A two-valued recurrent double sequence that is not automatic

Mihai Prunescu *

Abstract

A recurrent 2-dimensional (double) sequence $t(m, n)$ is given by fixing particular sequences $t(m, 0)$, $t(0, n)$ as initial conditions and a rule of recurrence $t(m, n) = t(a(m, n-1), t(m-1, n-1), t(m-1, n))$ for $m, n \geq 1$. We display such a sequence with values in the set $\{0, 1\}$ and show that all rows are periodic and that the minimal period of the n-th row has length 2^n. We conclude that this 2-dimensional sequence is not k, l-automatic for any $k, l \geq 2$. Other two-valued recurrent double sequences are shortly discussed. Minimal examples of non-trivial 2-automatic sequences are displayed. One case remains open.

Key Words: recurrent n-dimensional sequence, n-dimensional automatic sequence, rectangular substitution, n-dimensional uniform morphic sequence, Turing-complete models of computation.

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1 Introduction

The recurrent n-dimensional sequences over finite sets (alphabets) build a Turing-complete model of computation, see [7] for a proof. In comparison with other models of computation, the recurrent n-dimensional sequences are easier to be visualized - or at least small pieces of them in dimension 2 are so. Their interest is of purely theoretic nature: instead of complicated multi-sorted objects, like Turing or Register Machines, one only studies a finite function which has to be iterated. The neighborhood-wise recurrence makes them able to model some natural phenomena: crystals and quasi-crystals (see [5, 6]), mechanical behaviors (see [16]), some other processes described by solutions of differential equations (see [2]), and much more. Stephen Wolfram started in [17] the project to classify the behaviors in similar discrete dynamic processes. He also commented their possibility to model arbitrary natural, fractal or chaotic phenomena.

However, the recurrent double sequences over finite sets have the same disadvantages like most of the alternative models of computation, but in a very severe form: it is very hard to program or design them, and their output is difficult to be read (decoded).

Some of the recurrent n-dimensional sequences have a special algebraic structure (see [8, 9, 10, 11, 12]) and prove to be automatic (see [13]). A complete characterization of the automatic recurrent n-dimensional sequences is still unknown, but needed. Indeed, if we recall that this model of computation is Turing-complete, a natural question to pose is how to measure the complexity of different computations inside the model. The author believes that the differentiation in automatic and non-automatic recurrent double sequences is a first criterion separating less complex from more complex examples.

In [13] the author defined recurrent n-dimensional sequences for arbitrary systems of predecessors and arbitrary initial conditions. In the following lines the system of predecessors will always be $P = \{(1, 0), (1, 1), (0, 1)\}$ in dimension 2, and the initial conditions will be constant 1-dimensional

*Brain Products, Freiburg, Germany and Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit 5, P. O. Box 1-764, RO-014700 Bucharest, Romania. mihai.prunescu@math.uni-freiburg.de.

sequences. Recurrent double sequences with ultimately periodic non-constant initial conditions occur only in some examples.

Definition 1.1 Let A be a finite alphabet, $1 \in A$ be a fixed element and $f : A^3 \to A$ be a function, called rule of recurrence. This rule defines the recurrent 2-dimensional sequence $t : \mathbb{N}^2 \to A$ if $t(\mathbb{N}, 0) = t(0, \mathbb{N}) = 1$ and for all $i, j \geq 1$, $t(i, j) = f(t(i-1, j), t(i-1, j-1), t(i, j-1))$. We refer to this sequence as $(A, f, 1, 1)$.

Definition 1.2 Let $\vec{k} = (k_1, k_2, \dots, k_n)$ be a tuple of natural numbers $k_i \geq 2$. An n-dimensional deterministic finite $\vec{k}$-automaton with output (n-DFA) M consists of a finite nonempty set of states Q, an input alphabet $\Sigma = \{0, 1, \dots, k_1 - 1\} \times \{0, 1, \dots, k_2 - 1\} \times \cdots \times \{0, 1, \dots, k_n - 1\}$, a transition function $\delta : Q \times \Sigma \to Q$, an initial state q_0, an output alphabet A and an output mapping $\tau : Q \to A$. Any tuple $\vec{u} \in \mathbb{N}^n$ is written in the form $\vec{u} = \sum\limits_{0 \leq i \leq v} \vec{k}^i \vec{\sigma}_i$ with $\vec{\sigma}_0, \dots, \vec{\sigma}_v \in \Sigma$. We say that $\vec{\sigma}_v \vec{\sigma}_{v-1} \cdots \vec{\sigma}_0$ is the $\vec{k}$-*code* of $\vec{u}$. An n-dimensional sequence $a : \mathbb{N}^n \to A$ is produced by the $\vec{k}$-automaton M if for all $\vec{u} \in \mathbb{N}^n$, M stops in a state $q \in Q$ with output $\tau(q) = a(\vec{u})$ after reading the $\vec{k}$-code of $\vec{u}$. If there is a n-dimensional $\vec{k}$-automaton producing the n-dimensional sequence a we say that a is a $\vec{k}$-*automatic sequence*. See [1].

If $k_1 = k_2 = \cdots = k_n = k$, one speaks about k-automatic sequences. General properties of k-the automatic sequences were the object of classical research, like those of Cobham [3]. He pointed out that any k-automatic 1-dimensional sequence is a projection of a uniform morphic sequence, where morphic sequences are those completely produced by a family of uniform context free substitutions ($\sigma : A \to A^*$ such that for all $x \in A$, $|\sigma(x)| = k$). Another important topic in the history of 1-dimensional k-automatic sequences for $k = p$, a prime, was the work of Cristol, Kamae, Mendes-France and Rauzy, [4]. There has been proved that p-automatic sequences are algebraic formal series over the field of rational functions over some finite field of characteristic p. This has been later generalized in the n-dimensional case and for functions with n variables by Salon, see [14, 15].

In the monograph [1] is proven that the 2-dimensional sequence $t(m, n) = \binom{m}{n} \mod 6$, usualy known as Pascal Triangle modulo 6, is not k-automatic for any k. This sequence satisfies $t(m, n) = t(m, n-1) + t(m-1, n-1)$ in the group $\mathbb{Z}/6\mathbb{Z}$, so is a recurrent 2-dimensional sequence. The rule of recurrence uses a set of two predecessors $P = \{(1, 1), (0, 1)\}$. The sequence takes 6 values and is not automatic, because the sequence is the result of overlapping two automatic 2-dimensional sequences with incompatible substitution factors 2 and 3. In [13] the author gave another example of sequence that cannot be k-automatic for any k. In this case the recurrence uses a set of three predecessors and the sequence takes 4 values. It is not k-automatic basically because it interprets an arithmetic progression along its diagonal.

The present note displays an example of two-valued 2-dimensional sequence with constant initial conditions that cannot be k, l-automatic for any $k, l \geq 2$. From now on, 2-dimensional sequences will also be called double sequences.

We will concentrate on the case $A = \mathbb{F}_2$, the field with two elements. All functions $g : \mathbb{F}_2^n \to \mathbb{F}_2$ can be expressed by polynomials with coefficients in $\mathbb{F}_2$. For $x \in \mathbb{F}_2$ we write $\bar{x} = \neg x = x + 1$. $\mathbb{F}_2$ is also seen as a boolean algebra, so the boolean operation symbols will be freely used. The set $\{0, 1\}^*$ is the set of arbitrarily long finite binary words. For a word $w \in \{0, 1\}^*$, $w = \epsilon_1 \dots \epsilon_n$, let $\bar{w} = \bar{\epsilon_1} \dots \bar{\epsilon_n}$, where all $\epsilon_i \in \{0, 1\}$. We simply write $w_1 w_2$ for the concatenation of two words, and w^n for the repeated concatenation of w with itself. All matrices will be indexed starting with $i = 0$ and $j = 0$. Most of the time we use the term *double sequence* for 2-dimensional sequence. If U is some matrix, $U(i, j)$ denotes the corresponding element of U. The same conventions work for infinite matrices, i. e. double sequences.

Definition 1.3 The sequence $t : \mathbb{N}^2 \to \mathbb{F}_2$ is the recurrent double sequence satisfying $t(\mathbb{N}, 0) =$

$t(0, \mathbb{N}) = 1$ and the rule of recurrence $f : \mathbb{F}_2^3 \to \mathbb{F}_2$ given by $f(x, y, z) = 1 + x + z + yz$. Shortly, $t = (\mathbb{F}_2, 1 + x + z + yz, 1, 1)$.

Figure 1: $(\mathbb{F}_2, x + y + yz, (01), 0)$, 128×10.

The idea behind this example can be easily understood looking at Figure 1, where the first rows and columns of a related example are displayed. The recurrence $x + y + yz$ is applied here to initial conditions given by the periodic sequences (01) and (0) instead of (1) and (1). The result is a recurrent double sequence able to count binarilly. Indeed, if one recalls the initial conditions and applies a Lemma similar to Lemma 2.1, one proves that for $y \geq 0$, the row y is periodic and the period is $(0^{2^{y+1}} 1^{2^{y+1}})$. Consequently, for $x \geq 0$, the column x read upwards consists of an infinite number of zeros, preceding the binary representation of the number x.

Here we intend to describe the simplest recurrent double sequence that is not automatic. This means not only to use the minimal alphabet $\{0, 1\}$, but also to encode minimal information in the initial conditions. For this reason, the example with two constant margins (1) and (1) has been prefered, although it has a slightly more complicated behavior.

In the Section 2 is described the structure of the double sequence t. In the Section 3 we explore connections between general $\vec{k}$-automaticity and rectangular context-free substitution. Finally the structure of t is used to prove that t is not k, l-automatic for any $k, l \geq 2$. In the Section 4 all two-valued recurrent double sequences based on the predecessors $\{(0, 1), (1, 1), (1, 0)\}$ with constant initial conditions are shortly analysed. The conclusion is that the sequence t together with other three related sequences represent, up to another class of four sequences, the only examples of non-automatic double sequences in this class. The automatic character of the other four double sequences remains an open question. All other 248 sequences in this set are automatic. In a such poor family of recurrent double sequences, the occurrence of a small class of non-automatic examples is a remarkably lucky event.

2 Structure

Lemma 2.1 *The function $f : \mathbb{F}_2^3 \to \mathbb{F}_2$, $f(x, y, z) = 1 + x + z + yz$, satisfies:*

$$f(x, y, z) = x + 1 + \neg(z \to y) = \begin{cases} x & if \;\; y = 0 \;\wedge\; z = 1, \\ \bar{x} & else. \end{cases}$$

Double sequences are represented as matrices, such that the second index increases downwards. Using this convention, every application of the recurrence f in the double sequence t has the form:

$$\begin{pmatrix} y & z \\ x & f(x, y, z) \end{pmatrix}$$

According to Lemma 2.1 there are in fact only the rules: Repetition R and Alternation A.

$$R = \begin{pmatrix} 0 & 1 \\ x & x \end{pmatrix} \quad ; \quad A = \begin{pmatrix} y & z \\ x & \bar{x} \end{pmatrix} \mid (y, z) \neq (0, 1).$$

Definition 2.2 *Let $a, b : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ be two functions, given by $a(k) = (2^{2k-1} - 2)/3$ and $b(k) = (2^{2k} - 1)/3$.*

Theorem 2.3 *Let $L_n = (t(i,n) \,|\, i \geq 0)$ be the n-th row of the recurrent double sequence $t = (\mathbb{F}_2, 1+x+z+yz, 1, 1)$. For $n \geq 0$, all the rows L_n are periodic infinite words of period $p_n \in \{0,1\}^*$, where p_n is a word of length 2^n. For $n \geq 1$, the period p_n has the form $p_n = q_n\bar{q}_n$ for some $q_n \in \{0,1\}^*$. Consequently, p_n is always the minimal period for the row L_n. More exactly:*

$$k \geq 0 \;\to\; q_{2k+1} = (101)^{b(k)}1,$$

$$k \geq 1 \;\to\; q_{2k} = (100)^{a(k)}10.$$

Proof: By direct computation one sees that $p_0 = 1$, $p_1 = 10 = q_1\bar{q}_1$, $p_2 = 1001 = q_2\bar{q}_2$, and $p_3 = 10110100 = q_3\bar{q}_3$ with $q_3 = (101)^{b(1)}1$. The induction steps are as follows:

$2k+1 \rightsquigarrow 2k+2$: The first two periods of L_{2k+1}, followed by a 1, look like:

$$(101)^{b(k)}1(010)^{b(k)}0(101)^{b(k)}1(010)^{b(k)}01\ldots$$

L_{2k+2} starts with a 1. Apply the rules A and R and regroup. You get:

$$(100)^{b(k)}1(001)^{b(k)}0(011)^{b(k)}0(110)^{b(k)}11\ldots$$

But $2b(k) = a(k+1)$, so the last word is exactly:

$$(100)^{a(k+1)}10(011)^{a(k+1)}011\ldots$$

This means that $q_{2k+2} = (100)^{a(k+1)}10$, its occurrence is followed by $\bar{q}_{2k+2}$, and this will repeat periodically, because the next letter is a 1. Consequently, the row L_{2k+2} is periodic of period $p_{2k+2} = q_{2k+2}\bar{q}_{2k+2}$, whose length is $2|p_{2k+1}| = 2^{2k+2}$.

$2k \rightsquigarrow 2k+1$: The first two periods of L_{2k}, followed by a 1, look like:

$$(100)^{a(k)}10(011)^{a(k)}01(100)^{a(k)}10(011)^{a(k)}011\ldots$$

L_{2k+1} starts with a 1. Apply the rules A and R and regroup. You get:

$$(101)^{a(k)}(101)(101)^{a(k)}1(010)^{a(k)}(010)(010)^{a(k)}01\ldots$$

But $2a(k) + 1 = b(k)$, so:

$$(101)^{b(k)}1(010)^{b(k)}01\ldots$$

This means that $q_{2k+1} = (100)^{b(k)}1$, its occurrence is followed by $\bar{q}_{2k+1}$, and this will repeat periodically, because the next letter is a 1. Consequently, the row L_{2k+1} is periodic of period $p_{2k+1} = q_{2k+1}\bar{q}_{2k+1}$, whose length is $2|p_{2k}| = 2^{2k+1}$. □

See Figure 2.

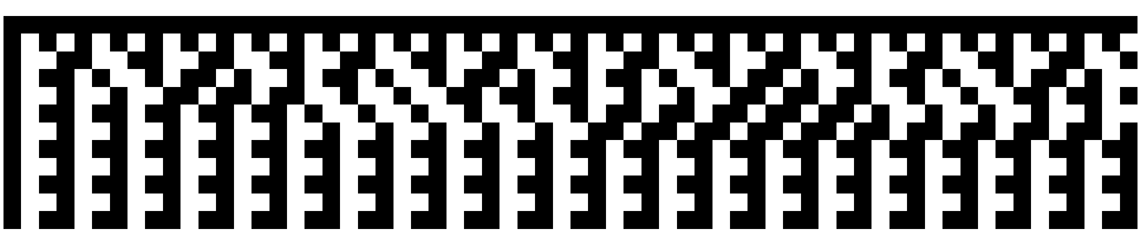

Figure 2: $(\mathbb{F}_2, 1+x+z+yz, 1, 1)$, 64×12.

3 Rectangular substitutions

Definition 3.1 Let B be a finite set, and $\lambda_1, \lambda_2 \geq 2$ be natural numbers. Let $1 \in B$ be a fixed element. Let $M_{\lambda_1,\lambda_2}(B)$ be the set of λ_1, λ_2-matrices with elements in B. We call λ_1, λ_2-morphism over B a function $\sigma : B \to M_{\lambda_1,\lambda_2}(B)$ such that the first element of the first row of the matrix $\sigma(1)$ is equal 1: $\sigma(1)(0,0) = 1$.

The morphism $\sigma : B \to M_{\lambda_1,\lambda_2}(B)$ can be extended to a morphism $\sigma : M_{\gamma_1,\gamma_2}(B) \to M_{\lambda_1\gamma_1,\lambda_2\gamma_2}(B)$ acting elementwise, in the sense that the image of an element $A(i,j)$ of the matrix $A \in M_{\gamma_1,\gamma_2}(B)$ is the corresponding block $\sigma(A(i,j))$ in the matrix $\sigma(A) \in M_{\lambda_1\gamma_1,\lambda_2\gamma_2}(B)$. Using this extended meaning of σ, we observe that $\sigma(1)(0,0) = 1$ implies:

$$\forall i \geq 1 \;\; \sigma^i(1) = \sigma^{i+1}(1)(\{0,\ldots,\lambda_1^i - 1\} \times \{0,\ldots,\lambda_2^i - 1\}).$$

Definition 3.2 A double sequence $u : \mathbb{N}^2 \to B$ is called a fix-point for the morphism σ if $u = \lim \sigma^i(1)$, in the sense that $u(m,n) = \sigma^i(1)(m,n)$ for an i that is big enough such that $\max(m,n) < \min(\lambda_1^i, \lambda_2^i)$. We say that the morphism σ produces the double sequence u.

The property $\sigma(1)(0,0) = 1$ mentioned above assures that the value $u(m,n)$ considered in Definition 3.2 does not depend on the choice of i.

Definition 3.3 Let A be a finite set, and let $a, b, \lambda_1, \lambda_2$ be natural numbers such that $a, b \geq 1$ and $\lambda_1, \lambda_2 \geq 2$. Let $B = M_{a,b}(A)$ be the set of a, b-matrices over A, and $1 \in B$ a fixed element. Let $t : \mathbb{N}^2 \to A$ be a double sequence. One says that t is produced by a rectangular substitution of type $(a,b) \to (\lambda_1 a, \lambda_2 b)$ if there is a λ_1, λ_2-morphism $\sigma : B \to M_{\lambda_1,\lambda_2}(B)$ producing a double sequence $u : \mathbb{N}^2 \to B$ (again, in the sense that $u = \lim \sigma^i(1)$) such that:

$$\forall m,n \in \mathbb{N} \;\; t(m,n) = u(m/\lambda_1, n/\lambda_2)(m \mod \lambda_1, \;\; n \mod \lambda_2).$$

Definition 3.4 Let $t : \mathbb{N}^2 \to A$ be a double sequence, and let m, n, a, b be natural numbers with $a, b \geq 1$. The rectangle starting in (m,n) with edges (a,b) is the set:

$$R(m,n,a,b) = t(\{m,\ldots,m+a-1\} \times \{n,\ldots,n+b-1\}).$$

Lemma 3.5 *Let $t : \mathbb{N}^2 \to A$ be a double sequence produced by a substitution of type $(a,b) \to (\lambda_1 a, \lambda_2 b)$. Then for all natural numbers u, v, w, s; if $R(ua, vb, a, b) = R(wa, sb, a, b)$, then:*

$$R(u\lambda_1 a, v\lambda_2 b, \lambda_1 a, \lambda_2 b) = R(w\lambda_1 a, s\lambda_2 b, \lambda_1 a, \lambda_2 b).$$

Proof: This follows from Definition 3.3 and Definition 3.4. □

Lemma 3.6 *The recurrent double sequence $t : \mathbb{N}^2 \to \mathbb{F}_2$, $t = (\mathbb{F}_2, 1 + x + z + yz, 1, 1)$, cannot be produced by any rectangular substitution.*

Proof: Suppose that t can be produced by a rectangular substitution of type $(a,b) \to (\lambda_1 a, \lambda_2 b)$. Without restricting the generality, we can assume that $3|a$ and $2|b$. This is true because if there exists a substitution of type $(a,b) \to (\lambda_1 a, \lambda_2 b)$ producing t then for all $m, n \geq 1$ there exist rectangular substitutions of type $(ma, nb) \to (\lambda_1 ma, \lambda_2 nb)$ producing the same sequence t. We are looking to a contradiction to Lemma 3.5.

First we choose $u = 0$ and $v \in \mathbb{N}$ big enough such that $a < 2^{vb-1} - 2$ and $\lambda_1 a < 2^{v\lambda_2 b - 1} - 2$. Recalling Theorem 2.3, this means that the rectangles $R(0, vb, a, b)$ and $R(0, v\lambda_2 b, \lambda_1 a, \lambda_2 b)$ consist only of blocks of the form:

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

On the other hand, for all $(m, n) \in \{0, \dots, (v+1)b - 1\} \times \mathbb{N}$ and for all $\theta \in \mathbb{N}$, one has that $t(m, n) = t(m + \theta 2^{(v+1)b-1}, n)$, because the length of the period in the row $L_{(v+1)b-1}$ is a multiple of the period lengths in all precedent rows. Taking $\theta = ka$ with $k \in \mathbb{N}$, one gets that:

$$R(0, vb, a, b) = R(2^{(v+1)b-1}ka, vb, a, b).$$

Now we are looking for a value of k such that

$$R_k := R(2^{(v+1)b-1}k\lambda_1 a, v\lambda_2 b, \lambda_1 a, \lambda_2 b) \neq R(0, v\lambda_2 b, \lambda_1 a, \lambda_2 b).$$

One can achieve this, for example, if the corresponding last rows in those rectangles are different. It would be sufficient if the last row of R_k intersects the first occurrence of the word $\bar{q}_{(v+1)\lambda_2 b-1}$ in the corresponding row of t. For this, it would be enough if:

$$\frac{1}{2}\, 2^{(v+1)\lambda_2 b-1} < 2^{(v+1)b-1}k\lambda_1 a < 2^{(v+1)\lambda_2 b-1},$$

which is equivalent to:

$$2^{M-1} < k\lambda_1 a < 2^M,$$

where $M = (v+1)(\lambda_2 - 1)b$. The last inequality can be always achieved if $2^{M-1} > \lambda_1 a$, which is equivalent to:

$$v + 1 > \frac{1 + \log(\lambda_1 a)}{(\lambda_2 - 1)b}.$$

One can satisfy this last condition, together with the conditions $a < 2^{vb-1} - 2$ and $\lambda_1 a < 2^{v\lambda_2 b-1} - 2$ stated above, only by choosing a value of v which is large enough. □

Lemma 3.7 *Let A be a finite set. If a double sequence $t : \mathbb{N}^2 \to A$ is k, l-automatic for some $k, l \geq 2$, then t can be constructed using a rectangular substitution of type $(k^u, l^u) \to (k^v, l^v)$ for some $0 \leq u < v$.*

Proof: The proof is an imitation of the proof done in [1] for the Lemma 6.9.1 at page 193. This Lemma states the corresponding result for 1-dimensional sequence.

First we recall Cobham's Theorem 14.2.3, page 410 of [1] (see also [3]): if t is k, l-automatic, then t is the image of a k, l-morphism under a coding. So there exists a finite set B, a fixed constant $1 \in B$, and an application $\varphi : B \to M_{k,l}(B)$, with $\varphi(1)(0, 0) = 1$, that produces a sequence $c : \mathbb{N}^2 \to B$, $c = \lim \varphi^i(1)$. Further, there exists a coding function $\tau : B \to A$ such that for all $m, n \in \mathbb{N}$, $t(m, n) = \tau(c(m, n))$.

For all $s \in \mathbb{N}$ and all $b, b' \in B$ we define the relation:

$$b \equiv_s b' \;\longleftrightarrow\; \tau(\varphi^s(b)) = \tau(\varphi^s(b')).$$

All relations $\equiv_s$ are equivalence relations. But there are only finitely many equivalence relations over the finite set B, so there are natural numbers $0 \leq u < v$ such that $\equiv_u$ and $\equiv_v$ coincide.

Now, if $R(xk^u, yl^u, k^u, l^u) = R(ik^u, jl^u, k^u, l^u)$, then $\tau(\varphi^u(c(x, y))) = \tau(\varphi^u(c(i, j)))$. This means that $c(x, y) \equiv_u c(i, j)$, so also $c(x, y) \equiv_v c(i, j)$. This is again equivalent with $\tau(\varphi^v(c(x, y))) = \tau(\varphi^v(c(i, j)))$, which means $R(xk^v, yl^v, k^v, l^v) = R(ik^v, jl^v, k^v, l^v)$.

So we have proved that:

$$R(xk^u, yl^u, k^u, l^u) = R(ik^u, jl^u, k^u, l^u) \to R(xk^v, yl^v, k^v, l^v) = R(ik^v, jl^v, k^v, l^v).$$

A substitution σ, of type $(k^u, l^u) \to (k^v, l^v)$ and generating t, can be now constructed in the following way. There are only finitely many $k^u \times l^u$- rectangles occurring in the sequence t in positions (xk^u, yl^u). We start with $R(0, 0, k^u, l^u)$ and write down $R(0, 0, k^v, k^v)$ in terms of $k^u \times l^u$-blocks. This corresponds to the image $\sigma(1) \in M_{k^{v-u}, l^{v-u}}(B)$. Now for every matrix in $M_{k^u, l^u}(A)$

occurring as $k^u \times l^u$-block $R(xk^u, yk^u, k^u, l^u)$ in $R(0,0,k^v,l^v)$ we write down the corresponding rectangle $R(xk^v, yl^v, k^v, l^v)$ in terms of $k^u \times l^u$-blocks. The implication proven above and the finity of $M_{k^u,l^u}(A)$ guarantee that one achieves in a finite number of steps a complete substitution of type $(k^u, l^u) \to (k^v, l^v)$ that generates the double sequence t. □

Theorem 3.8 *The recurrent double sequence $t : \mathbb{N}^2 \to \mathbb{F}_2$, $t = (\mathbb{F}_2, 1 + x + z + yz, 1, 1)$, is not k, l-automatic for any k and l with $k, l \geq 2$.*

Proof: This follows from Lemma 3.7 and Lemma 3.6. □

4 Other two-valued recurrent double sequences

Here we make a survey of the two-valued recurrent double sequences with constant initial conditions and predecessors $\{(1,0),(1,1),(1,0)\}$. The sequence t, together with other three related sequences, build the almost only one interesting (non-automatic) case of sucht double sequences. There is also another small case of four related sequences, whose automatic character is still open. The automaticity of the remaining 248 double sequences has been established by computer computations based on Theorem 3.13 in [13].

Definition 4.1 *Let $\mathcal{F}$ be the set of all functions $f : \mathbb{F}_2^3 \to \mathbb{F}_2$. The operator $\Phi : \mathcal{F} \to \mathcal{F}$ is defined by $\Phi(f)(x,y,z) = f(z,y,x)$.*

Definition 4.2 *Recall the convention established in Introduction and also used in [1]: if a double sequence is k,k-automatic, it will be simply called k-automatic. Similarly, 2-dimensional k,k-morphisms are called k-morphisms and matrix substitutions of type $(a,a) \to (\lambda a, \lambda a)$ are said to be of type $a \to \lambda a$.*

There are 256 many functions $f : \mathbb{F}_2^3 \to \mathbb{F}_2$. for every such function f, we consider the recurrent double sequence given by $(\mathbb{F}_2, f, 1, 1)$. According to their behavior, theese double sequences can be group in the following cases:

Constant Case: Exactly 128 of them are constantly equal 1. This is true if and only if the function f satisfies $f(1,1,1) = 1$. The recurrence continues in this way.

Ultimately Periodic Case: Further 103 sequences are periodic over a set $P = \{(i,j) \in \mathbb{N}^2 \mid i \geq M \wedge j \geq M\}$. Periodicity means here the repetition of one square matrix block. All the ultimately periodic sequences, together with the constant sequences, are trivially k-automatic for all $k \geq 2$.

Diagonal Case: 14 sequences are not ultimately periodic, but they are ultimately periodic in every of the two halfs of orthant defined by the diagonal $i = j$. Moreover, the diagonal subsequence $(t(i,i))_{i \in \mathbb{N}}$ is itself ultimately periodic. This is the case for the functions: $d_1 = 1 + y + z + xz$, $\Phi(d_1)$, $d_2 = 1 + y + yz + xz + xy + xyz$, $d_3 = 1 + y + xz + xyz$, $d_4 = 1 + y$, $d_5 = 1 + yz + xz + xy$, $d_6 = 1 + yz + xy + xyz$, $d_7 = 1 + x + z + xyz$, $d_8 = y + z + xz + xyz$, $\Phi(d_8)$, $d_9 = y + yz + xz + xyz$, $\Phi(d_9)$, $d_{10} = y + yz$, $\Phi(d_{10})$. We display here two examples: $d_1 = 1 + y + z + xz$ in Figure 3 and $d_2 = 1 + y + yz + xz + xy + xyz$ in Figure 4. All 14 double sequences in this case are k-automatic for all $k \geq 2$. This fact represents an important difference from the 1-dimensional case, because there only the ultimately periodic sequences are k-automatic for all $k \geq 2$. Now to come back to the displayed examles: (1) For all $\lambda \geq 2$, $(\mathbb{F}_2, d_1, 1, 1)$ is generated by a substitution of type $3 \to 3\lambda$ with 5 rules. Also, this sequence is generated by a substitution of type $6 \to 6\lambda$ with 3 rules. (2) For all $\lambda \geq 2$, $(\mathbb{F}_2, d_2, 1, 1)$ is generated by a substitution of type $16 \to 16\lambda$ with 10 rules. Also, for $\lambda = 2k + 1$ and $k \geq 1$, this sequence is generated by a substitution of type $8 \to 8\lambda$ with 11 rules.

Pascal's Triangle Case: This is the case of 3 functions: $t_1 = x + z$, $t_2 = x + y$ and $\Phi(t_2)$. These sequences are 2^i-automatic for all $i \geq 1$ and are not k-automatic for any k with odd divisors.

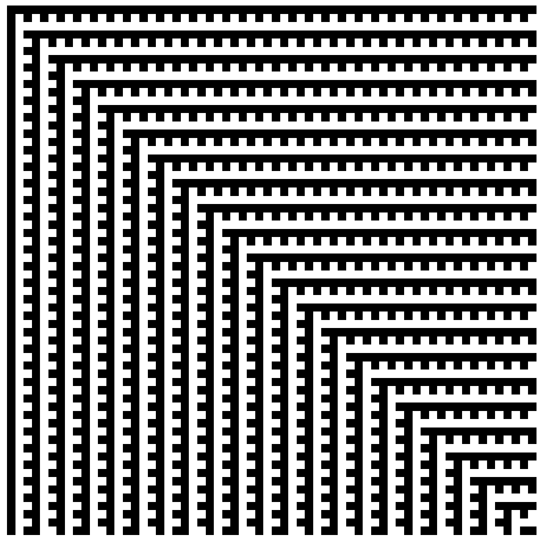

Figure 3: $(\mathbb{F}_2, d_1 = 1 + y + z + xz, 1, 1)$, 64×64.

$(\mathbb{F}_2, t_1, 1, 1)$ can be generated by a substitution of type $1 \to 2$ with 2 rules, so is a morphic sequence. $(\mathbb{F}_2, t_2, 1, 1)$ can be generated by a substitution of type $2 \to 4$ with 4 rules, but cannot be generated by any substitution of type $1 \to k$.

Logarithmic Case: This case contains 4 double sequences. They consist of periodic rows, such that the lengths of these periods form a geometric progression with ratio 2. They are $l_1 = 1 + x + z + yz$, $\Phi(l_1)$, $l_2 = 1 + x + y + yz$ and $\Phi(l_2)$. The sequence defined by l_1 is exactly the sequence studied in this article. The function $l_2(x, y, z) = 1 + x + \neg(y \to z)$ has a behavior similar with l_1. All 4 sequences in this case are not k, l-automatic for any $k, l \geq 2$.

We observe that all four recurrent double sequences, exactly like the recurrent double sequence shown in Figure 1, are *complete* in the sense that all eight possible tuples (x, y, z) occur in the sequence.

Exotic Case: All the 4 remaining sequences produce a similar pattern, see Figure 5. Although this pattern looks quite automatic, the author was not able to prove or to disprove their automaticity, and lets this question open. This is the case given by the functions: $e_1 = 1 + x + y + xyz$, $\Phi(e_1)$, and $e_2 = 1 + x + yz + xz + xy + xyz$, respectively $\Phi(e_2)$.

At this point we mention that the related recurrent double sequence $(\mathbb{F}_2, x + y + xyz, 1(0), (10))$ is 2-automatic, and can be generated by a substitution of type $4 \to 8$ with 9 rules. See Figure 6. As all four Exotic Case double sequences and Figure 6 are complete (all eight possible tuples (x, y, z) really occur in the respective double sequence), I see this fact as a hint that the Exotic Case could be 2-automatic as well. Because of the completeness, Figure 6 is a minimal example of p-automatic sequence defined by a recurrence that is *not* a homomorphism of finite abelian p-groups, for $p = 2$.

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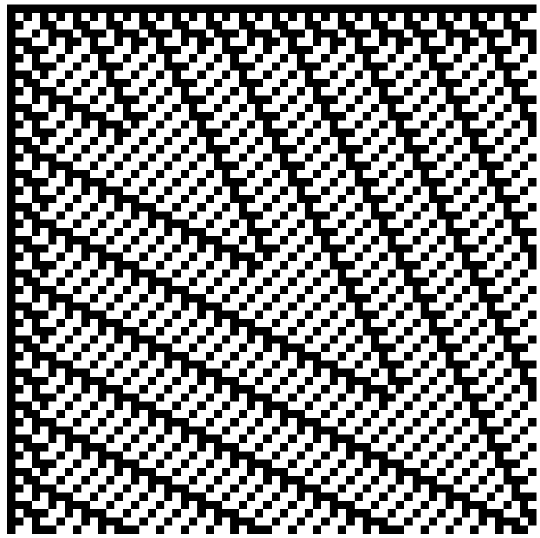

Figure 4: $(\mathbb{F}_2, d_2 = 1 + y + yz + xz + xy + xyz, 1, 1)$, 64×64.

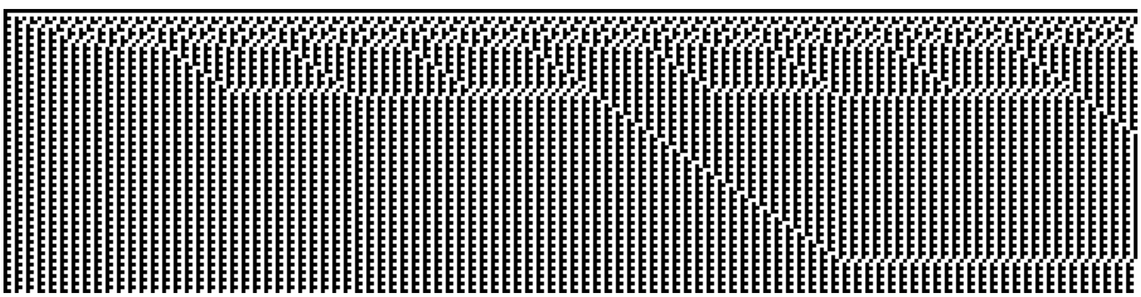

Figure 5: $(\mathbb{F}_2, e_1 = 1 + x + y + xyz, 1, 1)$, 600×80.

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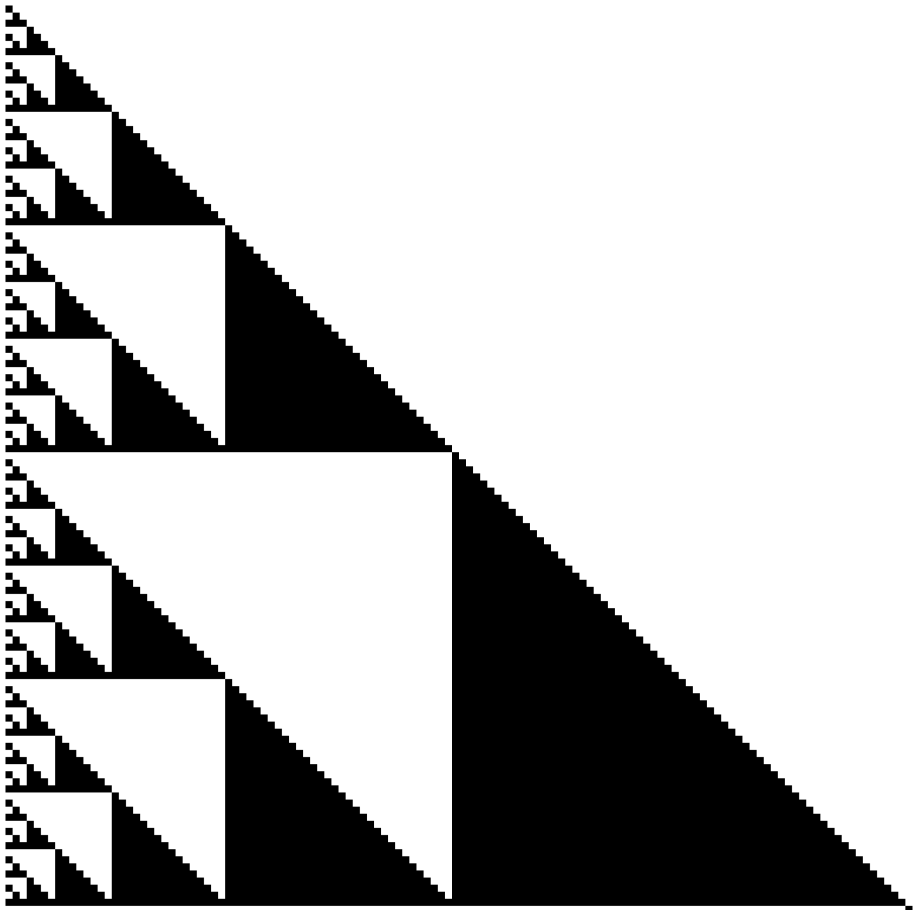

Figure 6: $(\mathbb{F}_2, x + y + xyz, 1(0), (10))$, 128×128.

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