal{F}$-machine in constant time.*
- *Σ cannot be decided by any deterministic $\mathcal{F}$-machine in constant time.*

Consequently, the set Σ is a trivial counterexample showing that $P_{\mathcal{F}} \neq NP_{\mathcal{F}}$.
Proof: We divide the proof in three cases.
Case 1: First we consider the case in which all members of $\mathcal{F}$ are **total**. We take $F = f : \mathbb{R}^m \to \mathbb{R}$, $t = m + 2 \geq 3$ and $\Sigma := \Sigma_F \subset \mathbb{R}^k$ as it has been defined before the Lemma. It is evident that Σ is the halting set of the following non-deterministic $\mathcal{F}$-machine:

input $x_1, \dots, x_{m+2}$ **; guess** $b_1, \dots, b_m$ **;**

if $x_{m+1} \leq 0$ **then neverstop**

else for $i = 1$ **to** m **do**

  if $x_{m+1} b_i - x_i \neq 0$ **then neverstop**

  end do ;

if $x_{m+2} - x_{m+1} f(b_1, \dots, b_m) \neq 0$ **then neverstop**

  else stop.

Let M be a deterministic $\mathcal{F}$-machine which decides the set Σ in a constant time T not depending of the input $\vec{x} \in \mathbb{R}^{m+2}$. M can build up by succesive composition of the members of $\mathcal{F}$ and of the arithmetical operations only finitely many analytic $\mathcal{F}$-terms $F_1, \dots, F_k : \mathbb{R}^{m+2} \to \mathbb{R}$ in the time T and can decide at most T tests $F_i > 0$ or $F_i = 0$. This means that M can decide Σ only if Σ belongs to the boolean algebra of subsets of $\mathbb{R}^{m+2}$ generated by the

sets $\{F_i > 0\}$ and $\{F_i = 0\}$. But this is impossible even for neighborhoods $\mathcal{O}$ of 0, as it has been provedin the Lemma. We had reached a contradiction.
Case 2: Now we cosider the case that f is still **total**, but there are members of the family $\mathcal{F}$ which are **partial**. In this case we choose again $F = f$ and Σ as in the first case. Σ will be recognised in constant time by the same non-deterministic machine as in the first case. Suppose again that Σ would be decided by a deterministic $\mathcal{F}$-machine M in a constant time. M can build up again only a finite number of analytic $\mathcal{F}$-terms $F_1, \dots, F_k$. Their maximal possible domains of definition are open and not necessarily connected sets and do not depend on the tests proceeded by M but only on the domains of definitions of the occurring functions and on the order in which they have been composed by the action of M. We denote this open sets dom $(F_1), \dots,$ dom (F_k). The following considerations are now relevant for choosing a neighborhood $\mathcal{O} \subset \mathbb{R}^{m+2}$ of 0. We rename the functions F_i so that for $i = 1$ to (say) l, $0 \in$ dom (F_i), for $i = l+1$ to (say) s, $0 \notin \overline{\text{dom } (F_i)}$, and for the remaining functions $i = s+1$ to k, if there are still some, $0 \in \overline{\text{dom } (F_i)} \setminus \text{dom } (F_i)$. Now we examinate the sets dom (F_i) in this order proceeding as follows:

$0 \in$ dom (F_i). If necessarily, we substitute $\mathcal{O}$ at each step with a smaller open neighborhood of 0, such that $\mathcal{O} \subset$ dom (F_i).

$0 \notin \overline{\text{dom } (F_i)}$. If necessarily, we substitute $\mathcal{O}$ at each step with a smaller neighborhood of 0, such that $\mathcal{O} \cap \overline{\text{dom } (F_i)} = \emptyset$ and we delete F_i from the list of relevant terms.

$0 \in \overline{\text{dom } (F_i)} \setminus \text{dom } (F_i)$. If this case occurs for at least one F_i we follow a more complicated procedure:

Suppose to have chosen $\mathcal{O}$ small enough in order to satisfy all occurrences of the other two situations. We define now a new family $\overline{\mathcal{F}}$ of tame analytic functions by substituting every function $f \in \mathcal{F}$ by the corresponding extended $\overline{f}$ according to the definition. Let $\overline{M}$ be the $\overline{\mathcal{F}}$-machine which has the same program as M but the instructions like $y := g(\vec{x})$ are substituted with the corresponding $y := \overline{g}(\vec{x})$. M was supposed to be deterministic and correct and to decide the belongness to Σ for all inputs. This means that to all input $\vec{x} \in \mathbb{R}^{m+2}$ corresponds a computation of M which takes a time less than a constant T, and $\overline{M}$ will make exactly the same computation for the same input.
So, if we get a contradiction from the fact that $\overline{M}$ decides Σ in constant time T, it will be enough to contradict also the assumption that the machine M decides Σ in constant time. If we know that for all analytic terms F_i in the

third situation $0 \in \text{dom }(\overline{F_i})$ which is an open set, we can choose $\mathcal{O}$ small enough so that for each F_i holds $\mathcal{O} \subset \text{dom }(\overline{F_i})$ and we can apply the Lemma directly.
In order to prove that if for an old term $0 \in \overline{\text{dom }(F_i)} \setminus \text{dom }(F_i)$ than for the corresponding new term $0 \in \text{dom }(\overline{F_i})$ we recall some elementary topological facts: if X and Y are topological spaces, $A, B \subset Y$ and $h : X \to Y$ continuous than $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, $\overline{A \times B} \subset \overline{A} \times \overline{B}$ and $\overline{h^{-1}(A)} \subset h^{-1}(\overline{A})$. In fact we will prove that $\overline{\text{dom }(F_i)} \subset \text{dom }(\overline{F_i})$. The proof will succeed by induction for all the analytic terms following the construction steps.
For the beginning we recall that for the functions $g \in \mathcal{F}$, $\overline{\text{dom }(g)} = \overline{U} \subset V = \text{dom }(\overline{g})$ as stated in the definition. The construction of new terms can be seen as the succesive application of two basic operations:
Composition Given n already built terms $f_i : \text{dom }(f_i) \subset \mathbb{R}^{k_i} \to \mathbb{R}$ with $k_1 + k_2 + \cdots + k_n = p$ and another already built term $f : \text{dom }(f) \subset \mathbb{R}^n \to \mathbb{R}$ we construct $g : \text{dom }(g) \subset \mathbb{R}^p \to \mathbb{R}$ which has the following definition and maximal possible domain:

$$g(x_1, \ldots, x_p) := f(f_1(x_1, \ldots, x_{k_1}), \ldots, f(x_{k_1+\cdots+k_{n-1}+1}, \ldots, x_p)) \text{ with}$$

$$\text{dom }(g) = \text{dom }(f_1) \times \cdots \times \text{dom }(f_n) \cap (f_1, \ldots, f_n)^{-1}(\text{dom } f).$$

The first term of the intersection is of course superfluos but helpful for the comprehension. The arithmetical operations $\{+, -, \cdot\}$ can occur in any step. Our induction hypothesis says that for all $i \in \{1, \ldots, n\}$ we have that $\overline{\text{dom }(f_i)} \subset \text{dom }(\overline{f_i})$ and $\overline{\text{dom }(f)} \subset \text{dom }(\overline{f})$. If we construct $\overline{g}$ from $\overline{f_i}$ and $\overline{f}$ in the same way as g from f_i and f, we observe that:

$$\overline{\text{dom }(g)} \subset \overline{\text{dom }(f_1)} \times \cdots \times \overline{\text{dom }(f_n)} \cap \overline{(f_1, \ldots, f_n)^{-1}(\text{dom } f)} \subset$$
$$\subset \text{dom }(\overline{f_1}) \times \cdots \times \text{dom }(\overline{f_n}) \cap (f_1, \ldots, f_n)^{-1}(\overline{\text{dom } f}) \subset$$
$$\subset \text{dom }(\overline{f_1}) \times \cdots \times \text{dom }(\overline{f_n}) \cap (\overline{f_1}, \ldots, \overline{f_n})^{-1}(\text{dom } \overline{f}) = \text{dom }(\overline{g})$$

The inclusion $\overline{\text{dom }(g)} \subset \text{dom }(\overline{g})$ is strict because one set is open and the other closed.
Restriction Given already built up $f : \text{dom }(f) \subset \mathbb{R}^n \to \mathbb{R}$ we define for a $k \leq n$ a new term $g : \text{dom }(g) \subset \mathbb{R}^k \to \mathbb{R}$ defined as $g(x_1, \ldots, x_k) := f(u_1, \ldots, u_n)$ where all $u_i \in \mathbb{R} \sqcup \{x_1, \ldots, x_k\}$ with:

$$\text{dom }(g) = (u_1, \ldots, u_n)^{-1}(\text{dom } f).$$

One sees immediately that the Restriction is just a particular case of the Composition, so we have nothing to prove. Before starting with the last case we remark that the crucial fact used here was the topological monotonicity of the dom $(\cdot)$ operator. It takes place because the occurrences of the set

variables in the set-theoretical formula of a domain are **positive**, i.e. without negations.
Case 3: It remained to discuss the following case: the essentially not semialgebraic function $f : U \subset \mathbb{R}^m \to \mathbb{R}$ is partial and there may be other partial functions in $\mathcal{F}$. We observe that the function

$$l : \mathbb{R} \to \mathbb{R} \text{ given by } l\,(x) = \frac{2x}{1+x^2}$$

is real-analytic, semialgebraic, total and that its image $l\,(\mathbb{R}) = [-1,1]$ is compact. For well chosen $r, q_1, \dots, q_m \in \mathbb{Q}$ with $m \neq 0$ is the function $F : \mathbb{R}^m \to \mathbb{R}$ given by

$$F(x_1, \dots, x_m) = f(q_1 + rl\,(x_1), \dots, q_m + rl\,(x_m))$$

real-analytic, total, and not semialgebraic. We construct Φ and $\Sigma := \Sigma_F$ as previously defined starting from this F.
As already proven, there is no deterministic $\mathcal{F}$-machine able to decide the belongness to this set Σ in a constant time. (The fact whether F is a member of $\mathcal{F}$ or not was not relevant for the proof.) For giving a non-deterministic $\mathcal{F}$-machine which recognizes Σ in a constant time, we fix first a common denominator $N \in \mathbb{N}$ for the $m+1$ rational numbers $r, q_1, \dots, q_m$ and write them as:

$$r = \frac{R}{N}\,,\ q_1 = \frac{Q_1}{N}\,, \dots\dots,\ q_m = \frac{Q_m}{N}.$$

Σ is a halting set for the following trivial non-deterministic $\mathcal{F}$-machine:

input $x_1, \dots, x_{m+2}$ **; guess** $b_1, \dots, b_m$ **;**

if $x_{m+1} \leq 0$ **then neverstop**

else for $i = 1$ **to** m **do**

$c_i := (1 + x_i^2)$ **;**

if $Nx_{m+1}b_ic_i - Q_ic_i - 2Rx_i \neq 0$ **then neverstop**

end do ;

if $x_{m+2} - x_{m+1}f(b_1, \dots, b_m) \neq 0$ **then neverstop**

else stop.

In our notation is the multiplication with an integer K a $|\,K\,|$ times iterated addition (substraction). This way we don't introduce real parameters for the rationals $r, q_1, \dots, q_n$ which would not be in $\mathcal{F}$. □□

4 Examples and discussion

In the following we will call a real machine **analytic** if it contains instructions permitting the evaluation of finitely many real-analytic functions with open connected domains, but not necessarily total or tame.
We compare our result with the result of Klaus Meer. In order to prove that $P_{\sin} \neq NP_{\sin}$ he showed that the set $[0, 2\pi] \cap 2\pi\mathbb{Q}$ can be recognized in a constant time by a non-deterministic sin-machine but cannot be decided (even without a time-bound) by any deterministic sin-machine. The last part seem to be a particular case of the following statement, which can be proved using an argumentation which is similar to his original one:
Remark 1: *A set $\Delta \subset \mathbb{R}$ which is dense in a real interval and whose complement is also dense in the same interval cannot be decided by any analytic machine, even if we do not pretend a limited time of computation.*
Proof: To see this we observe that a such machine can compute along the several branches of its computation tree just countably many different analytic functions in the input variable x and that every analytic function which is not identically 0 could have only countably many zeros. We consider an input x which does not belong to the countably many possible zeros of all these functions and we focuse on its finite branch from the input to the decision. All the tests "$f(x) \geq 0$?" have been strictly satisfied or refuted, so a whole neighborhood U of x will be computed along the same path as x, and this is a contradiction. □
Using this result we become an easy answer for the $P \neq NP$ problem for all family of analytic functions, not necessarily total or tame (for example, the division is allowed), but which must contain the total sin, the total cos, or other functions which can be used to define existentially $\mathbb{Z}$ (so also $\mathbb{Q}$) or somehow any other dense countable set. This is not the case of functions like exp. The definable sets in the theory of $(\mathbb{R}, +, -, \cdot, \exp)$ have always finitely many connected components; one says that the theory is **o-minimal** [DM]. With this occasion we cite also the paper [DD]. The autors study the theory of the reals expanded with the following huge language: for all $m \in \mathbb{N}$ one considers the real-analytic functions whose natural domains contains the compact box $[-1, 1]^m$ and for every such function adds a symbol interpreting the restriction of the respective function to the box. The division operation belongs also to this language. As proven in [DD], the new theory admits elimination of quantifiers, so we cannot give counterexamples in finite dimension for the $P \neq NP$ problem in the corresponding model of computation.
Using our Proposition we become the answer $P \neq NP$ for families like $\{\exp\}$, $\{\arctan\}$ or even $\{\sin|_I\}$, where I is any small interval. But as we will see in the next example, we cannot remove the tameness. We will consider the

wild family $\mathcal{F}_1 := \{\exp, \log\}$ and the set:

$$\Sigma_{\exp} := \{(x,y,z)\,|\, y>0 \text{ and } z = y\exp\frac{x}{y}\ \}.$$

Remark 2: $\Sigma_{\exp}$ *is decided by a deterministic* $\mathcal{F}_1$*-machine in constant time.*
Proof: This remark is trivial, exactly like the following program called $\mathfrak{A}\,(x,y,z)$. It computes the characteristic function χ of the set $\Sigma_{\exp}$ and it will be used further as a subroutine.

input (x,y,z) **;**

if $y \le 0$ **then** $\chi := 0$ **; stop else**

if $x < 0$ **then** $a := y\exp(-\exp(\log(-x) - \log y))$ **else**

if $x = 0$ **then** $a := y$ **else**

if $x > 0$ **then** $a := y\exp(\exp(\log x - \log y))$ **else**

if $z = a$ **then** $\chi := 1$ **else** $\chi := 0$ **; stop.** □

Our next example will show that even if the Proposition works, the set Σ may be decidable, but of course not in constant time. For this purpose we will consider again $\Sigma_{\exp}$ together with the tame family $\mathcal{F}_2 := \{\exp, \log|_{(1,+\infty)}\}$.
Remark 3: $\Sigma_{\exp}$ *is decided by an* $\mathcal{F}_2$*-machine, but not in a constant time.*
Proof: For the first part we observe that the program which solved Remark 2 is in the new situation no more correct. It becomes correct if it runs together with a small preambel:

input (x,y,z) **;**

if $x \ge 0$ **then** $w := x$ **else** $w := -x$ **;**

while $w \in (0,1]$ **or** $y \in (0,1]$ **do**

$$(x,y,z,w) := (2x, 2y, 2z, 2w)$$

end do ; call $\mathfrak{A}\,(x,y,z)$ **; stop.**

This program uses the homogenity of the function Φ from the definition of Σ_{exp}. The program $\mathfrak{A}(x,y,z)$ can solve the problem just for inputs with $x \in (-\infty,-1)\cup(1,+\infty)$ and $y \in (1,+\infty)$. If we call $\Sigma' := \Sigma_{\exp}\cap\{(-\infty,-1)\cup(1,+\infty)\}\times(1,+\infty)\times\mathbb{R}$, then

$$\Sigma_{\exp} = \bigcup_{n\ge 0} \frac{1}{2^n}\,\Sigma'.$$

It is clear that the time cannot be uniformly bounded. As proved in the Proposition, there is no $\mathcal{F}_2$-machine which decides Σ_{exp} in constant time. $\square$

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