$\mathbb{F}_p$-affine recurrent n-dimensional sequences over $\mathbb{F}_q$ are p-automatic

Mihai Prunescu *

Abstract

A recurrent 2-dimensional sequence $a(m,n)$ is given by fixing particular sequences $a(m,0)$, $a(0,n)$ as initial conditions and a rule of recurrence $a(m,n) = f(a(m,n-1), a(m-1,n-1), a(m-1,n))$ for $m,n \geq 1$. We generalize this concept to an arbitrary number of dimensions and of predecessors. We give a criterion for a general n-dimensional recurrent sequence to be alternatively produced by a n-dimensional substitution - i.e. to be an automatic sequence. We show also that if the initial conditions are p-automatic and the rule of recurrence is an $\mathbb{F}_p$-affine function, then the n-dimensional sequence is p-automatic. Consequently all such n-dimensional sequences can be also defined by n-dimensional substitution. Finally we show various positive examples, but also a 2-dimensional recurrent sequence which is not k-automatic for any k. As a byproduct we show that for polynomials $f \in \mathbb{Q}[X]$ with $\deg(f) \geq 2$ and $f(\mathbb{N}) \subset \mathbb{N}$, the characteristic sequence of the set $f(\mathbb{N})$ is not k-automatic for any k.

Key Words: recurrent n-dimensional sequence, automatic sequences, system of predecessors, transfinite induction, system of context-free substitutions, $\mathbb{F}_p((X,Y))$, formal series algebraic over $\mathbb{F}_p(X,Y)$, Christol's Theorem, finite fields, $\mathbb{F}_p$-vector space, $\mathbb{F}_p$-affine, Pascal's Triangle, tensor power carpets, patchwork carpets, $\mathbb{F}_4$, Prouhet-Thue-Morse sequence, Rudin-Shapiro sequence, Thue-Morse-Pascal 2-dimensional sequence, Rudin-Shapiro-Pascal 2-dimensional sequence.

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1 Introduction

This note reveals a new intersection between recurrence and substitution. Both notions occur in a field of interdisciplinary investigations unifying very heterogenous motivations and techniques. The recurrence - although a very classical task - is more and more present in studies concerning cellular automata, see [31, 13, 9] or the monograph [32]. Substitutions occur in various contexts such as automatic sequences [12, 1, 2], aperiodic tilings [30, 24, 15, 8, 7], various fractal constructions [23, 10, 25, 14] or mathematical quasicrystals [11, 5]. All objects and results introduced in this article seem to be most related with those studied in [3].

Definition 1.1 Let A be a finite set and $f : A^3 \to A$ a fixed function. We call the set A an *alphabet* and the function f a *recurrence*. We will refer to the function f as $f(x,y,z)$. We also fix two sequences $u, v : \mathbb{N} \to A$ with $u(0) = v(0)$, called *initial conditions*. We say that the tuple (A, f, u, v) defines a *recurrent* 2*-dimensional sequence* $a : \mathbb{N}^2 \to A$ if the following conditions are fulfilled:

1. $\forall\ k \in \mathbb{N}\ a(k,0) = u(k)$ and $a(0,k) = v(k)$.
2. $\forall\ m,n > 0\ a(m,n) = f(a(m,n-1), a(m-1,n-1), a(m-1,n))$.

*Brain Products, Freiburg, Germany and Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit 5, P. O. Box 1-764, RO-014700 Bucharest, Romania. mihai.prunescu@math.uni-freiburg.de.

In the case that $u = v$ we mention just one of them in the tuple. If u or v are periodic, we just write down the period.

The author proved in [17] that recurrent two-dimensional sequences are Turing-complete.

The special recurrence introduced in the Definition 1.1 means that the running element $a(m, n)$ of a 2-dimensional sequence depends of the predecessors $a((m, n) - (0, 1))$, $a((m, n) - (1, 1))$ and $a((m, n) - (1, 0))$. We say that the set $P = \{(0, 1), (1, 1), (1, 0)\} \subset \mathbb{Z}^2$ is the system of predecessors for the given recurrence. The domain of the initial condition $C_P = \{(x, y) \in \mathbb{N}^2 \,|\, x = 0 \vee y = 0\}$ depends of the system of predecessors.

In Section 2 we start by generalizing the notion of recurrence for n-dimensional sequences, and for arbitrary systems of predecessors. The technique needed is a rudiment of transfinite induction. This generalization has many motivations in the actual state of the art. To recall just one of them, the celebrated articles by Taylor [29] and by Socolar and Taylor [28] concerning the *Ein Stein* Tiling Problem - to find an aperiodic tiling set with one element for the plane - also uses matching conditions that do not concern only immediate but also further neighbors. In a narrow setting, such conditions can be modeled by the notion of recurrence introduced here, using predecessors which are not immediate neighbors of the element to be computed.

In Section 3 we define a notion of substitution which is appropriate for n-dimensional sequences $a : \mathbb{N}^n \to A$. In short, the most elementary tiles used to define the substitution are n-dimensional cubic matrices over the alphabet A with edge-length k, not necessarily equal 1. A substitution rule is the prescription to substitute such a cube with a n-dimensional cube over the alphabet A of edge length sk, with $s \geq 2$. Such a cube consists of s^n many cubes of edge k. For any of them there is a substitution rule to be applied in the next step, and so on.

After understanding the self-similar nature of a narrow class of recurrent two-dimensional sequences in [18], the author finally conjectured that all recurrent two-dimensional sequences given by homomorphisms of finite abelian p-groups and periodic initial conditions are produced by systems of substitution, see [18, 19, 20, 21]. In [22] the author considered some cases with initial conditions given by non-trivial automatic sequences. Excepting the results of [18], all other structures of substitution proved in these notes are based on ad-hoc computer computations done for particular cases. The main instruments used for these results were slightly weaker versions of our Theorem 3.13, Section 3.

At this stage the author realized that the right notion lying between the phenomenon of substitution in 2-dimensional sequences is their automaticity. For example, periodic and ultimately periodic sequences are k-automatic for all k and for all period-lengths. The successive constructions done before consisted of recurrent 2-dimensional sequences given by homomorphisms of p-groups as rules of recurrence, applied over constant, then periodic boreders, and finally morphic borders, like the Thue-Morse Sequence. The point was that all such borders were automatic sequences.

In Section 4 different results of the monograph [1] are tracked together in order to show that n-dimensional sequences defined by systems of substitutions given in Section 3 are exactly the automatic sequences. Other characterizations of automaticity prove to be extremely useful, as for example Christol's Theorem - a result connecting the automaticity with the property of a sequence to be an algebraic element over a field of rational functions, if seen as a formal series over a finite field.

In Section 5 is proved the main result, which is the Theorem 5.2. This Theorem states that $\mathbb{F}_p$-affine recurrences, given by non-negative systems of predecessors over arbitrary finite fields of characteristic p, always produce p-automatic n-dimensional sequences if they are applied on p-automatic initial conditions. Unhappily, the proofs using Christol's theorem are for the moment not constructive enough, and we still must use computational methods like one suggested by our Theorem 3.13 to obtain a concrete set of substitution rules generating the n-dimensional sequence in question.

In terms of finite abelian p-groups, Theorem 5.2 solves the question of p-automaticity only for the p-groups $G = \mathbb{F}_p \times \mathbb{F}_p \times \cdots \times \mathbb{F}_p$ with arbitrary homomorphisms $f : G^3 \to G$ and p-automatic sequences as initial conditions, and only for the so called moderate recurrence. It is also known that Pascal's Triangles modulo p^k are p^u-automatic; see, e.g., [1] for a proof. Several particular cases proved by the author suggest that homomorphic n-dimensional sequences over finite abelian p-groups with p-automatic initial conditions are automatic. A general proof of this fact is still missing.

In Section 6 we give examples of recurrent 2-dimensional sequences which are nonautomatic. This shows that the algebraic assumptions of the results above are necessary. As a byproduct we show that for polynomials $f \in \mathbb{Q}[X]$ with $\deg(f) \geq 2$ and $f(\mathbb{N}) \subset \mathbb{N}$, the characteristic sequence of the set $f(\mathbb{N})$ is not k-automatic for any k.

In Section 7 various examples are shown, and some shorter substitutions are concretely described. A less general form of Theorem 3.13 and example 7.7 were announced without proof in [22].

The Appendix called Section 8 provides some details for a counterexample stated in Section 6.

2 Recurrence

In this section a generalized notion of recurrence for n-dimensional sequences is introduced. An n-dimensional sequence over a finite alphabet A is a function $a : \mathbb{N}^n \to A$. In order to write down our definitions we need the notion of lexicographic order for the set $\mathbb{Z}^n$. We will always denote the tuple $(0, 0, \dots, 0)$ by $\vec{0}$.

Definition 2.1 Let $\mathbb{Z}$ be the set of integers and let $<$ be its relation of strict order. We extend the order relation $<$ to an ordering of the set $\mathbb{Z}^n$, also denoted by $<$, defined as follows: For all $\vec{x}, \vec{y} \in \mathbb{Z}^n$ we say that $\vec{x} < \vec{y}$ if and only if there is an i, $1 \leq i \leq n$, such that $x_1 = y_1, \dots, x_{i-1} = y_{i-1}$ and $x_i < y_i$. This relation is called *lexicographic ordering* of $\mathbb{Z}^n$. The restriction of the relation $<$ to the set $\mathbb{N}^n$ shall also be denoted by $<$ and shall be called *lexicographic ordering* of $\mathbb{N}^n$.

Lemma 2.2 *The ordered set* $(\mathbb{N}^n, <)$ *is a well-ordering.*

Proof: The ordered set $(\mathbb{N}^n, <)$ is order isomorphic with the ordinal ω^n. The isomorphism is given by $\iota : \omega^n \to \mathbb{N}^n$, $\iota(x_1 \omega^{n-1} + \cdots + x_n) = (x_1, \dots, x_n)$. □

Definition 2.3 Fix $m \geq 1$ and m many tuples $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{Z}^n$ such that all m tuples are pairwise distinct and lexicographically positive: $\vec{v}_1 > \vec{0}, \dots, \vec{v}_m > \vec{0}$. Such a collection of distinct lexicographically positive integral tuples is called a *system of predecessors* and will be denoted by $P = \{\vec{v}_1, \dots, \vec{v}_m\}$.

Definition 2.4 An n-dimensional *rectangle* is a subset of $\mathbb{Z}^n$ of the form:

$$R = \{x_1, \dots, x_1 + k_1 - 1\} \times \cdots \times \{x_n, \dots, x_n + k_n - 1\},$$

where $k_1, \dots, k_n \geq 1$ are the edges of R. If $k_1 = \cdots = k_n = k$, we call R an n-dimensional *cube*.

Definition 2.5 Let $P = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a system of predecessors. The smallest n-dimensional rectangle containing the set $-P \cup \{\vec{0}\} = \{-\vec{v}_1, \dots, -\vec{v}_m\} \cup \{\vec{0}\}$ shall be denoted by R_P.

Definition 2.6 Let $Y \subset \mathbb{Z}^n$ and $\vec{x} \in \mathbb{Z}^n$. The set $\vec{x} + Y$ is defined as set of all elements $\vec{x} + \vec{y}$, where $\vec{y} \in Y$, and is called the *translate* of Y by $\vec{x}$. If P is a system of predecessors and $\vec{x} \in \mathbb{N}^n$ we define $R_P(\vec{x}) := R_P + \vec{x}$. $R_P(\vec{x})$ is the smallest n-dimensional rectangle containing $\vec{x}$ and all the differences $\vec{x} - \vec{v}_i$.

Definition 2.7 Let $m \geq 1$ and let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{Z}^n$ such that $P = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a system of predecessors. Consider the set C defined as:

$$C_P = \{\vec{x} \in \mathbb{N}^n \,|\, \exists\, i \; 1 \leq i \leq m \wedge \vec{x} - \vec{v}_i \in \mathbb{Z}^n \setminus \mathbb{N}^n\} = \{\vec{x} \in \mathbb{N}^n \,|\, R_P(\vec{x}) \not\subset \mathbb{N}^n\}.$$

Any function $c : C_P \to A$ is called *initial condition* for the system of predecessors P. The set C_P is the *domain* of the initial condition.

Definition 2.8 Let $m \geq 1$ and let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{Z}^n$ such that $P = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a system of predecessors. For $j = 1, \ldots, m$ we write down the coordinates like $\vec{v}_j = (v_j^1, \ldots, v_j^n)$. For $i = 1, \ldots, n$ let $d_i = \max\limits_{j=1,\ldots,m} \max(v_j^i, 0)$ be the *depth of the recurrence for the direction* i. We observe that for at least one direction, the depth of the recurrence must be positive. The positive number $d = \max\limits_{i=1,\ldots,n} d_i$ is the *depth of the system of predecessors* P.

Lemma 2.9 *Let $m \geq 1$ and let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{Z}^n$ such that $P = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a system of predecessors. In this case the domain of the initial condition is:*

$$C_P = \{\vec{x} \in \mathbb{N}^n \,|\, \exists\, i \; x_i < d_i\}$$

Proof: Both inclusions are evident. □

We observe that the domain of the initial condition C_P is the union of some plane sections given by equations $x_i = k$ for some $i = 1, \ldots, n$ and all $0 \leq k < d_i$, where d_i is the given depth. If some $d_i = 0$ the set C_P does not contain sections parallel with the plane $x_i = 0$. According to the lexicographic order, the first point that does not belong to C_P, is $\vec{d} = (d_1, \ldots, d_i)$.

Definition 2.10 Let A be a finite alphabet. Fix $m \geq 1$, a function $f : A^m \to A$ called rule of recurrence and $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{Z}^n$ a system of m distinct predecessors: $\vec{v}_1 > \vec{0}, \ldots, \vec{v}_m > \vec{0}$, denoted by $P = \{\vec{v}_1, \ldots, \vec{v}_m\}$. Given an initial condition $c : C_P \to A$ for this system of predecessors, we say that an n-dimensional sequence $a : \mathbb{N}^n \to A$ *satisfies the recurrence* (A, f, P, c) if and only if the following two conditions are fulfilled:

1. For all $\vec{x} \in C_P$, $a(\vec{x}) = c(\vec{x})$.
2. For all $\vec{x} \in \mathbb{N}^n \setminus C_P$, $a(\vec{x}) = f(a(\vec{x} - \vec{v}_1), \ldots, a(\vec{x} - \vec{v}_m))$.

Lemma 2.11 *Let $\vec{w}, \vec{v} \in \mathbb{N}^n$ and $\vec{u} \in \mathbb{Z}^n$ such that $\vec{u} > \vec{0}$ and $\vec{v} = \vec{w} - \vec{u}$. Then $\vec{w} > \vec{v}$.*

Lemma 2.12 *Given a recurrence $(A, f, \vec{v}_1, \ldots, \vec{v}_m, c)$ as defined in Definition 2.10, then there exists a unique n-dimensional sequence $a : \mathbb{N}^n \to A$ satisfying this recurrence.*

Proof: For the proof we use the fact that $(\mathbb{N}^n, <)$ is a well-ordering (see Lemma 2.2) and we define the elements $a(\vec{x})$ by transfinite induction. The induction starts with $\vec{0}$. We observe that always $\vec{0} \in C_P$, so we define $a(\vec{0}) = c(\vec{0})$, which is also the only one possibility to define it. Suppose that we have already defined $a(\vec{y})$ for all $\vec{y} < \vec{x}$. If $\vec{x} \in C_P$ then we define $a(\vec{x}) = c(\vec{x})$ and this is again the unique possible value. If $\vec{x} \notin C_P$, all elements $\vec{x} - \vec{v}_1, \ldots, \vec{x} - \vec{v}_m$ are in $\mathbb{N}^n$ and according to Lemma 2.11 one has $\vec{x} - \vec{v}_1 < \vec{x}, \ldots, \vec{x} - \vec{v}_m < \vec{x}$. According to the hypothesis of induction all the values $a(\vec{x} - \vec{v}_j)$ have been already defined and were uniquely determined by the construction done so far. Then we define $a(\vec{x}) = f(a(\vec{x} - \vec{v}_1), \ldots, a(\vec{x} - \vec{v}_m))$ and that is again the unique possibility to define this value. □

Definition 2.13 Let $m \geq 1$ and let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{Z}^n$ such that $P = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a system of predecessors. We recall the notation $\vec{v}_j = (v_j^1, \ldots, v_j^n)$. Let $e_i = -\min\limits_{j=1,\ldots,m} \min(v_j^i, 0)$. We call e_i the *excess P in the direction i*. The number $e = \max\limits_{i=1,\ldots,n} e_i$ is the *excess* of P. If $e > 0$ we say that the system of predecessors P (the recurrence) *has excess* or is *excessive*. If $e = 0$ we say that the system of predecessors P (the recurrence) *lacks excess* or is *moderate*.

We observe that if a system of predecessors has excess, one can determine $a(\vec{x})$ only if one determines also some $a(\vec{y})$ with several coordinates $y_i > x_i$. However, for computing any $a(\vec{x})$ we must compute only finitely many $a(\vec{y})$ with $\vec{y} < \vec{x}$. This is a direct consequence of the fact that $(\mathbb{N}^n, <)$ is a well-ordering.

In the rest of this section some examples will be discussed.

- If $n = 2$ and $P = \{(0,1),(1,1),(1,0)\}$ we get again the recurrence introduced in Definition 1.1. In this case $d_1 = d_2 = 1$ and $C_P = \{x = 0\} \cup \{y = 0\}$. The excess is 0, so in order to compute the value $a(x,y)$ it is enough to have computed all the values in the rectangle $(0,0)\,(x,0)\,(x,y)\,(0,y)$. The recurrence works as shown in the following matrix:

$$\begin{pmatrix} c & c & c & c \\ c & v & w & \cdot \\ c & u & f(u,v,w) & \cdot \\ c & z & \cdot & \cdot \end{pmatrix}.$$

  Here elements determined by the initial condition are denoted by c, already computed elements are denoted by u, v, w, z , and the element that has been computed at this step of the recurrence is denoted $f(u,v,w)$. The elements marked with points will be computed in some future steps. In this case the rectangle $R_P(\vec{x})$ is exactly the rectangle with vertices marked v, w, $f(u,v,w)$ and u.

- If $n = 2$ and $P = \{(1,1),(1,0),(1,-1)\}$ we are facing a kind of recurrence used by many computer scientists to simulate the evolution in time for cellular automata and Turing machines, see for example [32]. In this case $d_1 = d_2 = 1$ and $C_P = \{x = 0\} \cup \{y = 0\}$. The excess is equal to 1, so in order to compute the value of (x,y) one must have computed all values of the elements in the sets $V(a)$, where $0 \leq a < x$ and $V(a) = \{(a,b) \,|\, 0 \leq b < y + (x-a)y\}$. The recurrence works as shown in the following matrix:

$$\begin{pmatrix} c & c & c & c \\ c & u & t & \cdot \\ c & v & f(u,v,w) & \cdot \\ c & w & \cdot & \cdot \end{pmatrix}.$$

  The notation is similar with those used in the precedent example. In this case the rectangle $R_P(\vec{x})$ is exactly the rectangle with three vertices marked u, t and w. The fourth vertex is marked in the matrix by a dot.

- If $n = 2$ and $P = \{(1,-2),(0,2)\}$, then $d_1 = 1$ and $d_2 = 2$. The domain of the initial condition is the set $\{x = 0\} \cup \{x = 1\} \cup \{y = 0\}$. The excess is equal to 2, so in order to compute the value of (x,y) one must have computed all values of the elements in the sets $V(a)$, where $0 \leq a \leq x$ and $V(a) = \{(a,b) \,|\, 0 \leq b < y + 2(x-a)y\}$. The recurrence works as shown in the following matrix:

$$\begin{pmatrix} c & c & c & c \\ c & c & c & c \\ c & t & v & \cdot \\ c & z & s & \cdot \\ c & q & f(u,v) & \cdot \\ c & w & \cdot & \cdot \\ c & u & \cdot & \cdot \end{pmatrix}.$$

  Here we let c denote again elements determined by the initial condition, by t, z, q, w, u, v, s already computed elements, and by $f(u,v)$ the element that has been computed at this step of the recurrence. The elements marked with points will be computed in some future steps. In this case the rectangle $R_P(\vec{x})$ is exactly the rectangle with three vertices marked t, v and u. The fourth vertex is marked in the matrix by a dot.

- If $n = 2$ and $P = \{(1,0),(1,-1)\}$, then $d_1 = 1$ and $d_2 = 0$. The domain of initial condition is the set $\{y = 0\}$. The excess is equal 1, and the recurrence works as follows:

$$\begin{pmatrix} c & x & y & \cdot \\ c & u & v & \cdot \\ c & t & f(t,z) & \cdot \\ c & z & \cdot & \cdot \end{pmatrix}.$$

3 Substitution

Definition 3.1 We recall that the set $\{0,\ldots,d-1\}^n$ is an n-dimensional cube. For some tuple $\vec{u} \in \mathbb{N}^n$, $\vec{u} = (u_1, u_2, \ldots, u_n)$ we recall that $\{u_1,\ldots,u_1+d-1\} \times \cdots \times \{u_n,\ldots,u_n+d-1\} = \vec{u} + \{0,\ldots,d-1\}^n$. For $d \geq 1$ we define the following two infinite sets:

$$\Delta_d = \{d\vec{u} + \{0,\ldots,d-1\}^n \mid \vec{u} \in \mathbb{N}^n\}.$$

$$\Gamma_d = \{d\vec{u} + \{0,\ldots,2d-1\}^n \mid \vec{u} \in \mathbb{N}^n\}.$$

We call Δ_d the *d-division* of $\mathbb{N}^n$ and Γ_d the *$2d$-covering* of $\mathbb{N}^n$.

Definition 3.2 Let A be a finite set (alphabet). An n-dimensional sequence is a function $a : \mathbb{N}^n \to A$. A *colored n-dimensional cube* is a function $D : \{0,\ldots,d-1\}^n \to A$. We say that D *occurs in a at $\vec{u} \in \mathbb{N}^n$* if $\forall\, \vec{x} \in \{0,\ldots,d-1\}^n$, $a(\vec{u}+\vec{x}) = D(\vec{x})$. We say that D *occurs in a* if there is a $\vec{u}$ such that D occurs in a at $\vec{u}$. We say that D *occurs at some d-position* in a if there $\vec{u} \in \mathbb{N}^n$ such that D occurs in a at $d\vec{u}$.

Definition 3.3 Let $s \in \mathbb{N}$ be a natural number ≥ 2 and let $E : \{0,\ldots,ds-1\}^n \to A$ be some n-dimensional cube over A. We define $D_d(E)$ as the set of all colored n-dimensional cubes $D : \{0,\ldots,d-1\}^n \to A$ occurring in E in some d-position. If $a : \mathbb{N}^n \to A$, $D_d(a)$ is the set of all colored n-dimensional cubes $D : \{0,\ldots,d-1\}^n \to A$ occurring in a in some d-position.

Definition 3.4 Let $E : [0, sd-1]^n \to A$ be a n-dimensional cube. Let $d \geq 1$ be a positive integer. We define $C_d(E)$ as set of all colored n-dimensional cubes $F : \{0,\ldots,2d-1\}^n \to A$ occurring in E in some d-position. If $a : \mathbb{N}^n \to A$, then $C_d(a)$ is the set of all colored n-dimensional cubes $F : \{0,\ldots,2d-1\}^n \to A$ occurring in a in some d-position.

We observe that the sets $C_d(a)$ and $D_d(a)$ are finite, and that copies of the elements in $C_d(a)$ cover a with overlappings. Moreover, $C_d(a)$ and $D_d(a)$ are the set of traces of the covering Γ_d and respectively of the division Δ_d on the n-dimensional sequence a.

Definition 3.5 Let $d \geq 1$ and $s \geq 2$ two natural numbers. A *n-dimensional system of substitutions* (for short, *n-dimensional substitution*) of type $d \to sd$ over the finite set A is a tuple of finite sets $(A, \mathcal{D}, \mathcal{E}, D_1, \Sigma)$, as follows:

$\mathcal{D}$ is a set of colored n-dimensional cubes $D : \{0,\ldots,d-1\}^n \to A$,

$\mathcal{E}$ is a set of colored n-dimensional cubes $E : \{0,\ldots,2d-1\}^n \to A$, such that for every $E \in \mathcal{E}$, $D_d(E) \subset \mathcal{D}$, and

$D_1 \in \mathcal{D}$ is a special element called *start-symbol*.

Finally, Σ is a function $\Sigma : \mathcal{D} \to \mathcal{E}$, called the set of substitution *rules*, or simply the *substitution*. The function Σ has a natural extension defined on the set of cubes F such that $D_d(F) \subseteq \mathcal{D}$. We remark that if $D_d(F) \subseteq \mathcal{D}$ then $D_d(\Sigma(F)) \subseteq \mathcal{D}$, so Σ can be applied again to $\Sigma(F)$. Moreover, Σ must fulfill the following condition:

$$\Sigma(D_1) \mid \{0,\ldots,d-1\}^n = D_1.$$

(In this case, we say that the substitution Σ is *expansive*.) The number $s \geq 2$ is called the *factor of substitution.*

Definition 3.6 A n-dimensional substitution $(A, A, \mathcal{E}, a_1, \Theta)$ of type $1 \to s$ is called n-dimensional uniform morphism over the alphabet A.

Indeed, in the last definition the n-dimensional cubes of edge 1 over A are identified with the elements of A, $\mathcal{E}$ consists of n-dimensional cubes of edge s, and $a_1 \in A$ is the start symbol for the substitution $\Theta : A \to \mathcal{E}$. The only one condition to fulfill now is that $E_1(\vec{0}) = a_1$.

As one immediately can prove by induction, the expansivity of Σ means that for all $m \in \mathbb{N}$ one has that $\Sigma^m(D_1) \,|\, \{0, \dots, ds^{m-1}\}^n = \Sigma^{m-1}(D_1)$. So we can define the n-dimensional sequence b:

Definition 3.7

$$b := \lim_{i \to \infty} \Sigma^i(D_1)$$

We say that the n-dimensional sequence b is *defined by substitution.*

Substitution in multi-dimensional sequences and many aspects of this tool can be also found in [26], [27], [16], [4] and in the references therein.

Lemma 3.8 *Let $R = \vec{x} + \{0, \dots, k_1 - 1\} \times \dots \times \{0, \dots, k_n - 1\}$ be some rectangle and $k = \max(k_1, \dots, k_n)$. Consider the covering Γ_d of $\mathbb{N}^n$ consisting of all cubes $d\vec{y} + \{0, \dots, 2d-1\}^n$, where $\vec{y} \in \mathbb{N}^n$. If ($d = 1$ and $k = 2$) or if $k \leq d$, then there exists $E \in \Gamma_d$ such that $R \subseteq E$.*

Proof: It is enough to prove the lemma for the cube $W = \vec{x} + \{0, \dots k-1\}^n$, because $R \subset W$.

- Case $k \leq d$: According to the principle of division with remainder, for all $x \in \mathbb{N}$ there exists $r, y \in \mathbb{N}$ with $0 \leq r < d$ such that $x = dy + r$. If we apply the division by d for all the coordinates x_i of x, we get a point $d\vec{y}$ with the property that $W = \vec{x} + \{0, \dots, k-1\}^n \subset E = d\vec{y} + \{0, \dots 2d-1\}^n$, because $0 \leq r_i + k \leq (d-1) + d = 2d - 1$, for all $i = 1, \dots, n$.

- Case $d = 1$ and $k = 2$: In this case we can always take $E = W$. $\square$

Definition 3.9 Let $P = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a system of predecessors and $f : A^m \to A$ be a function. We say that a colored cube $D : \vec{z} + \{0, \dots, m-1\}^n \to A$ *satisfies the recurrence* given by P and f, if for all $\vec{x} \in \vec{z} + \{0, \dots, m-1\}^n$ such that $R_P + \vec{x} \subset \vec{z} + \{0, \dots, m-1\}^n$, $D(\vec{x}) = f(D(\vec{x} - \vec{v}_1), \dots, D(\vec{x} - \vec{v}_m))$. We say that a n-dimensional sequence *satisfies the recurrence* given by P and f, if for all $\vec{x} \in \mathbb{N}^n$ such that $R_P(\vec{x}) \subset \mathbb{N}^n$, $a(\vec{x}) = f(a(\vec{x} - \vec{v}_1), \dots, a(\vec{x} - \vec{v}_m))$.

Lemma 3.10 *Let A be a finite set and $a : \mathbb{N}^n \to A$ a n-dimensional sequence. Let $P = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a system of predecessors and $f : A^m \to A$ a function. Let $R_P = \vec{u} + \{0, \dots, k_1 - 1\} \times \dots \times \{0, \dots, k_n - 1\}$ and let $k = \max(k_1, \dots, k_n)$. Consider a natural number d such that ($d = 1$ and $k = 2$) or $k \leq d$ holds, and the covering Γ_d consisting of all cubes $d\vec{y} + \{0, \dots, 2d-1\}^n$, where $\vec{y} \in \mathbb{N}^n$. Then $a : \mathbb{N}^n \to A$ satisfies the recurrence given by P and f if and only if for all $E \in \Gamma_d$, the restriction $a : E \to A$ satisfies the recurrence given by P and f.*

Proof: The property to satisfy the recurrence given by P and f is a universal property (can be formalized using a universal quantifier), so is always inherited by subsets. If the n-dimensional sequence a satisfies the recurrence given by P and f, so does any colored cube in its covering Γ_d. Suppose now that any colored cube $E \in \Gamma_d$ satisfies the recurrence given by P and f. According to Lemma 3.8, for every $\vec{x} \in \mathbb{N}^n$, if $\vec{x} + R_P \subset \mathbb{N}^n$ then there is some $E \in \Gamma_d$ such that $\vec{x} + R_P \subset E$. But E satisfies the recurrence given by P and f by assumption, so the relation $a(\vec{x}) = f(a(\vec{x} - \vec{v}_1), \dots, a(\vec{x} - \vec{v}_m))$ holds. $\square$

Remark 3.11 Given $a : \mathbb{N}^n \to A$, the sequence of sets $(C_d(a\,|\,\{0, \ldots, ds^M - 1\}^n))_{M\in\mathbb{N}}$ is always ultimately constant, because for all $M \in \mathbb{N}$, $C_d(a\,|\,\{0, \ldots, ds^M - 1\}^n) \subseteq C_d(a\,|\,\{0, \ldots, ds^{M+1} - 1\}^n)$ and the set of all possible colored cubes $Y : \{0, \ldots, 2d - 1\}^n \to A$ is finite.

For n-dimensional sequences defined by substitution one can prove more:

Lemma 3.12 *Let $(A, \mathcal{D}, \mathcal{E}, D_1, \Sigma)$ be an n-dimensional substitution of type $d \to sd$ and $M \in \mathbb{N}$ such that $C_d(\Sigma^M(D_1)) = C_d(\Sigma^{M-1}(D_1))$. We denote this finite set by C_d. Then for all $i \geq M-1$, $C_d(\Sigma^i(D_1)) = C_d$, and also $C_d(\lim_{i\to\infty} \Sigma^i(D_1)) = C_d$.*

Proof: The conclusion is true for $i = M$ by hypothesis. For the proof we will use the shorter notation $C_d(\Sigma^i)$ for $C_d(\Sigma^i(D_1))$. Suppose that we have already proven that $C_d(\Sigma^i) = C_d$ for some $i \geq M$. Let U be a n-dimensional cube of edge $2d$ occurring in d-position somewhere in Σ^{i+1}. If we consider the elements V of the covering $\Gamma_d(\Sigma^i)$ and forget the colors of their elements, we see that their images $\Sigma(V)$ build up together a covering $\Gamma_{sd}(\Sigma^{i+1})$. But $2d \leq sd$, so we may apply Lemma 3.8 with k substituted by $2d$ and with d substituted by $2sd$, and so we conclude that there exists a cube V with edge $2d$ occurring in Σ^i in some d-position, such that U is covered by $\Sigma(V)$. We know that $V \in C_d(\Sigma^i) = C_d$, the last equality being the hypothesis of induction. But as we know that $C_d(\Sigma^{i-1}) = C_d$, it follows that $V \in C_d(\Sigma^{i-1})$, so that U already occurs in Σ^i in some d-position, as a sub-block in an occurrence of the block $\Sigma(V)$. This means that $C_d(\Sigma^{i+1}) = C_d(\Sigma^i) = C_d$. $\square$

Theorem 3.13 *Let A be a finite set, let (A, f, P, c) be an n-dimensional recurrence (see Definition 2.10) and let $(A, \mathcal{D}, \mathcal{E}, D_1, \Sigma)$ be an n-dimensional substitution (see Definition 3.5). Suppose that the recurrence generates an n-dimensional sequence $a : \mathbb{N}^n \to A$ and that the substitution generates an n-dimensional sequence $b : \mathbb{N}^n \to A$. Finally, suppose that the substitution is of type $d \to sd$ and that the following conditions are satisfied:*

1. *If R_P is the minimal rectangle containing $-P$ and $\{\vec{0}\}$, $k_1, \ldots, k_n$ are the edge-lengths of R_P and $k = \max(k_1, \ldots, k_n)$, then ($d = 1$ and $k = 2$) or $k \leq d$.*
2. *For all $\vec{x} \in C_P$, $a(\vec{x}) = b(\vec{x})$.*
3. *There exists $M \in \mathbb{N}$ such that $a\,|\,\{0, \ldots, ds^M - 1\}^n = \Sigma^M(D_1)$ and $C_d(\Sigma^{M-1}(D_1)) = C_d(\Sigma^M(D_1))$.*

Then $a = b$.

Proof: According to Lemma 2.12, the sequence $b = \lim_{i\to\infty} \Sigma^i(D_1)$ is identical with the sequence a if and only if b satisfies the initial conditions of a, which is true by the second assumption, and b satisfies the recurrence given by P and f. By Lemma 3.10 and by the first assumption of the statement, it would be enough to prove that for all cubes W in the $2d$-covering Γ_d, $b\,|\,W$ satisfies the recurrence given by P and f. The restrictions $b\,|\,W$ with $W \in \Gamma_d$, translated in $\vec{0}$, build together the finite set of colored cubes $C_d(b)$. According to Lemma 3.12, the third assumption implies that $C_d(b) = C_d(\Sigma^M(D_1))$, and $C_d(\Sigma^M(D_1)) = C_d(a\,|\,\{0, \ldots, ds^M - 1\}^n)$, so $C_d(b) = C_d(a\,|\,\{0, \ldots, ds^M - 1\}^n)$. That means that all cubes in $C_d(b)$ satisfies the recurrence defined by P and f, so b satisfies this recurrence in all cubes of the $2d$-covering Γ_d. Lemma 3.10 closes the argument. $\square$

4 Automatic n-dimensional sequences

The content presented is based on the monograph [1].

Definition 4.1 An *n-dimensional deterministic finite k-automaton with output (n-DFA) M* consists of a finite nonempty set of states Q, an input-alphabet $\Sigma = [0, k-1]^n$, a transition function $\delta : Q \times \Sigma \to Q$, an initial state q_0, an output-alphabet A and an output mapping $\tau : Q \to A$. Any tuple $\vec{u} \in \mathbb{N}^n$ is written in the form $\vec{u} = \sum_{0 \le i \le v} k^i \vec{\sigma}_i$ with $\vec{\sigma}_0, \ldots, \vec{\sigma}_v \in \Sigma$. We say that $\vec{\sigma}_v \vec{\sigma}_{v-1} \cdots \vec{\sigma}_0$ is the *k-code* of $\vec{u}$. An n-dimensional sequence $a : \mathbb{N}^n \to A$ is *produced by the k-automaton M* if for all $\vec{u} \in \mathbb{N}^n$, M stops in a state $q \in Q$ with output $\tau(q) = a(\vec{u})$ after reading the k-code of $\vec{u}$. If there is a n-dimensional k-automaton producing the n-dimensional sequence a we say that a is a *k-automatic sequence*.

Definition 4.2 The *k-kernel* of a n-dimensional sequence $a : \mathbb{N}^n \to A$ is the set of n-dimensional sequences:

$$K_k(a) = \{(a(k^s u_1 + v_1, k^s u_2 + v_2, \ldots, k^s u_n + v_n))_{\vec{u}} \,|\, s \ge 0,\ 0 \le v_i < k^s\}.$$

In the next lines we recall some notions of algebra.

Let R be some domain. A formal series in n variables over R is a expression of the form

$$S = \sum_{\vec{x} \ge 0} a(\vec{x}) X_1^{x_1} \ldots X_n^{x_n},$$

where all $a : \mathbb{N}^n \to R$. The formal series build together a domain $R[[X_1, \ldots, X_n]]$. From now on we will denote the tuple of variables $X_1, \ldots, X_n$ by $\vec{X}$ and the monomial $X_1^{x_1} \cdots X_n^{x_n}$ by $\vec{X}^{\vec{x}}$. Let K be the field of quotients of R. The field of quotients of the domain $R[[\vec{X}]]$ is denoted by $K((\vec{X}))$ and can be identified with the set of all series

$$S = \sum_{\vec{x} \ge k} b(\vec{x}) X_1^{x_1} \ldots X_n^{x_n} = \sum_{\vec{x} \ge k} b(\vec{x}) \vec{X}^{\vec{x}},$$

where $k \in \mathbb{Z}$, $b : [k, +\infty)^n \to K$ and $\vec{x} \ge k$ means that all coordinates $x_i \ge k$.

If K is a field, the field of power series in n variables $K((\vec{X}))$ contains the field of rational functions $K(\vec{X})$. In order to make this embedding transparent, recall that polynomials are power series with finite support, and rational functions are formal quotients of polynomials.

A formal series $A \in K((\vec{X}))$ is said to be algebraic over the field of rational functions $K(\vec{X})$ if there exist polynomials $P_0, \ldots P_s \in K[\vec{X}]$, $P_0 \ne 0$, such that:

$$\sum_{j=0}^{s} P_j A^j = 0.$$

The elements of $K((\vec{X}))$ algebraic over $K(\vec{X})$ build a subfield of $K((\vec{X}))$, called the relative algebraic closure of $K(\vec{X})$ in $K((\vec{X}))$.

We recall now that according to Definition 3.6 a substitution $(B, B, \mathcal{Y}, x_1, \Sigma)$ of type $1 \to s$ is called n-dimensional uniform morphism.

Definition 4.3 Let $(B, B, \mathcal{E}, b_1, \Theta)$ be a uniform morphism. Given an alphabet A, some function $g : B \to A$ will be called a *coding* of B.

Theorem 4.4 *Let $a : \mathbb{N}^n \to A$ be an n-dimensional sequence with values in a finite set A. Let p be a prime. Then the following are equivalent:*

1. *a is p-automatic.*
2. *$K_p(a)$ is finite.*

3. *There exists an n-dimensional uniform morphism $(B, B, \mathcal{E}, b_1, \Theta)$ of type $1 \to p$ that produces an n-dimensional sequence b by substitution, i.e. $\Theta(b_1)(\vec{0}) = b_1$ and $b = \lim_{i\to\infty} \Theta^i(b_1))$, $g : B \to A$ is a coding of B, and $a = g(b)$.*

4. *For every embedding $\iota : A \to K$ in a sufficiently large finite field K of characteristic p, the corresponding series $S = \sum \iota(a(\vec{x}))\vec{X}^{\vec{x}}$ is an element of $K((\vec{X}))$ algebraic over $K(\vec{X})$.*

5. *There exists an embedding $\iota : A \to K$ in a sufficiently large finite field K of characteristic p, the corresponding series $S = \sum \iota(a(\vec{x}))\vec{X}^{\vec{x}}$ is an element of $K((\vec{X}))$ algebraic over $K(\vec{X})$.*

The equivalences between 1, 2 and 3 are true also without the assumption that p is a prime.

The item 3 in Theorem 4.4 is known as Cobham's Theorem. The item 4 is known as Christol's Theorem. The whole Theorem 4.4 seems to have been formulated and proven for the first time in this form by Salon, see [26] and [27]. Cobham's Theorem highlights the fact that automaticity and substitution are related up to a coding. The next two results represent a slight generalization: for a coding one can use n-dimensional cubes instead of individual letters.

Theorem 4.5 *Let A be some finite set and $u : \mathbb{N}^n \to A$ an n-dimensional sequence. For $i = 1, \dots, n$ let $a_i \geq 1$ be natural numbers. If all $a_1 a_2 \cdots a_n$ many n-dimensional sequences:*

$$u^{b_1,\dots,b_n} = (u(a_1 x_1 + b_1, \dots, a_n x_n + b_n))_{(x_1,\dots,x_n)},$$

where $0 \leq b_i < a_i$, are s-automatic for some $s \geq 2$, then u is s-automatic.

Proof: Proven as Theorem 14.2.7 in [1] for $n = 2$. The general proof works in the same way. □

Corollary 4.6 *Let $k \geq 1$ and $s \geq 2$ be two natural numbers. If the n-dimensional sequence $u : \mathbb{N}^n \to A$ is the result of a substitution of type $k \to sk$ then u is s-automatic.*

Proof: Let $(A, \mathcal{D}, \mathcal{E}, D_1, \Sigma)$ be the substitution of type $k \to sk$, such that $u = \lim_{i\to\infty} \Sigma^i(D_1)$. We define now a uniform n-dimensional morphism $(\mathcal{F}, \mathcal{F}, \mathcal{V}, f_1, \Theta)$ as follows:

- $\mathcal{F} = \{f_1, \dots, f_r\}$ is an abstract finite set and has the same number of elements as $\mathcal{D} = \{D_1, \dots, D_n\}$. Let $\varphi : \mathcal{D} \to \mathcal{F}$ the bijection given by $\varphi(D_i) = f_i$. In particular the start symbol $f_1 = \varphi(D_1)$. This bijection extends naturally to words and to n-dimensional cubes. The extension will be also called φ.

- $\mathcal{V}$ is a set of n-dimensional cubes of edge s over $\mathcal{F}$. For any $E : \{0, \dots, sk-1\}^n \to A$ with the k-division $D_k(E) \subset \mathcal{D}$, if $E \,|\, k\vec{y} + \{0, \dots, k-1\}^n = D_i$, then $\varphi(E)(\vec{y}) = f_i$.

- $\Theta(f_i) := \varphi(\Sigma(D_i))$.

Let $v = \lim_{i\to\infty} \Theta^i(f_1)$ be the n-dimensional sequence produced by the uniform n-dimensional morphism $(\mathcal{F}, \mathcal{F}, \mathcal{V}, f_1, \Theta)$. In order to apply Theorem 4.5, take $a_1 = \cdots = a_n = k$ and choose arbitrarily $b_1, \dots, b_i$ with $0 \leq b_i < k$. Consider the coding $\pi^{b_1,\dots,b_i} : \mathcal{F} \to A$ given by $\pi^{b_1,\dots,b_n}(f_i) := D_i(b_1, \dots, b_n)$. According to Theorem 4.4, point 3, the k^n-many sequences:

$$v^{b_1,\dots,b_n} := \pi^{b_1,\dots,b_i}(v),$$

are all s-automatic. On the other side $v^{b_1,\dots,b_n} = u^{b_1,\dots,b_n}$, the sequence defined in the hypothesis of Theorem 4.5, hence the n-dimensional sequence u is itself s-automatic. □

The next result is a converse:

Lemma 4.7 *Let $u : \mathbb{N}^n \to A$ be an s-automatic sequence. Then there are natural numbers r and q such that $1 \leq r < q$ and an n-dimensional substitution $(A, \mathcal{E}, \mathcal{F}, D_1, \Sigma)$ of type $s^r \to s^q$ such that $u = \lim_{i\to\infty} \Sigma^i(D_1)$.*

Proof: Proven as Lemma 6.9.1 in [1] for $n = 1$. The general proof works in the same way. $\square$

The remaining results will be used in the next section:

Lemma 4.8 *If $b : \mathbb{N} \to B$, $b = (b(n))_{n\in\mathbb{N}}$ is k-automatic, A is some set and $g : B \to A$ is some function, then the sequence $a = g(b) = (g(b(n)))_{n\in\mathbb{N}}$ is k-automatic.*

Proof: Let $\tau : Q \to B$ be the output function of a deterministic finite k-automaton with output producing $(b(n))$. Replacing $\tau : Q \to B$ with $g \circ \tau : Q \to A$ we obtain a deterministic finite automaton with output producing $(g(b(n)))$. The proof works also in more dimensions. $\square$

Lemma 4.9 *Let p be a prime, $q = p^s$, $\mathbb{F}_q$ the finite field with q elements. Let $\{b_1, \dots, b_s\}$ be a fixed basis of $\mathbb{F}_q$ seen as a vector space over $\mathbb{F}_p$, and let $\pi_j : \mathbb{F}_q \to \mathbb{F}_p$ be the projection on the coordinate j: for all $\lambda_1, \dots, \lambda_s \in \mathbb{F}_p$, $\pi_j(\sum \lambda_i b_i) = \lambda_j$. For a formal series $A \in \mathbb{F}_q((X))$, $A = \sum a_n X^n$, let $\pi_j(A) \in \mathbb{F}_p((X))$ be defined by $\pi_j(A) = \sum \pi_j(a_n)X^n$. If A is algebraic over $\mathbb{F}_q(X)$ then all $\pi_j(A)$ are algebraic over $\mathbb{F}_p(X)$.*

Proof: If A is algebraic over $\mathbb{F}_q(X)$ then the sequence (a_n) is p-automatic, by Theorem 4.4. By Lemma 4.8 for $k = p$ the sequence $\pi_j((a_n))$ is p-automatic. Applying again Theorem 4.4 for the field $\mathbb{F}_p$, $\pi_j(A)$ must be algebraic over $\mathbb{F}_p(X)$. This proof also works equally in more dimensions. $\square$

5 Main result

Let p be a prime, $q = p^s$, and let $\mathbb{F}_q$ be the field with q elements. We recall that a $\mathbb{F}_p$-linear function $g : \mathbb{F}_q \to \mathbb{F}_q$ is a polynomial

$$g(x) = a_0 x + a_1 x^p + a_2 x^{p^2} + \cdots + a_{s-1} x^{p^{s-1}} = a_0 \varphi^0(x) + a_1 \varphi^1(x) + \cdots + a_{s-1}\varphi^{s-1}(x)$$

where $a_0, \dots, a_{s-1} \in \mathbb{F}_q$, $\varphi(x) = x^p$ is the Frobenius automorphism, $\varphi^0(x) = x$ and $\varphi^{k+1}(x) = \varphi(\varphi^k(x))$. This polynomial representation of the $\mathbb{F}_p$-linear applications of $\mathbb{F}_q$ in itself has the advantage to be independent of the choice of a $\mathbb{F}_p$-basis of $\mathbb{F}_q$. However, in many concrete cases it is more helpful to choose and fix an $\mathbb{F}_p$-basis of $\mathbb{F}_q$ and to work with matrices. Such a situation occurs in the proof of the main result.

Definition 5.1 Let p be a prime and $\mathbb{F}_q$ be the field with $q = p^s$ elements. A function $f : \mathbb{F}_q^m \to \mathbb{F}_q$ is called $\mathbb{F}_p$-*affine* if there are $\mathbb{F}_p$-linear functions $g_1, \dots, g_m : \mathbb{F}_q \to \mathbb{F}_q$ and a constant $t \in \mathbb{F}_q$ such that for all $a_1, \dots, a_m \in \mathbb{F}_q$ one has $f(a_1, \dots, a_m) = g_1(a_1) + \cdots + g_m(a_m) + t$.

Theorem 5.2 *Let p be a prime, and let $\mathbb{F}_q$ be the field with $q = p^s$ elements. Let $(\mathbb{F}_q, f, \vec{v}_1, \dots, \vec{v}_m, c)$ be an n-dimensional recurrence, such that the following conditions are fulfilled:*

1. *The system of predecessors $P = \{\vec{v}_1, \dots, \vec{v}_m\} \subset \mathbb{Z}^n$ is moderate (has excess $e_P = 0$).*
2. *The function $f : \mathbb{F}_q^m \to \mathbb{F}_q$ is $\mathbb{F}_p$-affine.*
3. *$c : C_P \to \mathbb{F}_q$ is an initial condition, such that for all $i = 1, \dots, n$ and for all $a \in \mathbb{N}$, if $(x_i = a) \cap \mathbb{N}^n \subset C_P$, then $c\,|\,(x_i = a) \cap \mathbb{N}^n$ is a p-automatic $(n-1)$-dimensional sequence.*

Then the recurrence $(\mathbb{F}_q, f, \vec{v}_1, \dots, \vec{v}_m, c)$ produces a p-automatic n-dimensional sequence.

Proof: Let $(a(\vec{x}))$ be the n-dimensional sequence produced by the recurrence $(\mathbb{F}_q, f, \vec{v}_1, \dots, \vec{v}_m, c)$ and let $S = \sum a(\vec{x})\vec{X}^{\vec{x}}$ be the corresponding formal series over $\mathbb{F}_q$. We want to prove that S is algebraic over $\mathbb{F}_q(\vec{X})$.

Fix a $\mathbb{F}_p$-basis $\{b_1, \ldots, b_s\}$ of $\mathbb{F}_q$. According to this basis the elements given by initial condition have a linear decompositions $c(\vec{x}) = \sum b_j c^j(\vec{x})$ and the n-dimensional sequence a has a linear decomposition $a(\vec{x}) = \sum b_j a^j(\vec{x})$, for short $a = \sum b_j a^j$. The functions c^j, a^j are defined in the following way: $c^j : C_P \to \mathbb{F}_p$ and $a^j : \mathbb{N}^n \to \mathbb{F}_p$. The formal series $L = \sum d(\vec{x})\vec{X}^{\vec{x}}$, element of $\mathbb{F}_q((\vec{X}))$, has a similar linear decomposition $L = \sum b_j L^j$, where $L^j \in \mathbb{F}_p((\vec{x}))$. According to Lemma 4.9, if L is algebraic over $\mathbb{F}_q(\vec{X})$, then all L^j are algebraic over $\mathbb{F}_p(\vec{X})$.

The $\mathbb{F}_p$-affine function $f : \mathbb{F}_q^k \to \mathbb{F}_q$ is given by $f(a_1, \ldots, a_k) = g_1(a_1) + \cdots + g_m(a_m) + t$, where every $g_r : \mathbb{F}_q \to \mathbb{F}_q$ is given by a square $s \times s$ matrix α^r with elements in $\mathbb{F}_p$. We recall that $S = \sum b_j S^j$. We denote the monomial $a^j(\vec{x})\vec{X}^{\vec{x}}$ by $S^j(\vec{x})$. Let us concentrate our attention on the monomial $S^j(\vec{x})$ for some $\vec{x} \in \mathbb{N}^n \setminus C_P$. The recurrence means that for $\vec{x} \in \mathbb{N}^n \setminus C_P$, the following identity holds:

$$S^j(\vec{x}) = \sum_{\vec{v} \in P} \vec{X}^{\vec{v}} \sum_{i=1}^{s} \alpha_i^r S^i(\vec{x} - \vec{v}) + t^j \vec{X}^{\vec{x}}.$$

Here the constants α_i^r build the row number j in the matrix α^r. Consider the formal series:

$$T^j = S^j - \sum_{\vec{v} \in P} \vec{X}^{\vec{v}} \sum_{i=1}^{s} \alpha_i^r S^i - t^j \sum_{\vec{x} \in \mathbb{N}^n \setminus C_P} \vec{X}^{\vec{x}}.$$

The last term belongs indeed to $\mathbb{F}_p(\vec{X})$:

$$\sum_{\vec{x} \in \mathbb{N}^n \setminus C_P} \vec{X}^{\vec{x}} = \vec{X}^{\vec{d}} \sum_{\vec{y} \in \mathbb{N}^n} \vec{X}^{\vec{y}} = \vec{X}^{\vec{d}} \prod_{i=1}^{n} \frac{1}{1 - X_i},$$

where $\vec{d}$ is the first point to be computed by applying the recurrence, as shown in the Lemma 2.9.

Case 1: $\vec{x} \in \mathbb{N}^n \setminus C_P$. Then $T^j(\vec{x}) = 0$, as shown above.

Case 2: $\vec{x} \in C_P$. In this case:

$$T^j(\vec{x}) = S^j(\vec{x}) - \sum_{\{\vec{v} \in P \,|\, \vec{x} - \vec{v} \in \mathbb{N}^n\}} \vec{X}^{\vec{v}} \sum_{i=1}^{s} \alpha_i^r S^i(\vec{x} - \vec{v}).$$

The system of predecessors lacks excess (i.e. $e_P = 0$), so all $\vec{v} \in P$ have only non-negative coordinates. This means that if $\vec{x} \in C_P$ and $\vec{x} - \vec{v} \in \mathbb{N}^n$ then $\vec{x} - \vec{v} \in C_P$, because C_P is a union of plane slices of the form $K_{i,m} := (x_i = m) \cap \mathbb{N}^n$ containing $K_{i,m}$ for all $m \leq d_i$ - and this is true for all $i = 1, \ldots, n$. Also, we know from the second assumption of the Theorem to prove, combined with Lemma 4.9, that every restriction $S^j \,|\, K_{i,m} = c \,|\, K_{i,m}$ is algebraic over $\mathbb{F}_p(\vec{X})$. Of course there are only finitely many slices $K_{i,m}$.

As P is finite, there are only finitely many possible substets of P able to arise as $\{\vec{v} \in P \,|\, \vec{x} - \vec{v} \in \mathbb{N}^n\}$. It follows that the expression:

$$T^j = \sum_{\vec{x} \in C_P} T^j(\vec{x}) + \sum_{\vec{x} \in \mathbb{N}^n \setminus C_P} T^j(\vec{x}) = \sum_{\vec{x} \in C_P} T^j(\vec{x})$$

is algebraic over $\mathbb{F}_p(\vec{X})$ as linear combination of series which are algebraic over $\mathbb{F}_p(\vec{X})$ with coefficients, which are elements of $\mathbb{F}_p(\vec{X})$. If we rewrite the definition of all series T^j as linear combinations of the series S^j with coefficients in $\mathbb{F}_p(\vec{X})$, we get the following expression:

$$D(\vec{X})\begin{pmatrix} S^1 \\ S^2 \\ \vdots \\ S^s \end{pmatrix} = \begin{pmatrix} C^1 \\ C^2 \\ \vdots \\ C^s \end{pmatrix}.$$

Here $C^1, C^2, \dots C^s \in F_p((\vec{X}))$ are algebraic over $\mathbb{F}_p(\vec{X})$ and the matrix $D(\vec{X})$ is an $s \times s$ matrix over $\mathbb{F}_p(\vec{X})$. But from Lemma 2.12 we know that there exists a unique solution $S^1, \dots, S^s$ of this linear system of equations, so the matrix $D(\vec{X})$ is non-singular. Consequently all series $S^1, \dots, S^s$ are algebraic over $\mathbb{F}_p(\vec{X})$, and $S = \sum b_i S^i$ is algebraic over $\mathbb{F}_q(\vec{X})$. □

Remark 5.3 For all primes p, both constant sequences and ultimately periodic sequences are p-automatic. This explains many examples given by the author with ad-hoc proofs in [19], [20], [21]. Moreover if $\mathbb{F}_q = \mathbb{F}_p$ and the borders consist of constant sequences, the formal series is a rational function over $\mathbb{F}_p$.

Remark 5.4 Theorem 5.2 guarantees the correctness of an algorithmic method to find out the structure of the substitution, which is based on Theorem 3.13. Indeed, one orders all pairs (u, v) of natural numbers in one sequence. For every pair one constructs the initial square $puv \times puv$ of the recurrent n-dimensional sequence and checks the conditions of Theorem 3.13 to see if the n-dimensional sequence is the result of a substitution of type $u \to pu$. For p-affine recurrent sequences this algorithm always stops and outputs a substitution.

At the end of Section 7 are listed some open problems. Three of them concern possible generalizations of Theorem 5.2.

Example: Set $n = 2$, $f(a_1, a_2, a_3) = ua_1 + va_2 + wa_3 + t$ for some fixed elements $u, v, w, t \in \mathbb{F}_q$, $P = \{(0, 1), (1, 1), (1, 0)\}$ and p-automatic initial conditions $(x(n))$, $(y(m))$ as given in Definition 1.1. Let $S(m, n) = a(m, n)X^mY^n$ be the corresponding monomial of the 2-dimensional series S. The recurrence is expressed for monomials with $m, n \geq 1$ by:

$$S(m, n) = uXS(m-1, n) + vXYS(m-1, n-1) + wYS(m, n-1) + tX^mY^n.$$

In order to find an algebraic relation for S we compute

$$T = S(1 - uX - vXY - wY) - t \sum_{m,n\geq 1} X^mY^n.$$

Let $T(m, n)$ be the corresponding monomial of T. For $m, n \geq 1$ one has:

$$T(m, n) = S(m, n) - uXS(m-1, n) - vXYS(m-1, n-1) - wYS(m, n-1) - tX^mY^n = 0, \text{ so}$$

$$S(1 - uX - vXY - wY) - t \sum_{m,n\geq 1} X^mY^n = a(0, 0)(1 - uX - wY) + (1 - uX) \sum_{m\geq 1} x_mX^m +$$
$$+(1 - wY) \sum_{m\geq 1} y_mY^m.$$

We observe that $\sum\limits_{m,n\geq 1} X^mY^n = XY(1 - X)^{-1}(1 - Y)^{-1}$ and that $A(X) = (1 - uX) \sum x_mX^m \in \mathbb{F}_q((X, Y))$ and $B(Y) = (1 - wY) \sum y_mY^m \in \mathbb{F}_q((X, Y))$ are algebraic over $\mathbb{F}_q(X, Y)$ because of Theorem 4.4 and the fact that the sequences (x_i) and (y_j) are p-automatic. This implies that:

$$S = [tXY(1 - X)^{-1}(1 - Y)^{-1} + a(0, 0)(1 - uX - wY) + A(X) + B(Y)](1 - uX - vXY - wY)^{-1}$$

is algebraic over $\mathbb{F}_q(X, Y)$.

6 Nonautomatic recurrent 2-dimensional sequences

It is natural to ask whether Theorem 5.2 remains true for general functions $f : A^m \to A$. The answer is negative, even for systems of predecessors P with $e_P = 0$. One example of nonautomatic 2-dimensional sequence is given in [1]: Pascal's Triangle modulo 6. This 2-dimensional sequence can be defined as a recurrent sequence by $A = \mathbb{Z}/6\mathbb{Z}$, $P = \{(0,1),(1,0)\}$, $f(x,y) = x + y$ and initial conditions $c(\{0\} \times \mathbb{N}) = c(\mathbb{N} \times \{0\}) = 1$. According to the Chinese Remainder Theorem $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, so the 2-dimensional sequence is an overlapping of two 2-dimensional sequences, one of them being 2- and the other one 3-automatic. However, the proof of this non-automaticity is not trivial (see Theorem 14.6.2 in [1]). The counterexample given here is based upon a different phenomenon.

Theorem 6.1 *There is no $k \in \mathbb{N}$ such that the recurrent 2-dimensional sequence $a : \mathbb{N}^2 \to \mathbb{F}_5$ defined by $P = \{(0,1),(1,1),(1,0)\}$, $f(x,y,z) = 2x^3y^3z^3 + 2xy^2 + 2y^2z + y$ and initial conditions $c(\{0\} \times \mathbb{N}) = c(\mathbb{N} \times \{0\}) = 1$ is k-automatic. See Figure 10.*

Proof: Suppose that the 2-dimensional sequence $a(m,n)$ is k-automatic for some $k \geq 2$. Define the 1-dimensional sequence $b(n) := a(n+1, n+1)$ for all $n \in \mathbb{N}$. According to Lemma 8.4, by projecting the set $\{b, B, a\}$ onto $\{0,1\}$ such that b and B are replaced by 0 and a is replaced by 1, $b(n)$ has the following structure:

$$0 \,|\, 0,1 \,|\, 1,0,1 \,|\, 1,0,1,0 \,|\, 0,1,0,1,0 \,|\, 0,1,0,1,0,1 \,|\, 1,0,1,0,1,0,1 \,|\, 1 \ldots$$

The sequence $b(n)$ consists of an alternate word of length 1, followed by an alternate word of length 2 starting with the same letter in which the last alternate word ends, and so on. After an alternate word of length n follows an alternate word of length $n+1$, whose first letter repeats the last letter of the precedent word.

In [1] a *finite-state transducer* is defined as an automaton $T = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ where Q is a finite set of states, Σ an input alphabet, $\delta : Q \times \Sigma \to Q$ a transition function, q_0 an initial state, Δ an output alphabet and $\lambda : Q \times \Sigma \to \Delta^*$ an output function. In the case that there is an integer t such that for all $q \in Q$ and $a \in \Sigma$ one has $|\,\lambda(q,a)\,| = t$ one says that the transducer is t-uniform. We construct an 1-uniform transducer T as follows: $Q = \{u_1, u_2, z_1, z_2\}$, where $q_0 = z_1$ plays the role of initial state. We take $\Sigma = \Delta = \{0,1\}$ and we define the transition function as follows: $\delta(z_1, 0) = z_2$, $\delta(z_1, 1) = u_1$, $\delta(u_1, 0) = z_1$, $\delta(u_1, 1) = u_2$, $\delta(z_2, 0) = z_1$, $\delta(z_2, 1) = u_1$, $\delta(u_2, 0) = z_1$, $\delta(u_2, 1) = u_1$. The output function λ is defined such that $\lambda(u_1, x) = \lambda(z_1, x) = 0$ and $\lambda(u_2, x) = \lambda(z_2, x) = 1$, for all $x \in \{0,1\}$. The image of a sequence $x : \mathbb{N} \to \Sigma$ by the transducer T is defined as:

$$T(x) = \lambda(q_0, x(0))\lambda(\delta(q_0, x(0)), x(1))\lambda(\delta(\delta(q_0, x(0)), x(1)), x(2)) \ldots.$$

We apply the transducer T constructed above to the sequence b, let $c := T(b)$:

$$0 \,|\, 1,0 \,|\, 1,0,0 \,|\, 1,0,0,0 \,|\, 1,0,0,0,0 \,|\, 1,0,0,0,0,0 \,|\, 1,0,0,0,0,0,0 \,|\, 1 \ldots$$

We see by induction that $c(n) = 1$ if and only if there exists $m \geq 1$ such that $n = m(m+1)/2$. The sequence $c : \mathbb{N} \to \{0,1\}$ is the characteristic function of the set of triangular numbers. In fact we can modify the first term to 1 and delete the condition $k \geq 1$; such modifications of finitely many terms have no influence on the automaticity of the sequence. According to Theorem 6.9.2 in [1] the image of a k-automatic sequence under a 1-uniform transducer is k-automatic. It remains to show that the characteristic function of the triangular numbers cannot be an automatic sequence. This follows from a more general statement:

Lemma 6.2 *Let $f \in \mathbb{Q}[X]$ be a polynomial such that $f(\mathbb{N}) \subset \mathbb{N}$ and $\deg(f) \geq 2$. Let $x : \mathbb{N} \to \{0,1\}$ be the characteristic function of the set $f(\mathbb{N})$. Then the 1-dimensional sequence $(x(n))$ is not k-automatic for any k.*

Proof: This is a direct consequence of the following dichotomy: An infinite set of natural numbers $S = \{s_1, s_2, \dots\}$ with $s_1 < s_2 < \cdots$ is k-automatic then either is *syndetic*, i.e. there is some $C > 0$ such that $s_{i+1} - s_i < C$ for all i, or there is some $\epsilon > 0$ such that $s_{i+1}/s_i > 1 + \epsilon$ for infinitely many i. The first occurrence of this result was in A. Cobham's clasical paper [6] as Theorem 10, p. 184. Moreover, if $\deg(f) \le 1$ then the sequence is k-automatic for all k. $\Box$

7 Examples

All examples are 2-dimensional sequences. Excepting the last subsection, all examples given uses the system of predecessors $(0, 1)$, $(1, 1)$, $(1, 0)$ or one of its subsets. Also, we look now only to $\mathbb{F}_p$-affine functions, neglecting other homomorphisms of p-groups.

7.1 Tensor power carpets and patchwork carpets

Definition 7.1 Let $\mathbb{F}_p$ be a prime finite field. By the term *tensor power carpet* we mean a recurrent 2-dimensional sequence $(\mathbb{F}_p, (1, 0), (1, 1), (0, 1), x + my + z, 1)$ where $m \in \mathbb{F}_p$. By the term *patchwork carpet* we mean a recurrent 2-dimensional sequence $(\mathbb{F}_p, (1, 0), (1, 1), (0, 1), ax + by + cz + d, e)$ where $a, b, c, d, e \in \mathbb{F}_p$.

In [18] the author proved that the recurrent 2-dimensional sequences $(\mathbb{F}_p, (1, 0), (1, 1), (0, 1), x + my + z, 1)$ are indeed tensor powers of their starting $p \times p$ left-upper minor, called there a fundamental block. Their geometry can be roughly classified according to the parameter m. Those recurrent 2-dimensional sequences contain big square regions which are constantly equal to 0, called the holes. The name *carpets* comes from the previously known example of this kind, $(\mathbb{F}_3, (1, 0), (1, 1), (0, 1), x + y + z, 1)$ - the Sierpinski Carpet. For $m = 0$ one gets Pascal's Triangle modulo p. For both examples, see [18].

On the other hand the name of the *patchwork carpets* comes fom the empiric observation that those carpets have in general the same geometric content as the tensor power carpets, up to the fact that the holes are filled with periodic domains giving the illusion of a patchwork with different kinds of tissues. Some exceptions are given by some 2-dimensional sequences which are periodic or which present only two tissues, one over the main diagonal and one below it.

In Figure 1 we see the carpet ($\mathbb{F}_3$,(1,0),(1,1),(0,1), $x + y$, 1), which is a patched Pascal Triangle modulo 3. In Figure 3 we see $(\mathbb{F}_5, (1, 0), (1, 1), (0, 1), 3x + 4y + 3z + 3, 1)$. This carpet has the geometric behavior of the tensor power carpet $(\mathbb{F}_5, (1, 0), (1, 1), (0, 1), x+y+z, 1)$, Figure 2 in the sense that every hole of $(\mathbb{F}_5, (1, 0), (1, 1), (0, 1), x+y+z, 1)$ (see [18] for some properties) is covered by some homogeneous patch.

Applying the Theorem 3.13 we get rid of:

Example 7.2 *The 2-dimensional sequence $(\mathbb{F}_3, (1, 0), (1, 1), x + y, 1)$ is 3-automatic and can be generated by a substitution of type $3 \to 9$ with 9 rules. See Figure 1.*

The system of substitutions $(\{0, 1\}, \mathcal{D}, \mathcal{E}, D_1, \Sigma)$ consists of the following sets. The set $\mathcal{D} = \{D_1, \dots, D_9\}$:

$$D_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad D_3 = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$D_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad D_5 = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix} \quad D_6 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

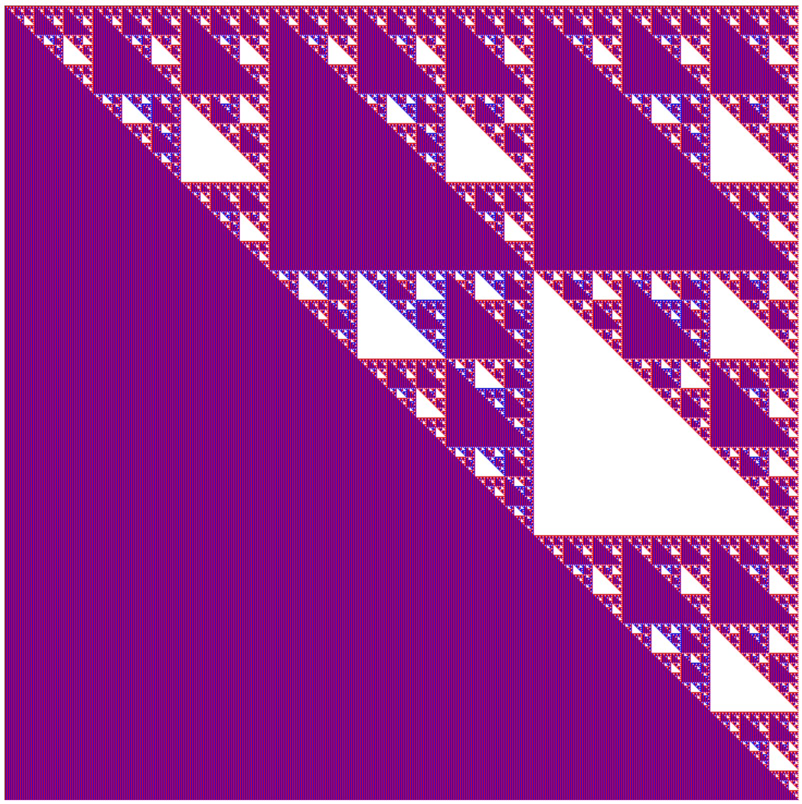

Figure 1: $\mathbb{F}_3$, $(1,0)$, $(1,1)$, $x+y$, 1 , 729×729.

$$D_7 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \; D_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; D_9 = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

The set $\mathcal{E} = \{E_1, \ldots, E_9\}$ as given below. The function $\Sigma : \mathcal{D} \to \mathcal{E}$ is defined such that $\Sigma(D_i) = E_i$ for all $i = 1, \ldots, 9$.

$$E_1 = \begin{pmatrix} D_1 & D_1 & D_1 \\ D_2 & D_3 & D_4 \\ D_2 & D_5 & D_1 \end{pmatrix} \; E_2 = \begin{pmatrix} D_2 & D_5 & D_2 \\ D_2 & D_5 & D_2 \\ D_2 & D_5 & D_2 \end{pmatrix} \; E_3 = \begin{pmatrix} D_3 & D_6 & D_3 \\ D_5 & D_1 & D_7 \\ D_5 & D_2 & D_3 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} D_4 & D_7 & D_4 \\ D_8 & D_4 & D_1 \\ D_8 & D_8 & D_4 \end{pmatrix} \; E_5 = \begin{pmatrix} D_5 & D_2 & D_5 \\ D_5 & D_2 & D_5 \\ D_5 & D_2 & D_5 \end{pmatrix} \; E_6 = \begin{pmatrix} D_6 & D_3 & D_6 \\ D_8 & D_6 & D_9 \\ D_8 & D_8 & D_6 \end{pmatrix}$$

$$E_7 = \begin{pmatrix} D_7 & D_4 & D_7 \\ D_2 & D_9 & D_3 \\ D_2 & D_5 & D_7 \end{pmatrix} \; E_8 = \begin{pmatrix} D_8 & D_8 & D_8 \\ D_8 & D_8 & D_8 \\ D_8 & D_8 & D_8 \end{pmatrix} \; E_9 = \begin{pmatrix} D_9 & D_9 & D_9 \\ D_5 & D_7 & D_6 \\ D_5 & D_2 & D_9 \end{pmatrix}$$

Example 7.3 *The 2-dimensional sequence $(\mathbb{F}_5, (1,0), (1,1), (0,1), 3x+4y+3z+3, 1)$ is 5-automatic and can be generated by a substitution of type $5 \to 25$ with 125 rules. See Fig. 3. One remarks that this 2-dimensional sequence has the same geometric behavior like the tensor power carpet $(\mathbb{F}_5, x+y+z, 1)$ shown in Figure 3. The last is a substitution of type $1 \to 5$ with 5 rules, see [18] for the proof.*

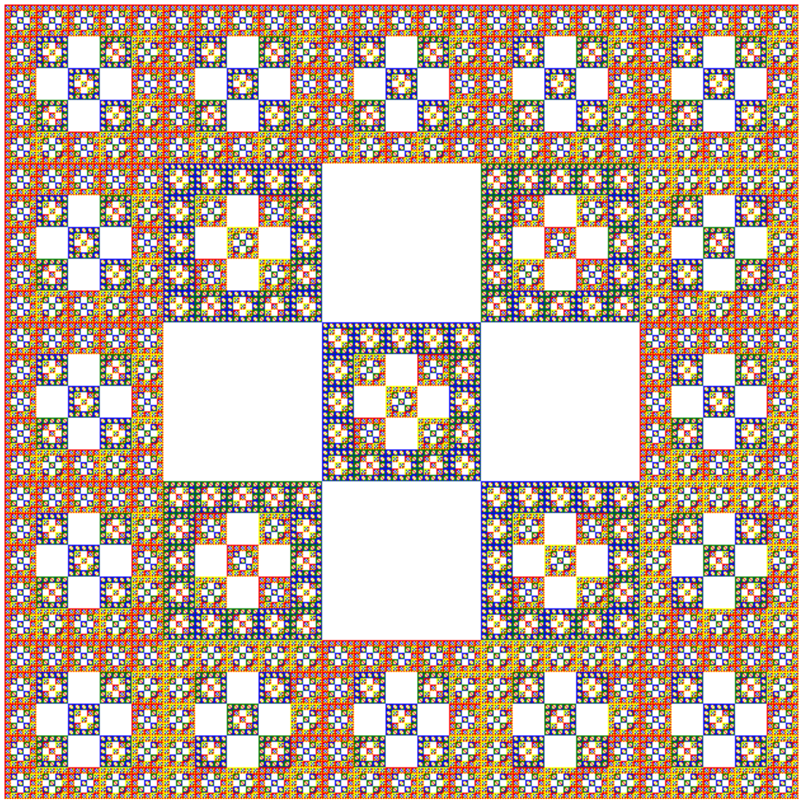

Figure 2: $\mathbb{F}_5$, $(1,0)$, $(1,1)$, $(0,1)$, $x+y+z$, 1 , 625×625.

7.2 The free term

There are seldom cases where by adding a free term to a linear rule of recurrence one obtains some geometric behavior that cannot be obtained using only linear recurrences.

In order to give an example we shortly recall the structure of the field $\mathbb{F}_4$. $\mathbb{F}_4 = \mathbb{F}_2[\omega]$ where ω is a solution of the equation $\omega^2 + \omega + 1 = 0$ over $\mathbb{F}_2$. The elements of $\mathbb{F}_4$ are $\{0, 1, \omega, \omega + 1\}$ and the set $\{1, \omega\}$ is a $\mathbb{F}_2$-basis of the $\mathbb{F}_2$-vector field $\mathbb{F}_4$.

Now consider the $\mathbb{F}_2$-linear function $f(x, y, z) : \mathbb{F}_4^3 \to \mathbb{F}_4$ given by $g(x, y, z) = \omega x + y + \omega z$. The recurrent 2-dimensional sequence $(\mathbb{F}_4, (1, 0), (1, 1), (0, 1), g, 1)$ is constant, because $g(1, 1, 1) = 1$. But if we consider instead $f(x, y, z) = g(x, y, z) + k$ with $k \in \{1, \omega, \omega + 1\}$ we get a totally new geometric behavior, which cannot be found by any linear recurrence over $\mathbb{F}_4$. See [21] for a classification of all those recurrent 2-dimensional sequences, and for other properties of this 2-dimensional sequence, last but not least that it can be generated by a primitive substitution.

Example 7.4 *The 2-dimensional sequence* $(\mathbb{F}_4, (1, 0), (1, 1), (0, 1), \omega x+y+\omega z+\omega, 1)$ *is 2-automatic and can be generated by a substitution of type* $2 \to 4$ *with 12 rules. See Figure 4.*

7.3 Non-trivial occurrences of Frobenius

The non-trivial occurrences of Frobenius in the rule of recurrence enlarge sensibly the set of examples. For a systematic review of all linear recurrent 2-dimensional sequences over $\mathbb{F}_4$ we refer again to [21].

Example 7.5 *The 2-dimensional sequence* $(\mathbb{F}_4, (1, 0), (1, 1), (0, 1), \omega x + \omega x^2 + \omega y + \omega z + \omega z^2, 1)$

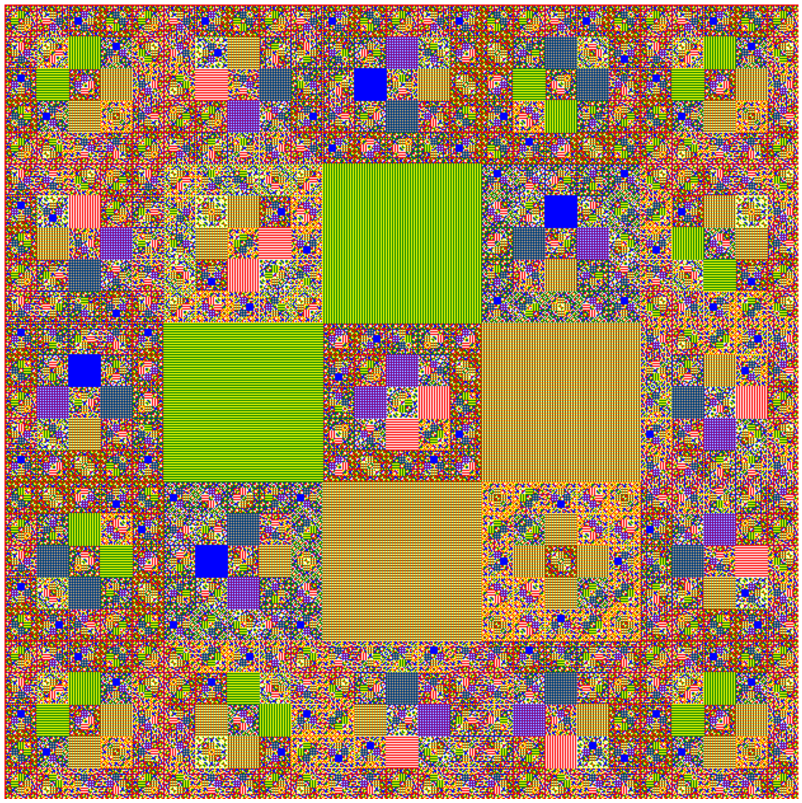

Figure 3: $\mathbb{F}_5$, $(1,0)$, $(1,1)$, $(0,1)$, $3x+4y+3z+3$, 1 , 625×625.

is 2-automatic and can be generated by a substitution of type $2\to 4$ with 41 rules. See Figure 5.

7.4 Non-constant borders

The first examples of non-constant borders studied by the author were periodic borders, see [20]. It is known that all ultimately periodic sequences are k-automatic, for all $k\geq 2$. Here, we prefer to present two examples with nonperiodic automatic borders. The first example uses the Prouhet-Thue-Morse sequence, which is a morphic sequence; the second one uses the Rudin-Shapiro sequence, which can be generated by a one-dimensional substitution of type $2\to 4$.

Definition 7.6 The *Prouhet-Thue-Morse Sequence* $t(n)$ is a 2-automatic sequence $t(n)$ produced by the following uniform morphism (substitution of type $1\to 2$): $0\to 01$, $1\to 10$ with start symbol 0. The *Thue-Morse-Pascal 2-dimensional Sequence* is the recurrent 2-dimensional sequence $(\mathbb{F}_2,(1,0),(0,1),x+y,t,t)$.

Example 7.7 *The Thue-Morse-Pascal 2-dimensional sequence is 2-automatic and can be generated by a substitution of type $4\to 8$ with 15 rules. See Figure 6.*

The system of substitutions $(\{0,1\},\mathcal{D},\mathcal{E},D_1,\Sigma)$ consists of the following sets. The set $\mathcal{D}=\{D_1,\dots,D_{15}\}$ consists of the matrices:

$$D_1=\begin{pmatrix}0&1&1&0\\1&0&1&1\\1&1&0&1\\0&1&1&0\end{pmatrix}\quad D_2=\begin{pmatrix}1&0&0&1\\0&0&0&1\\1&1&1&0\\1&0&1&1\end{pmatrix}\quad D_3=\begin{pmatrix}1&0&0&1\\0&0&0&1\\0&0&0&1\\1&1&1&0\end{pmatrix}\quad D_4=\begin{pmatrix}0&1&1&0\\1&0&1&1\\0&0&1&0\\0&0&1&1\end{pmatrix}$$

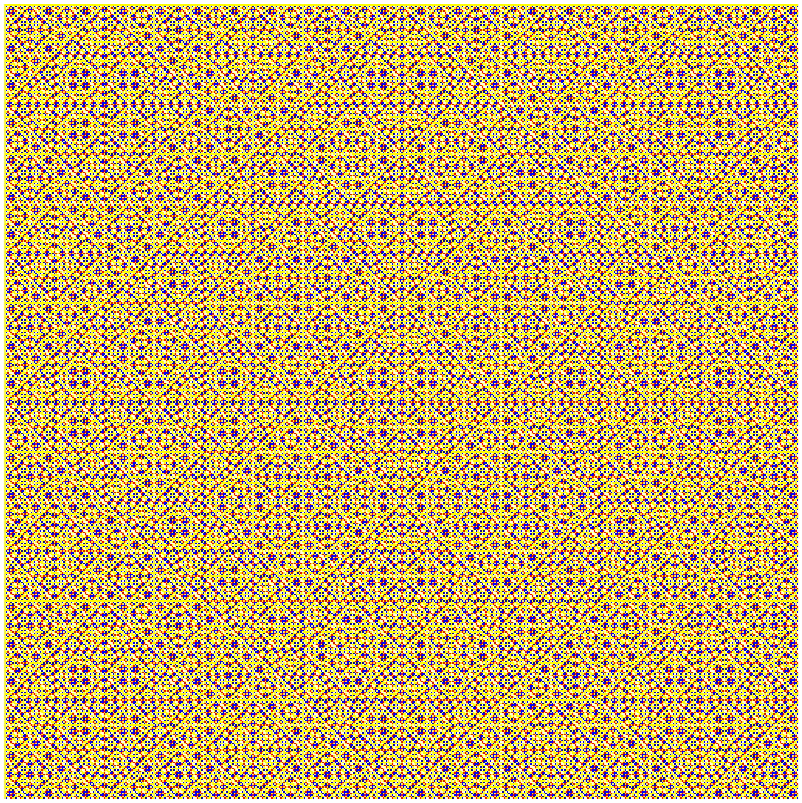

Figure 4: $\mathbb{F}_4$, $(1,0)$, $(1,1)$, $(0,1)$, $\omega x+y+\omega z+\omega$, 1, 512×512.

$$D_5=\begin{pmatrix}1&0&1&1\\0&0&1&0\\0&0&1&1\\1&1&0&1\end{pmatrix}\quad D_6=\begin{pmatrix}0&0&1&0\\0&0&1&1\\1&1&0&1\\0&1&1&0\end{pmatrix}\quad D_7=\begin{pmatrix}1&0&1&1\\0&0&1&0\\1&1&0&0\\1&0&0&0\end{pmatrix}\quad D_8=\begin{pmatrix}1&1&0&1\\1&0&0&1\\1&1&1&0\\1&0&1&1\end{pmatrix}$$

$$D_9=\begin{pmatrix}0&1&0&0\\1&0&0&0\\1&1&1&1\\0&1&0&1\end{pmatrix}\quad D_{10}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad D_{11}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad D_{12}=\begin{pmatrix}1&1&1&1\\1&0&1&0\\1&1&0&0\\1&0&0&0\end{pmatrix}$$

$$D_{13}=\begin{pmatrix}0&0&1&0\\0&0&1&1\\0&0&1&0\\0&0&1&1\end{pmatrix}\quad D_{14}=\begin{pmatrix}1&1&1&1\\1&0&1&0\\0&0&1&1\\1&1&0&1\end{pmatrix}\quad D_{15}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&1&1&1\\0&1&0&1\end{pmatrix}$$

The set $\mathcal{E}=\{E_1,\ldots,E_{15}\}$ consists of the matrices:

$$E_1=\begin{pmatrix}D_1&D_2\\D_5&D_6\end{pmatrix}\qquad E_2=\begin{pmatrix}D_3&D_4\\D_7&D_8\end{pmatrix}\qquad E_3=\begin{pmatrix}D_3&D_4\\D_9&D_6\end{pmatrix}\qquad E_4=\begin{pmatrix}D_1&D_2\\D_{10}&D_8\end{pmatrix}$$

$$E_5=\begin{pmatrix}D_3&D_7\\D_9&D_{14}\end{pmatrix}\qquad E_6=\begin{pmatrix}D_{11}&D_8\\D_{14}&D_6\end{pmatrix}\qquad E_7=\begin{pmatrix}D_3&D_7\\D_7&D_{11}\end{pmatrix}\qquad E_8=\begin{pmatrix}D_{12}&D_{13}\\D_{12}&D_8\end{pmatrix}$$

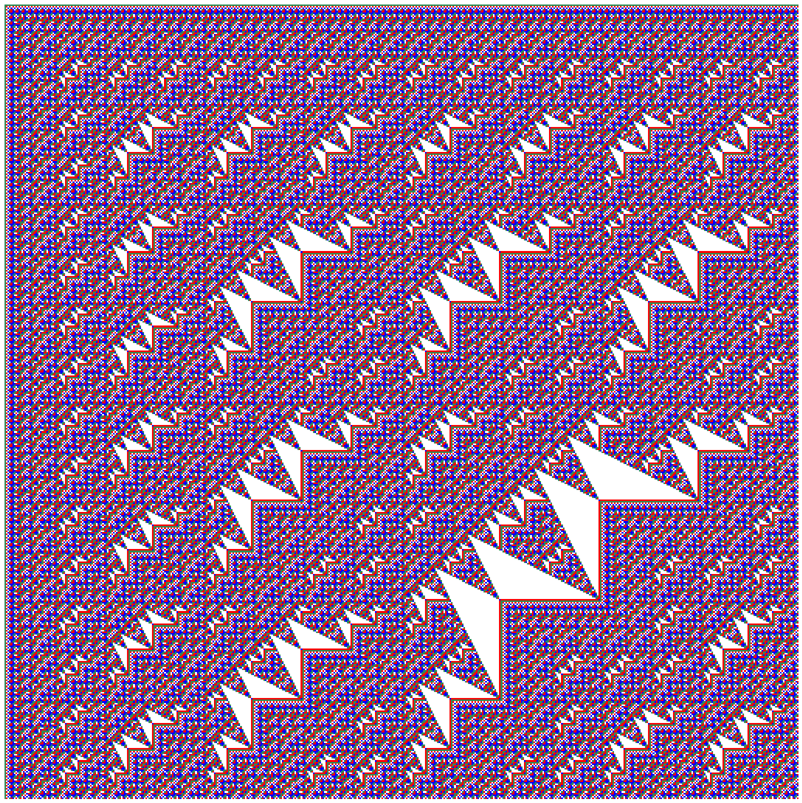

Figure 5: $\mathbb{F}_4$, $(1,0)$, $(1,1)$, $(0,1)$, $\omega x + \omega x^2 + \omega y + \omega z + \omega z^2$, 1, 512×512.

$$E_9 = \begin{pmatrix} D_1 & D_{10} \\ D_5 & D_{14} \end{pmatrix} \quad E_{10} = \begin{pmatrix} D_1 & D_{10} \\ D_{10} & D_{11} \end{pmatrix} \quad E_{11} = \begin{pmatrix} D_{11} & D_{11} \\ D_{11} & D_{11} \end{pmatrix} \quad E_{12} = \begin{pmatrix} D_{12} & D_{12} \\ D_{12} & D_{11} \end{pmatrix}$$

$$E_{13} = \begin{pmatrix} D_{11} & D_8 \\ D_{11} & D_8 \end{pmatrix} \qquad E_{14} = \begin{pmatrix} D_{12} & D_{12} \\ D_{15} & D_{14} \end{pmatrix} \qquad E_{15} = \begin{pmatrix} D_{11} & D_{11} \\ D_{14} & D_{14} \end{pmatrix}$$

The matrix D_1 is the start symbol, and $\forall\, i\ \Sigma(D_i) = E_i$.

We can verify by hand the first condition of Theorem 3.13. Only the substitution rules for D_1, D_2, D_3 and D_4 touch the horizontal border. Let $M_i = D_i \cap (y = 0)$. The relevant parts of the substitution rules read $M_1 \to M_1M_2$, $M_2 \to M_3M_4$, $M_3 \to M_3M_4$ and $M_4 \to M_1M_2$, where $M_1 = M_4 = 0110$ and $M_2 = M_3 = 1001$. So the horizontal border is the sequence given by start word 0110 and by the rules α : $0110 \to 01101001$ and β : $1001 \to 10010110$. Let $h : \{0,1\}^* \to \{0,1\}^*$ be the homomorphism of monoids h whose fixed point is the Prouhet-Thue-Morse sequence. The homomorphism h is defined by $h(0) = 01$ and $h(1) = 10$. The start word $0110 = h^2(0)$ and every further complete substitution step using both rules α and β has the same effect as applying h^2. It follows by induction that the horizontal border is exactly the Prouhet-Thue-Morse sequence. The proof is similar for the vertical border.

The second condition of Theorem 3.13 has been checked using a computer program. The program generated a 8000×8000 initial square of the recurrent 2-dimensional sequence, checked that this square was identical with the corresponding square produced by substitution, and checked the fact that all 8×8-squares occurring in 4-position have already occurred in some 4-positions in the 4000×4000 left-upper quarter of this initial square. This way the conditions of Theorem 3.13 are fulfilled. $\square$

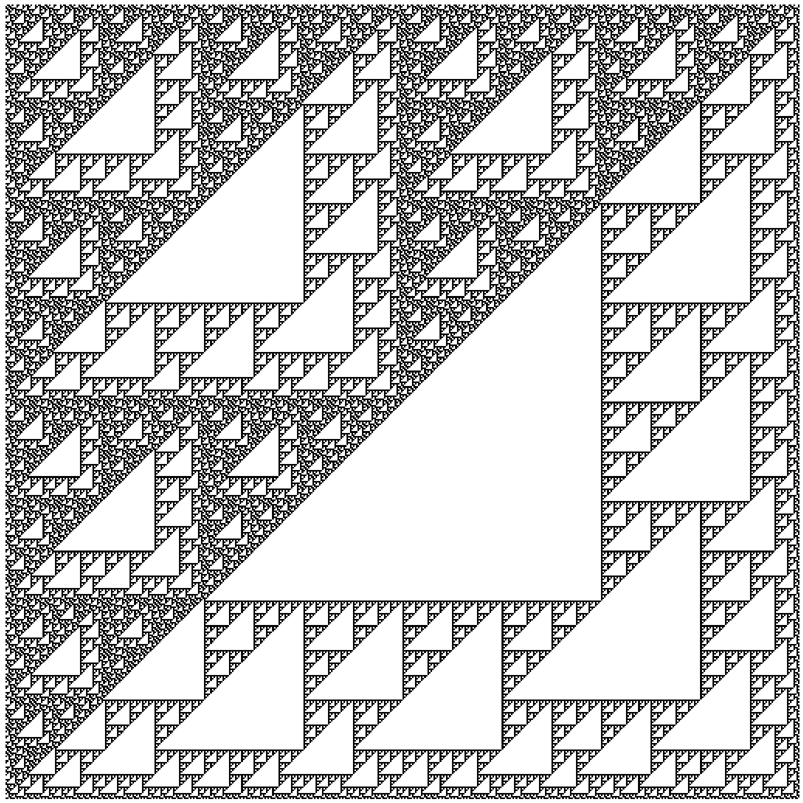

Figure 6: $\mathbb{F}_2$, $(0,1)$, $(1,0)$, $x+y$, Thue-Morse, 512×512.

For the next example we use the Rudin-Shapiro sequence, which is originally a sequence in the alphabet $\{+1, -1\}$. Because -1 does not make sense in $\mathbb{F}_2$ we use here an isomorphic sequence over the alphabet $\{0, 1\}$.

Definition 7.8 The *Rudin-Shapiro Sequence* $r(n)$ is a 2-automatic sequence produced by the following substitution of type $2 \to 4$: $00 \to 0001$, $01 \to 0010$, $10 \to 1101$, $11 \to 1110$ with start word 00. The *Rudin-Shapiro-Pascal* 2*-dimensional Sequence* is the recurrent 2-dimensional sequence $(\mathbb{F}_2, (1,0), (0,1), x+y, r, r)$.

Example 7.9 *The Rudin-Shapiro-Pascal* 2*-dimensional sequence is* 2*-automatic and can be generated by a substitution of type* $4 \to 8$ *with* 44 *rules. See Figure 7.*

7.5 Excessive recurrence

Extremely interesting are the examples of recurrence with positive excess, where the alphabet is some finite abelian p-group and the recurrence is a homomorphism of p-groups. Although they are not proven to be automatic by Theorem 5.2, in all examples computed by the author one can guess a substitution and then prove the identity of the 2-dimensional sequence obtained by substitution with the 2-dimensional sequence got by recurrence, using Theorem 3.13. We show two examples:

Example 7.10 *The* 2*-dimensional sequence defined by a recurrence with system of predecessors* $(0,1)$*,* $(1,-1)$ *and the rule* $x+y$ *over the field* $\mathbb{F}_2$*, with initial condition* $a(i,0) = 1$*,* $a(0,j) = 1$*, is* 2*-automatic and can be generated by a substitution of type* $4 \to 8$ *with* 16 *rules. See Figure 8.*

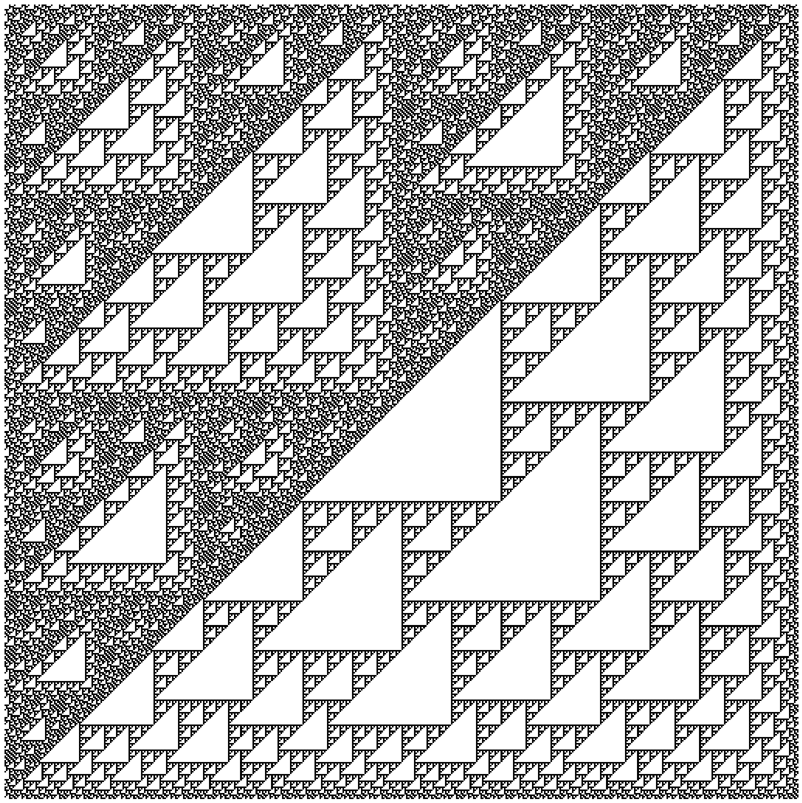

Figure 7: $\mathbb{F}_2$, $(0,1)$, $(1,0)$, $x+y$, Rudin-Shapiro, 512×512.

Example 7.11 *The 2-dimensional sequence defined by a recurrence with system of predecessors* $(1,0)$, $(1,-2)$, $(0,1)$ *and the rule* $x+y+z$ *over the field* $\mathbb{F}_2$, *with initial condition* $a(i,0) = t(i)$, $a(0,j) = t(j)$, *where* t *is the Thue-Morse sequence starting with* 0, *is* 2*-automatic and can be generated by a substitution of type* $4 \to 8$ *with* 32 *rules. See Figure 9.*

Open problems:

1. Prove that every non-periodic and non-diagonal patchwork carpet has the geometric behavior of some tensor power carpet.
2. Find a connection between the coefficients of a patchwork carpet and its geometric behavior.
3. Does Theorem 5.2 hold true for excessive recurrence?
4. Does Theorem 5.2 remain true for arbitrary homomorphisms of groups $f : G^m \to G$, where G is any finite abelian p-group?
5. Does Theorem 5.2 remain true in the most general setting in which we dare to formulate that question: general recurrence and homomorphisms of finite abelian p-groups?

Up to now all examples computed by the author confirm positive conjectures for the questions 1, 3, 4 and 5.

8 Appendix: The anatomy of Stairway

The recurrent two-dimensional sequence Stairway (see Figure 10) has the system of predecessors $\{(1,0),(1,1),(0,1)\}$, constant borders $a(i,0) = a(0,j) = 1$ and is given by the polynomial

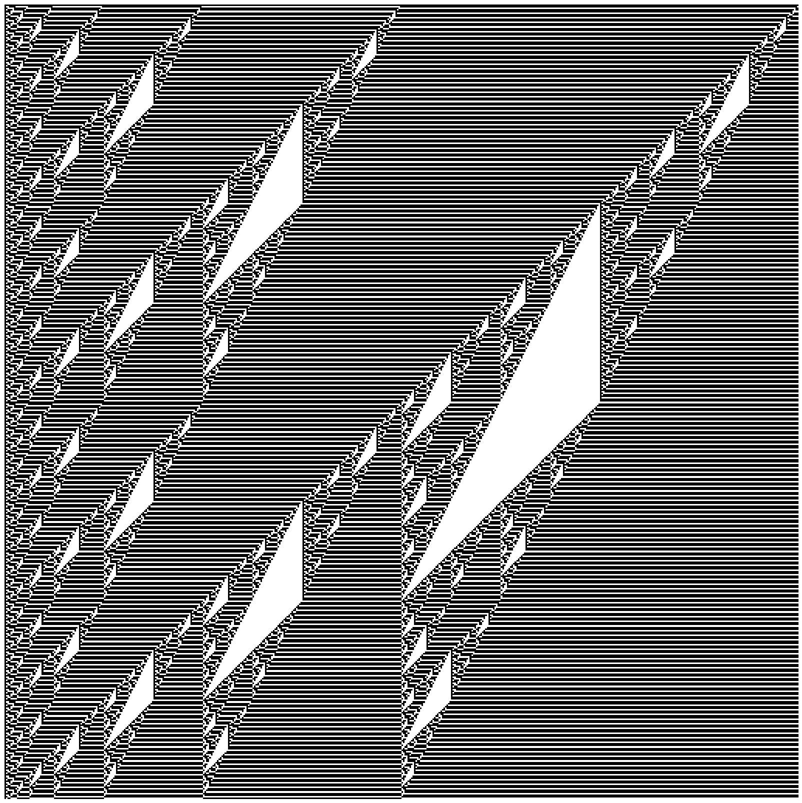

Figure 8: $\mathbb{F}_2$, $(0,1)$, $(1,-1)$, $x+y$, 1, 512×512.

$f(x,y,z) = 2x^3y^3z^3 + 2xy^2 + 2y^2z + y$ over $\mathbb{F}_5$. The goal of this Appendix is a short study of this recurrent 2-dimensional sequence, in order to complete the proof of Theorem 6.1.

The function $f(x,y,z)$ has the property $f(x,y,z) = f(z,y,x)$. This implies that the recurrent 2-dimensional sequence fulfills $a(m,n) = a(n,m)$, i.e. is diagonally symmetric.

In spite of the fact that there are 125 triples $(a,b,c) \in \mathbb{F}_5^3$, we will see that only 32 triples consisting of coordinates different from 0 really appear in the sequence. In order to understand how the sequence works, it is useful to enumerate them here. All rules of recurrence will be represented in the form:

$$R = \begin{pmatrix} y & z \\ x & f(x,y,z) \end{pmatrix}$$

Because of the diagonal symmetry stated above, if a rule R does concretely occur in the recurrent 2-dimensional sequence, its transposed R^T also occurs in the sequence. This fact help us for a faster and easier enumeration of the rules.

We change the notation of the elements of $\mathbb{F}_5 \setminus \{0\}$ in the following way: $1 = a$, $2 = B$, $3 = A$ and $4 = b$. In Figure 10 you see 1 = red, 2 = green, 3 = blue and 4 = yellow. According to this notation, there are:

6 **Sporadic Rules**

$$S_1 = \begin{pmatrix} a & a \\ a & B \end{pmatrix},\ S_2 = \begin{pmatrix} a & a \\ B & A \end{pmatrix},\ S_2^T,\ S_3 = \begin{pmatrix} B & A \\ A & b \end{pmatrix},\ S_4 = \begin{pmatrix} A & A \\ b & a \end{pmatrix},\ S_4^T.$$

2 **Square Start Rules**

$$N_1 = \begin{pmatrix} a & A \\ A & a \end{pmatrix},\ N_2 = \begin{pmatrix} b & B \\ B & b \end{pmatrix}.$$

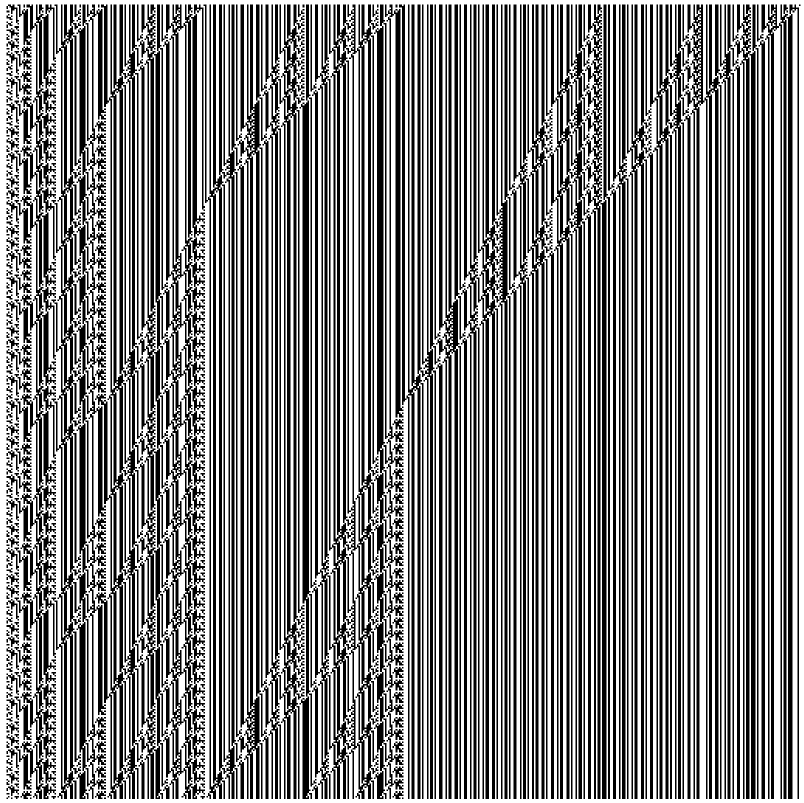

Figure 9: $\mathbb{F}_2$, $(1,0)$, $(1,-2)$, $(0,1)$, $x+y+z$, Thue-Morse, 512×512.

4 **Alternate Square Margin Rules**

$$M_1 = \begin{pmatrix} A & B \\ a & b \end{pmatrix},\ M_1^T,\ M_2 = \begin{pmatrix} B & A \\ b & a \end{pmatrix},\ M_2^T.$$

4 **Exterior Stripe Rules**

$$E_1 = \begin{pmatrix} A & A \\ B & B \end{pmatrix},\ E_1^T,\ E_2 = \begin{pmatrix} B & B \\ A & A \end{pmatrix},\ E_2^T.$$

4 **Interior Stripe Rules**

$$I_1 = \begin{pmatrix} a & a \\ b & b \end{pmatrix},\ I_1^T,\ I_2 = \begin{pmatrix} b & b \\ a & a \end{pmatrix},\ I_2^T.$$

2 **Wave Rules**

$$W_1 = \begin{pmatrix} A & A \\ A & B \end{pmatrix},\ W_2 = \begin{pmatrix} B & B \\ B & A \end{pmatrix}.$$

4 **Square Corner Rules**

$$C_1 = \begin{pmatrix} A & A \\ a & A \end{pmatrix},\ C_1^T,\ C_2 = \begin{pmatrix} B & B \\ b & B \end{pmatrix},\ C_2^T.$$

2 **Diagonal Rules**

$$D_1 = \begin{pmatrix} a & b \\ b & b \end{pmatrix},\ D_2 = \begin{pmatrix} b & a \\ a & a \end{pmatrix}.$$

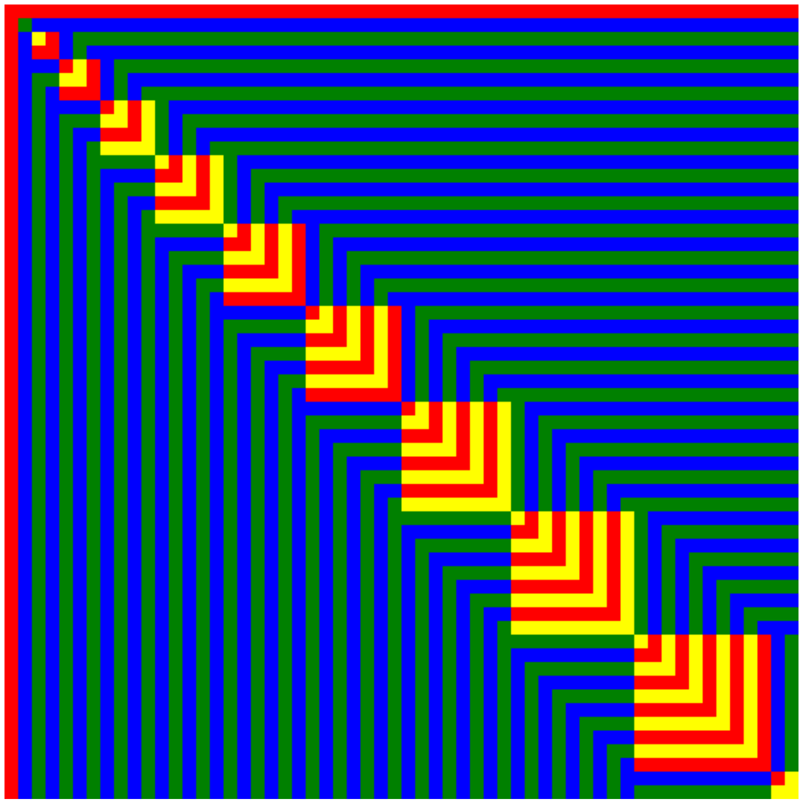

Figure 10: $\mathbb{F}_5$, $(0,1)$, $(1,1)$, $(1,0)$, $2x^3y^3z^3 + 2xy^2 + 2y^2z + y$, 1, 58×58.

4 **Monochrome Square Margin Rules**

$$Q_1 = \begin{pmatrix} a & a \\ A & A \end{pmatrix},\ Q_1^T,\ Q_2 = \begin{pmatrix} b & b \\ B & B \end{pmatrix},\ Q_2^T.$$

This enumeration of matrices must not be confounded with seemingly similar lists written down in Section 7. All those sets of matrices are left-hand sides of substitution rules, so they will finally build together the set $D_d(a)$ for some 2-dimensional sequences a. In other words, those $d \times d$ matrices will arise in the respective 2-dimensional sequences in d-positions. On the other side, the rules of recurrence listed here build the covering $C_2(a)$ for the 2-dimensional sequence a called Stairway.

Definition 8.1 Let $t(n) = n(n+1)/2$ be the n-th triangular number. Let T_n be the square starting at $(t(n)+1, t(n)+1)$ of edge-length $n+1$, that is $T(n) = \{t(n)+1, \dots, t(n)+n+1\} \times \{t(n)+1, \dots, t(n)+n+1\}$. The eck-points of $T(n)$ are: $e(n) = (t(n)+1, t(n)+1)$, $v(n) = (t(n)+n+1, t(n)+1)$, $f(n) = (t(n)+n+1, t(n)+n+1)$ and $w(n) = (t(n)+1, t(n)+n+1)$. The square $T(n)$ contains its eck-points.

Definition 8.2 The wave $V(n)$ consists of the union of the sets $\{v(n) + (1,-1) + (k,k) \,|\, 0 \le k \le n+1\}$ and its mirrored image along the diagonal $\{w(n) + (-1,1) + (k,k) \,|\, 0 \le k \le n+1\}$. Every segment is parallel to the diagonal and consists of $n+2$ elements. Let $i(n) = v(n) + (1,-1)$ and $j(n) = w(n) + (-1,1)$ be the two initial points of $V(n)$. Let $x(n) = v(n) + (n+2, n)$ and $y(n) = w(n) + (n, n+2)$ be the two final points of $V(n)$.

Definition 8.3 A stripe is a vertical or horizontal finite or infinite subword of the 2-dimensional sequence Stairway, consisting of at least two equal letters.

Lemma 8.4 *Only the elements* $1,2,3,4 \in \mathbb{F}_5$ *(that have been renamed as a, B, A, b) really occur in Stairway and only the given* 32 *rules of recurrence are needed to constuct it. The Sporadic Rules S are applied only once respectively, all other rules occur infinitely often. Every triangle* $(e(n)-(0,1))i(n-1)x(n-1)$ *consists of* $n+1$ *alternating vertical stripes in A and B. Every segment of the wave* $V(n)$ *consists of alternating letters A and B. The first letter of a segment of* $V(n)$ *is always the same as the last letter of a segment of* $V(n-1)$*. The squares* $T(n)$ *in Stairway consist only of the letters a and b. The letter occurring in* $e(n)$ *is always the same as the letter occurring in* $f(n-1)$*. The diagonal of Stairway starts with a, B, b, a and continues as follows:*

$$a \,|\, B \,|\, b, a \,|\, a, b, a \,|\, a, b, a, b \,|\, b, a, b, a, b \,|\, b, a, b, a, b, a \,|\, a, \dots$$

Proof: The proof works by induction over the sets $M(i)$, where $M(1) = \{0,1,2,3\} \times \mathbb{N} \cup \mathbb{N} \times \{0,1,2,3\}$ and for $n \geq 2$, $M(n) = \{t(n)+1, \dots, t(n)+n+1\} \times \{x \in \mathbb{N} \,|\, x \geq t(n)+1\} \cup \{x \in \mathbb{N} \,|\, x \geq t(n)+1\} \times \{t(n)+1, \dots, t(n)+n+1\}$. This is possible because the system of predecessors $\{(0,1),(1,1),(1,0)\}$ has excess $= 0$ and makes possible both a row-wise and a column-wise recurrence.

Induction start. By constructing $M(1)$ we apply every Sporadic Rule once and the Rule D_2 once. The result is the square $T(1)$ of the form:

$$\begin{pmatrix} b & a \\ a & a \end{pmatrix}.$$

On the left border (respectively under the bottom border) of the square $T(1)$ we apply Q_1 (respectively Q_1^T). We get that $T(1)$ is bordered by A's. At $a(4,1)$ and $a(1,4)$ we apply W_1, at $a(5,2)$ and $a(2,5)$ we apply W_2, at $a(6,3)$ and $a(3,6)$ we apply again W_1 - and so we produced the wave $V(1)$. Both branches of $V(1)$ have colors A, B, A respectively. Left and downwards from $V(1)$ there are only monochrome stripes in colors A, B, A, parallel to the axes, so the triangle $e(2)-(0,1), i(1), x(1)$ consists of 3 alternating vertical stripes in A and B. The segments of the wave $V(1)$ bend an A-stripe and a B-stripe with $90°$. Finally we observe that the points $i(2)$ and $j(2)$ are also colored in $M(1)$ by the rule E_1 and respectively E_1^T, an that they are neighbors with $x(1)$ and respectively $y(1)$ on the same line (column) so they all get the same color: $a(x(1)) = a(i(2)) = a(y(1)) = a(j(2))$. The diagonal sequence constructed so far is a, B, b, a.

Induction Step. Suppose that we have already constructed the set $M(n-1)$ and we are about to construct $M(n)$. The first point of $M(n)$ is $e(n)$, which is constructed using an appropriate Square Start Rule and consequently has the same color as $f(n-1)$. Remember that $M(n-1)$ contained the triangle $e(n)-(0,1), i(n-1), x(n-1)$, consisting of $n+1$ vertical stripes in A and B. From $e(n)$ we continue horizontally with the rules M_1 and M_2, since we arrive at $v(n)$ (we continue vertically with the rules M_1^T, M_2^T since we arrive at $w(n)$). Here one has got the configurations:

$$\begin{pmatrix} a(x(n-1)) = C & a(i(n)) = C \\ v(n) & C \end{pmatrix}, \quad \begin{pmatrix} a(y(n-1)) = C & a(w(n)) = C \\ j(n) & C \end{pmatrix}.$$

The points $x(n-1)$, $i(n)$, $y(n-1)$, $j(n)$ have already been constructed in $M(n-1)$ according to the hypothesis of induction and have all the same color $C \in \{A, B\}$. By the rules C_i the new point to be constructed is in both cases a C. The construction of the square $T(n)$ is closed now by applying the rules $Q_i^{(T)}$ for the last two edges, and the rules $I_i^{(T)}$, D_i inside $T(n)$. We check that all the letters used for $T(n)$ are a and b, and that the edge-length of $T(n)$ equals the length of one segment of the wave $V(n-1)$, which is $(n-1)+2 = n+1$. Finally, in the rest of $M(n)$ we apply the Exterior Stripe Rules, then the Wave Rules, and finally again the Exterior Strip Rules. By the first application of the Exterior Stripe Rules we get the triangle $e(n+1)-(0,1), i(n), x(n)$ consisting of $n+2$ many alternating stripes in A and B, and its mirrored image along the diagonal. Then we construct the wave $V(n)$, and the infinite stripes starting by the wave segments. In particular we observe that $a(i(n+1)) = a(x(n)) = a(j(n+1)) = a(y(n))$ because the points $i(n+1)$ is the right neighbor of $x(n)$ on a horizontal stripe (because $j(n+1)$ is the downward neighbor of $y(n)$ in a vertical stripe). At the diagonal sequence constructed so far we append an alternated word of length $n+1$ in a and b which starts with the same letter in which the last appended word of length n has ended. □

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