The Annals of Probability
2004, Vol. 32, No. 1B, 1140–1167

THE LIMIT SHAPE OF THE ZERO CELL IN A STATIONARY POISSON HYPERPLANE TESSELLATION

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In the early 1940s, D. G. Kendall conjectured that the shape of the zero cell of the random tessellation generated by a stationary and isotropic Poisson line process in the plane tends to circularity given that the area of the zero cell tends to ∞. A proof was given by I. N. Kovalenko in 1997. This paper generalizes Kovalenko's result in two directions: to higher dimensions and to not necessarily isotropic stationary Poisson hyperplane processes. In the anisotropic case, the asymptotic shape of the zero cell depends on the direction distribution of the hyperplane process and is obtained from it via an application of Minkowski's existence theorem for convex bodies with given area measures.

1. Introduction and main results. Let X be a stationary and isotropic Poisson line process in $\mathbb{R}^2$. It induces a random tessellation of $\mathbb{R}^2$ into convex polygons, the cells of the tessellation. The cell containing the origin of $\mathbb{R}^2$ is almost surely unique; it is called the *zero cell* or *Crofton cell* of the tessellation and denoted by Z_0. In his foreword to the first edition of [11], Kendall formulated his conjecture, made in the early 1940s, that the conditional law for the shape of Z_0, given the area $A(Z_0)$ of Z_0, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the circular shape. Strong support for the truth of this conjecture came from papers of Miles [7] and Goldman [2]. A proof was given by Kovalenko [4], and a simplified version in [5]. We will extend the methods and results of the latter paper to higher dimensions and, adding a geometrically interesting new aspect, to anisotropic hyperplane processes.

Let X be a stationary Poisson hyperplane process in d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$, with intensity $\lambda > 0$. The induced tessellation $T(X)$ and its zero cell Z_0 are defined in the obvious way. We assume that X is nondegenerate, in the sense that there is no line with the property that almost surely all hyperplanes of the process are parallel to this line. Under this assumption, the zero cell Z_0 is bounded almost surely.

The direction distribution φ of the stationary hyperplane process X is an even measure on the unit sphere which describes the distribution of the unit normals of

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Received October 2002; revised April 2003.

[1] Supported by the Austrian Science Foundation Project J1940-MAT.

AMS 2000 subject classifications. Primary 60D05; secondary 52A22.

Key words and phrases. Poisson hyperplane process, hyperplane tessellation, zero cell, Crofton cell, typical cell, asymptotic shape, D. G. Kendall's conjecture.

the hyperplanes of X. By Minkowski's theorem, there exists a centrally symmetric convex body B for which φ is the area measure (for explanations and details, see Section 2). We call B the *direction body* of X. In the following, the *shape* of a convex body $K \subset \mathbb{R}^d$ is understood as a homothetic shape: two convex bodies K, M have the same shape if they are homothetic, which means that $K = rM + z$ with suitable $r > 0$ and $z \in \mathbb{R}^d$. In order to measure the deviation of the shape of a convex body K from the shape of B, we put

$$r_B(K) := \inf\{s/r - 1 : rB + z \subset K \subset sB + z,\ z \in \mathbb{R}^d,\ r, s > 0\}.$$

Note that $r_B(K)$ is invariant under dilatations of B and homotheties of K. The convex body K is homothetic to B if and only if $r_B(K) = 0$. Now we can formulate our main result. By $\mathbb{P}$ we denote probability, and V is the volume in $\mathbb{R}^d$. We consider intervals of the form $I = [a, b)$ with $0 < a < b$, where $b = \infty$ is allowed.

THEOREM 1. *Let X be a nondegenerate stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity $\lambda > 0$ and direction body B; let Z_0 be the zero cell of the tessellation $T(X)$ induced by X. There is a positive constant c_0 depending only on B such that the following is true. If $\varepsilon \in (0, 1)$ and $I = [a, b)$ is any interval with $a^{1/d}\lambda \geq \sigma_0 > 0$, then*

$$\mathbb{P}\big(r_B(Z_0) \geq \varepsilon | V(Z_0) \in I\big) \leq c \exp\{-c_0 \varepsilon^{d+1} a^{1/d} \lambda\},$$

where c is a constant depending on B, ε, σ_0.

As a consequence, leaving aside part of the more precise information contained in the theorem, we may formulate that

$$\lim_{a\to\infty} \mathbb{P}\big(r_B(Z_0) \geq \varepsilon | V(Z_0) \geq a\big) = 0$$

for every $\varepsilon > 0$. This shows that the conditional law for the shape of Z_0, given a lower bound for the volume $V(Z_0)$, converges weakly, as that lower bound tends to ∞, to the law concentrated at the shape of the direction body of the process X. Here the intensity λ of the process X was kept fixed. Alternatively, one may fix a lower bound a for $V(Z_0)$ and let λ tend to ∞.

We may also formulate the following consequences of Theorem 1.

COROLLARY 1. *Under the assumptions of Theorem* 1, *and for any intervals $I_a \subset [a, \infty)$,*

$$\limsup_{a\to\infty} a^{-1/d} \log \mathbb{P}\big(r_B(Z_0) \geq \varepsilon | V(Z_0) \in I_a\big) \leq -c_0 \varepsilon^{d+1} \lambda,$$

uniformly for $\lambda \geq \lambda_0 > 0$.

COROLLARY 2. *Under the assumptions of Theorem* 1,

$$\limsup_{\lambda\to\infty} \lambda^{-1} \log \mathbb{P}\big(r_B(Z_0) \geq \varepsilon | V(Z_0) \in I\big) \leq -c_0 \varepsilon^{d+1} a^{1/d},$$

uniformly for intervals $I \subset [a, \infty)$ with $a \geq a_0 > 0$.

In addition to the zero cell, one may also consider the typical cell Z of the Poisson hyperplane tessellation $T(X)$. By a result of Mecke [6], it is known that Z_0 is larger than Z in a precise sense: There exists a suitable random vector ξ such that $Z + \xi \subset Z_0$ almost surely. However, this strong relationship does not seem to be useful for deriving an analogue of Theorem 1 with the typical cell replaced by the zero cell. Instead, our approach of such an analogue will be based on the main estimates used for the proof of Theorem 1 and on a relationship between the distribution of the zero cell and the volume weighted distribution of the typical cell.

THEOREM 2. *The assertion of Theorem* 1 *remains verbally true if the zero cell Z_0 is replaced by the typical cell Z.*

A reader who has studied Kovalenko's [5] paper will notice that the principal ideas of that work are still present in our proof. However, the extension to higher dimensions and to the anisotropic case requires not only more elaborate techniques, but also the application of additional geometric tools, such as results on the stability of Minkowski's inequality for mixed volumes or on approximation of convex bodies by polytopes with a given number of vertices. We have further generalized Kovalenko's result to the extent that the intervals in Theorem 1 need not be sufficiently small. For the reader's convenience, we also wanted to make some of the arguments more explicit.

2. Preliminaries. We work in d-dimensional Euclidean vector space $\mathbb{R}^d$, with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Its unit ball, $\{x \in \mathbb{R}^d : \|x\| \le 1\}$, is denoted by B^d, and $S^{d-1} = \partial B^d$ (∂ is the boundary operator) is the unit sphere. Hyperplanes can be written as $H(u,t) := \{x \in \mathbb{R}^d : \langle x, u \rangle = t\} = u^\perp + tu$, where $u \in S^{d-1}$, $t \in \mathbb{R}$, and $u^\perp$ denotes the orthogonal complement of $\operatorname{lin}\{u\}$. The set $H^-(u,t) := \{x \in \mathbb{R}^d : \langle x, u \rangle \le t\}$ is a closed halfspace. The space of convex bodies (nonempty, compact, convex subsets) in $\mathbb{R}^d$ is denoted by $\mathcal{K}^d$, and $\mathcal{K}_0^d$ is the subset of bodies with interior points. $\mathcal{K}^d$ is equipped with the Hausdorff metric δ. By $\mathcal{P}^d \subset \mathcal{K}^d$ we denote the subset of convex polytopes, and we set $\mathcal{P}_0^d := \mathcal{P}^d \cap \mathcal{K}_0^d$.

For basic facts from stochastic geometry which are not explained in the following, we refer to [10] and [11]. The employed notions and results from the theory of convex bodies are found in [9].

Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$, with intensity $\lambda \in (0, \infty)$. Its intensity measure $\mathbb{E}\, X(\cdot)$ ($\mathbb{E}$ denotes mathematical expectation) has a unique representation in the form

$$\mathbb{E}\, X(\cdot) = 2\lambda \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\}\, dt\, \varphi(du), \tag{1}$$

where φ is an even probability measure on the sphere S^{d-1} (cf. [10], 4.1.2 and page 115). The measure φ is called the *direction distribution* of X. We assume that X is nondegenerate; this is equivalent to the assumption that the measure φ is not concentrated on a great subsphere of S^{d-1}, and it implies that the cells of the induced tessellation $T(X)$ are almost surely bounded (cf. [10], page 272). The zero cell of the tessellation induced by a hyperplane process X is denoted by $Z_0 = Z_0(X)$.

By Minkowski's existence and uniqueness theorem from the geometry of convex bodies (cf. [9], Section 7.1), there exists a unique convex body $B \in \mathcal{K}_0^d$ with $B = -B$ such that

$$\varphi = S_{d-1}(B, \cdot).$$

Here $S_{d-1}(B, \cdot)$ is the area measure of B, which means that, for a Borel set $\omega \subset S^{d-1}$, the number $S_{d-1}(B, \omega)$ is the area [the $(d-1)$-dimensional Hausdorff measure] of the set of boundary points of B at which there exists an outer normal vector belonging to ω. We call B the *direction body* of X. The dilated body $B(X) := \lambda^{1/(d-1)} B$ has been called the *Blaschke body* of X (see [10], page 172). In the following, it seems preferable to consider λ and B separately. The usefulness of Minkowski's existence theorem for certain questions in stochastic geometry was first noticed in [8], there in connection with finitely many random hyperplanes. Later, similar constructions were applied to hyperplane processes, particle processes and special random closed sets; see [10], page 158, and the notes on pages 178 and 179.

Using the direction body, we can rewrite (1) in the form

$$\mathbb{E}\, X(\cdot) = 2\lambda \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\}\, dt\, S_{d-1}(B, du). \tag{2}$$

For $K \in \mathcal{K}^d$, we write $\mathcal{H}_K$ for the set of all hyperplanes $H \subset \mathbb{R}^d$ with $H \cap K \neq \varnothing$. Then

$$\mathbb{E}\, X(\mathcal{H}_K) = 2\lambda \int_{S^{d-1}} h(K,u)\, S_{d-1}(B, du) = 2d V_1(B,K)\lambda, \tag{3}$$

where $h(K, \cdot)$ denotes the support function of K and

$$V_1(B,K) := V(K, B[d-1]) := \frac{1}{d} \lim_{\alpha \to 0+} \frac{V(B + \alpha K) - V(B)}{\alpha}$$

is the mixed volume of K and $d-1$ copies of B (see [9] for an introduction to mixed volumes). With this notation, the assumption that X is a Poisson process implies that, for $K \in \mathcal{K}^d$ and $k \in \mathbb{N}_0$,

$$\mathbb{P}\big(X(\mathcal{H}_K) = k\big) = \frac{[2d V_1(B,K)\lambda]^k}{k!} \exp\{-2d V_1(B,K)\lambda\}. \tag{4}$$

At this point, we want to give a rough description of the idea that leads to revealing the direction body B as the limit shape. We are interested (in the simplest case) in the conditional probabilities

$$\mathbb{P}(Z_0 \in \mathcal{C} | V(Z_0) \geq a)$$

for large a, where $\mathcal{C}$ is a Borel set in $\mathcal{K}_0^d$, closed under homotheties (since we are asking for the shapes of the zero cells with large volume). A lower bound for $\mathbb{P}(V(Z_0) \geq a)$ is easily obtained. In fact, for $K \in \mathcal{K}_0^d$ with $0 \in K$ and $V(K) = a$, we have

$$\mathbb{P}(V(Z_0) \geq a) \geq \mathbb{P}(X(\mathcal{H}_K) = 0) = \exp\{-2dV_1(B, K)\lambda\}.$$

Choosing $K = (a/V(B))^{1/d} B$, we get

$$\mathbb{P}(V(Z_0) \geq a) \geq \exp\{-2dV(B)^{(d-1)/d} a^{1/d} \lambda\}. \tag{5}$$

For an upper bound, we can estimate the mixed volume $V_1(B, K)$ by using Minkowski's inequality ([9], page 317)

$$V_1(B, K) \geq V(B)^{(d-1)/d} V(K)^{1/d}. \tag{6}$$

Here equality holds if and only if K is homothetic to B. Let $\mathcal{C} \subset \mathcal{K}^d$ be a closed set which is also closed under homotheties. Suppose that $B \notin \mathcal{C}$. Then there is a number $\tau > 0$ such that

$$V_1(B, K) \geq (1 + \tau) V(B)^{(d-1)/d} V(K)^{1/d} \qquad \text{for } K \in \mathcal{C}. \tag{7}$$

[Otherwise, for every $\tau = 1/n$, $n \in \mathbb{N}$, there is a convex body $K_n \in \mathcal{C}$ violating (7), without loss of generality with $0 \in K_n$ and $D(K_n) = 1$, where D denotes the diameter. The Blaschke selection theorem (see [9]) yields the existence of some $K \in \mathcal{C}$ with $D(K) = 1$ for which equality holds in (6). This implies $B \in \mathcal{C}$, a contradiction.] If now $K \in \mathcal{C}$ and $V(K) \geq a$, then

$$\begin{aligned}\mathbb{P}(X(\mathcal{H}_K) = 0) &= \exp\{-2dV_1(B, K)\lambda\} \\ &\leq \exp\{-2d(1 + \tau)V(B)^{(d-1)/d} a^{1/d} \lambda\}.\end{aligned}$$

Since a convex body contained in the interior of the zero cell $Z_0(X)$ does not meet any hyperplane of X, one might now hope, with a bold heuristic analogy, that something like

$$\mathbb{P}(Z_0 \in \mathcal{C}, V(Z_0) \geq a) \leq c' \exp\{-2d(1 + c''\tau)V(B)^{(d-1)/d} a^{1/d} \lambda\}, \tag{8}$$

with positive constants c', c'', might be true. If that holds, then dividing (8) by (5) immediately gives

$$\lim_{a \to \infty} \mathbb{P}(Z_0 \in \mathcal{C} | V(Z_0) \geq a) = 0.$$

Since this holds whenever $B \notin \mathcal{C}$, we see that the shapes of the zero cells with large volume concentrate at the shape of B.

A large part of the proof to follow (besides aiming at greater generality) is devoted to the replacement of the heuristic estimate (8) by solid arguments.

The proofs of the theorems, to be given in Section 7, require a number of preparations of different types. We divide them into groups of lemmas giving lower bounds for probabilities (Section 3), geometric tools (Section 4) and upper bounds for probabilities (Section 5). Section 6 provides and uses an auxiliary transformation.

We finish these preliminaries with two abbreviations which will be used throughout the paper and should, therefore, be well memorized. Since intersections of parameterized halfspaces will frequently occur, we put

$$\bigcap_{i=1}^{n} H^-(u_i, t_i) =: P(u_1, \dots, u_n; t_1, \dots, t_n) =: P\big(u_{(n)}, t_{(n)}\big)$$

for $u_1, \dots, u_n \in S^{d-1}$, $t_1, \dots, t_n \in (0, \infty)$. Further, for $n \in \mathbb{N}$, we write

$$E^n := (S^{d-1})^n \times (0, \infty)^n$$

and

$$\mu_n := S_{d-1}(B, \cdot)^{\otimes n} \otimes L^{\otimes n},$$

where L is Lebesgue measure on $(0, \infty)$. Thus, the repeatedly needed integral

$$\int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_0^\infty \cdots \int_0^\infty f(u_1, \dots, u_n, t_1, \dots, t_n) \\ \times dt_1 \cdots dt_n \, S_{d-1}(B, du_1) \cdots S_{d-1}(B, du_n),$$

where f is nonnegative and measurable, will appear in the form

$$\int_{E^n} f(u_1, \dots, u_n, t_1, \dots, t_n) \, d\mu_n(u_1, \dots, u_n, t_1, \dots, t_n).$$

3. A lower bound. In the following, $c_1, c_2, \dots$ are positive constants. They depend on various parameters, as indicated, and only on these. If they depend on B and the dimension d, the latter dependence will not be mentioned, since B determines d. If the existence of these constants is not explicitly substantiated, it will be clear from the context.

For the proof of a lower bound, we need a geometric auxiliary result on the approximation of B by polytopes with normal vectors taken from the support of $S_{d-1}(B, \cdot)$. This is stated in the following lemma. The condition on the normal vectors of the facets is needed in order to obtain a positive lower bound at a crucial step.

LEMMA 3.1. *Let $\beta > 0$ be given. There exists a polytope $P \in \mathcal{K}^d$ with $0 \in \operatorname{int} P$, $P \subset (1+\beta)B$, $V(P) = V(B)$, and such that the exterior unit normal vectors of the facets of P are contained in the support of $S_{d-1}(B, \cdot)$.*

PROOF. The set $\operatorname{reg} B$ of regular boundary points of B is dense in ∂B. Let $\{x_1, x_2, \dots\}$ be a countable dense subset of $\operatorname{reg} B$. For $i \in \mathbb{N}$, let $H^-(x_i)$ be the (unique) supporting halfspace of B which contains x_i in its boundary; then

$$B = \bigcap_{i=1}^{\infty} H^-(x_i).$$

For $n \in \mathbb{N}$, let

$$P_n := \bigcap_{i=1}^{n} H^-(x_i).$$

Then $P_n \downarrow B$ in the Hausdorff metric as $n \to \infty$. Hence, there is some $n \in \mathbb{N}$ such that

$$B \subset P_n \subset (1+\beta)B.$$

Let u be an exterior unit normal vector of a facet of the polytope P_n. Since u is an exterior normal vector of B at a regular boundary point of B, it is an extreme normal vector of B and hence belongs to the support of $S_{d-1}(B, \cdot)$ (cf. [9], Theorem 4.6.3). The polytope $P_n' := [V(B)/V(P_n)]^{1/d} P_n$ satisfies $P_n' \subset (1+\beta)B$, $V(P_n') = V(B)$, and $0 \in \operatorname{int} P_n'$. This completes the proof. □

The following lemma provides a lower bound for the probability that $V(Z_0)$ lies in a prescribed interval. In contrast to the easily obtained lower bound (5), we will need the following more delicate estimate in order to be able to deal with small intervals in the statement of our main theorems.

LEMMA 3.2. *For each $\beta > 0$, there are constants $h_0 > 0$, $N \in \mathbb{N}$ and $c_3 > 0$, depending only on β and B, such that, for $a > 0$ and $0 < h < h_0$,*

$$\mathbb{P}\big(V(Z_0) \in a[1, 1+h]\big) \geq c_3 h (a^{1/d}\lambda)^N \exp\{-2d(1+\beta)V(B)^{(d-1)/d} a^{1/d}\lambda\}.$$

PROOF. Let $\beta > 0$ be given. By Lemma 3.1, there are a number $N \in \mathbb{N}$, distinct unit vectors $u_1^0, \dots, u_N^0$ in the support of $S_{d-1}(B, \cdot)$ and numbers $t_1^0, \dots, t_N^0 > 0$ such that the polytope

$$P^0 := P(u_1^0, \dots, u_N^0; t_1^0, \dots, t_N^0)$$

has N facets (and thus normal vectors $u_1^0, \dots, u_N^0$) and satisfies

$$P^0 \subset (1+\beta/2)B \quad \text{and} \quad V(P^0) = V(B).$$

We choose one such polytope (for given B and β); its facet number, N, then depends only on β and B.

We can choose a positive number $\alpha = \alpha(\beta, B)$ with $t_i^0 - \alpha > 0$, $i = 1, \dots, N$, such that, for all $u_1, \dots, u_N \in S^{d-1}$ and $t_1, \dots, t_N \in \mathbb{R}$ with

$$\|u_i - u_i^0\| < \alpha, \qquad |t_i - t_i^0| < \alpha, \qquad i = 1, \dots, N, \tag{9}$$

the following condition is satisfied:

(i) $P := P(u_1, \dots, u_N; t_1, \dots, t_N)$ is a polytope with N facets and satisfying $P \subset (1+\beta)B$.

For the following, it is important to note that the values

$$V\big(P(u_1^0, \dots, u_N^0; t_1^0, \dots, t_{N-1}^0, t)\big), \qquad \text{where } |t - t_N^0| < \alpha,$$

cover an interval containing $V(B)$ in its interior. Therefore, a continuity argument shows that, if α has been chosen sufficiently small, we can choose a number $h_0 > 0$ such that (9) implies condition (i) together with the following condition:

(ii) If $0 < h < h_0$, then

$$V(B)[1, 1+h] \subset \big\{V\big(P(u_1, \dots, u_N; t_1, \dots, t_{N-1}, t)\big) : |t - t_N^0| < \alpha\big\}.$$

Let $\rho > 0$ be a given number. If $u_1, \dots, u_N, t_1, \dots, t_N$ satisfy (9), then the following conditions also hold:

(i_ρ) $P_\rho := P(u_1, \dots, u_N; \rho t_1, \dots, \rho t_N)$ is a polytope with N facets and satisfying $P_\rho \subset (1+\beta)\rho B$.

(ii_ρ) If $0 < h < h_0$, then

$$V(\rho B)[1, 1+h] \subset \big\{V\big(P(u_1, \dots, u_N; \rho t_1, \dots, \rho t_{N-1}, t)\big) : |t - \rho t_N^0| < \rho\alpha\big\}.$$

Let $u_1, \dots, u_N \in S^{d-1}$ and $t_1, \dots, t_{N-1} \in \mathbb{R}$ satisfy (9). For $t \in \rho[t_N^0 - \alpha, t_N^0 + \alpha]$, we define

$$v(t) := V\big(P(u_1, \dots, u_N; \rho t_1, \dots, \rho t_{N-1}, t)\big).$$

The function v is strictly increasing and differentiable. The derivative $v'(t)$ is equal to the $(d-1)$-dimensional volume of the facet of $P_{\rho,t} := P(u_1, \dots, u_N; \rho t_1, \dots, \rho t_{N-1}, t)$ with exterior normal vector u_N, and this can be bounded from above by $V_{d-1}(P_{\rho,t}|u_N^\perp)$, the $(d-1)$-volume of the orthogonal projection of $P_{\rho,t}$ on to $u_N^\perp$. Since $P_{\rho,t} \subset (1+\beta)\rho B$, we obtain

$$0 < v'(t) \le (1+\beta)^{d-1}\rho^{d-1} V_{d-1}(B|u_N^\perp) \le c_1(\beta, B)\rho^{d-1}$$

for $t \in \rho[t_N^0 - \alpha, t_N^0 + \alpha]$. Defining

$$I := \big\{t \in \rho[t_N^0 - \alpha, t_N^0 + \alpha] : v(t) \in V(\rho B)[1, 1+h]\big\}$$

and denoting by τ the inverse function of v, we deduce that the length $|I|$ of I satisfies

$$\begin{aligned}
|I| &= \int \mathbf{1}_I(\tau(w))\tau'(w)\,dw \\
&= \int \mathbf{1}\{w \in V(\rho B)[1, 1+h]\}\tau'(w)\,dw \\
&\ge c_1(\beta, B)^{-1}\rho^{1-d}\rho^d V(B)h \\
&= c_2(\beta, B)\rho h.
\end{aligned}$$

To complete the argument, we set

$$\mathcal{P} := \{P(u_{(N)}, t_{(N)}) : \|u_i - u_i^0\| < \alpha, |t_i - \rho t_i^0| < \rho\alpha \text{ for } i = 1, \ldots, N,$$
$$V(P(u_{(N)}, t_{(N)})) \in V(\rho B)[1, 1+h]\}.$$

In the following, the symbol $\llcorner$ denotes the restriction of a measure; in particular, $(X \llcorner \mathcal{H}')(\mathcal{A}) := X(\mathcal{A} \cap \mathcal{H}')$ for Borel subsets $\mathcal{H}'$ and $\mathcal{A}$ of $\mathcal{H}$. In the subsequent computation (as well as at several places below), we use (4) and a fundamental property of Poisson processes (Theorem 3.2.3(b) in [10]). It implies that

$$\mathbb{P}(X \llcorner \mathcal{H}_K \in \cdot | X(\mathcal{H}_K) = N) = \mathbb{P}_{\sum_{i=1}^N \delta_{H_i}},$$

where δ denotes a Dirac measure and $H_1, \ldots, H_N$ are independent, identically distributed random hyperplanes with distribution $\Theta \llcorner \mathcal{H}_K / \Theta(\mathcal{H}_K)$ ($\Theta = \mathbb{E} X$ is the intensity measure of X). By (2), for any Borel set $\mathcal{A} \subset \mathcal{H}$,

$$(\Theta \llcorner \mathcal{H}_K)(\mathcal{A}) = 2\lambda \int_{E^1} \mathbf{1}_{\mathcal{H}_K \cap \mathcal{A}}(H(u,t))\, d\mu_1(u,t),$$

and by (3) we have $\Theta(\mathcal{H}_K) = 2\lambda\, d V_1(B, K)$. If $K = (1+\beta)\rho B$, then $V_1(B, K) = (1+\beta)\rho V(B)$. Thus, we get

$$\begin{aligned}
&\mathbb{P}(V(Z_0) \in V(\rho B)[1, 1+h]) \\
&\quad \geq \mathbb{P}(X(\mathcal{H}_{(1+\beta)\rho B}) = N, Z_0(X \llcorner \mathcal{H}_{(1+\beta)\rho B}) \in \mathcal{P}) \\
&\quad = \frac{[2d(1+\beta)\rho V(B)\lambda]^N}{N!} \exp\{-2d(1+\beta)\rho V(B)\lambda\} \\
&\qquad \times \mathbb{P}(Z_0(X \llcorner \mathcal{H}_{(1+\beta)\rho B}) \in \mathcal{P} | X(\mathcal{H}_{(1+\beta)\rho B}) = N) \\
&\quad \geq \frac{(2\lambda)^N}{N!} \exp\{-2d(1+\beta)\rho V(B)\lambda\} \\
&\qquad \times \int_{E^N} \mathbf{1}\{\|u_i - u_i^0\| < \alpha, |t_i - \rho t_i^0| < \rho\alpha \text{ for } i = 1, \ldots, N, \\
&\qquad\qquad V(P(u_{(N)}, t_{(N)})) \in V(\rho B)[1, 1+h]\} \\
&\qquad\qquad \times d\mu_N(u_1, \ldots, u_N, t_1, \ldots, t_N) \\
&\quad \geq \frac{(2\lambda)^N}{N!} \exp\{-2d(1+\beta)\rho V(B)\lambda\} c_2(\beta, B)\rho h (2\rho\alpha)^{N-1} \\
&\qquad \times \prod_{i=1}^N S_{d-1}(B, U_i(\alpha)),
\end{aligned} \tag{10}$$

where $U_i(\alpha)$ is the open spherical cap with center u_i^0 and radius α. Since u_i^0 is in the support of $S_{d-1}(B, \cdot)$, we have $S_{d-1}(B, U_i(\alpha)) > 0$. Finally, for given $a > 0$, we choose $\rho > 0$ such that $V(\rho B) = a$. This gives

$$\begin{aligned}\mathbb{P}\big(V(Z_0) \in a[1, 1+h]\big) \\ \geq c_3(\beta, B) h (a^{1/d}\lambda)^N \exp\big\{-2d(1+\beta)V(B)^{(d-1)/d} a^{1/d}\lambda\big\}\end{aligned}$$

and thus proves the assertion. □

REMARK. In spite of the seemingly crude estimate (10), the lower bound given by Lemma 3.2 is of the right order. In fact, from Lemmas 6.3 and 5.1 one can deduce, for $0 < h < 1/2$ and $a^{1/d}\lambda \geq \sigma_0 > 0$, the upper bound

$$\mathbb{P}\big(V(Z_0) \in a[1, 1+h]\big) \leq c_4(B, \sigma_0) h \exp\big\{-c_5(B) a^{1/d}\lambda\big\}. \tag{11}$$

From this upper bound, one can conclude, in particular, that the distribution of $V(Z_0)$ is absolutely continuous with respect to the Lebesgue measure.

4. Some geometric tools. As explained in Section 2, Minkowski's inequality (6) (for $K \in \mathcal{K}_0^d$) plays an important role. In the case where a lower bound $V(K) \geq a > 0$ is prescribed, we can choose $\rho > 0$ with $V(\rho B) = a$ and deduce from (6) that

$$V_1(\rho B, K) \geq V(\rho B). \tag{12}$$

If equality holds here, then ρB and K are homothetic, hence $r_B(K) = 0$. The subsequent lemma improves (12) if $r_B(K) \geq \varepsilon > 0$.

Let $[0, u]$ be the closed line segment with endpoints 0 and u. In the following, we denote by U_0 an interval of unit length for which

$$V_1(B, U_0) = \min\{V_1(B, [0, u]) : u \in S^{d-1}\}.$$

LEMMA 4.1. *There is a constant* $c_8 = c_8(B)$ *such that, for* $K \in \mathcal{K}^d$, $\rho > 0$ *and* $\varepsilon \in (0, 1)$, *the following is true. If* $V(K) \geq V(\rho B)$ *and* $r_B(K) \geq \varepsilon$, *then*

$$V_1(\rho B, K) \geq (1 + c_8 \varepsilon^{d+1}) V(\rho B).$$

PROOF. It is sufficient to prove the result for $\rho = 1$. From this the general case follows. In fact, let $K \in \mathcal{K}^d$ and $\rho > 0$ satisfy $V(K) \geq V(\rho B)$ and $r_B(K) \geq \varepsilon$. Then we have $V(\rho^{-1}K) \geq V(B)$ and $r_B(\rho^{-1}K) \geq \varepsilon$. Hence, $V_1(B, \rho^{-1}K) \geq (1 + c_8\varepsilon^{d+1})V(B)$ and, therefore, $V_1(\rho B, K) \geq (1 + c_8\varepsilon^{d+1})V(\rho B)$.

We consider the case $\rho = 1$. Assume that $V(K) \geq V(B)$ and $r_B(K) \geq \varepsilon$ and set $D_0 := 2V(B)/V_1(B, U_0)$. The constants c_6, c_7, c_8 below depend only on B.

Recall that by $D(K)$ we denote the diameter of a convex body K. If $D(K) \geq D_0$, then there is some $u_1 \in S^{d-1}$ such that

$$\begin{aligned}V_1(B, K) &\geq D(K) V_1(B, [0, u_1]) \geq D_0 V_1(B, U_0) \\ &= 2V(B) \geq (1 + \varepsilon^{d+1}) V(B).\end{aligned}$$

Now we assume that $D(K) < D_0$. Then

$$D\big(V(K)^{-1/d}K\big) \le D\big(V(B)^{-1/d}K\big) \le V(B)^{-1/d}D_0,$$

so that

$$\max\{D\big(V(K)^{-1/d}K\big), D\big(V(B)^{-1/d}B\big)\}$$

can be estimated from above by a constant depending only on B. From Corollary 1 of [3] (assuming, without loss of generality, that K has centroid 0), we obtain the estimate

$$\frac{V_1(B,K)^d}{V(B)^{d-1}V(K)} \ge 1 + c_6\delta(\tilde{K},\tilde{B})^{d+1},$$

where δ denotes the Hausdorff distance, $\tilde{K} = V(K)^{-1/d}K$ and $\tilde{B} = V(B)^{-1/d}B$.

Let μ denote the inradius of $V(B)^{-1/d}B$. If $\delta(\tilde{K},\tilde{B}) \ge \mu/2$, then

$$\left(\frac{V_1(B,K)}{V(B)}\right)^d \ge \frac{V_1(B,K)^d}{V(B)^{d-1}V(K)} \ge 1 + \left(\frac{\mu}{2}\right)^{d+1} c_6 \ge 1 + \left(\frac{\mu}{4}\right)^{d+1} c_6\varepsilon^{d+1}.$$

If $\delta(\tilde{K},\tilde{B}) < \mu/2$, then $\delta(\tilde{K},\tilde{B}) \ge r_B(K)\mu/4$, as we will check below, and therefore again

$$\left(\frac{V_1(B,K)}{V(B)}\right)^d \ge 1 + \left(\frac{\mu}{4}\right)^{d+1} c_6\varepsilon^{d+1}.$$

Setting $c_7 := c_6(\mu/4)^{d+1}$, we thus get

$$V_1(B,K) \ge (1 + c_7\varepsilon^{d+1})^{1/d}V(B) \ge (1 + c_8\varepsilon^{d+1})V(B),$$

where we choose $c_8 \le 1$.

It remains to prove the stated estimate for the Hausdorff distance $\tilde{\delta} := \delta(\tilde{K},\tilde{B})$. By definition,

$$\frac{K}{V(K)^{1/d}} \subset \frac{B}{V(B)^{1/d}} + \tilde{\delta}B^d \subset \left(1 + \frac{\tilde{\delta}}{\mu}\right)\frac{B}{V(B)^{1/d}} \tag{13}$$

and

$$\frac{B}{V(B)^{1/d}} \subset \frac{K}{V(K)^{1/d}} + \tilde{\delta}B^d \subset \frac{K}{V(K)^{1/d}} + \frac{\tilde{\delta}}{\mu}\frac{B}{V(B)^{1/d}};$$

thus,

$$\left(1 - \frac{\tilde{\delta}}{\mu}\right)\frac{B}{V(B)^{1/d}} \subset \frac{K}{V(K)^{1/d}}, \tag{14}$$

since $\tilde{\delta}/\mu < 1$ by assumption. Relations (13) and (14) yield that

$$r_B(K) \le \frac{(1 + \tilde{\delta}/\mu)V(K)^{1/d}V(B)^{-1/d}}{(1 - \tilde{\delta}/\mu)V(K)^{1/d}V(B)^{-1/d}} - 1 = \frac{2\tilde{\delta}/\mu}{1 - \tilde{\delta}/\mu} \le \frac{4}{\mu}\tilde{\delta},$$

which completes the proof. □

For a polytope $P \in \mathcal{P}^d$, we write $f_0(P)$ for the number of vertices and $\operatorname{ext} P$ for the set of vertices of P. The subsequent lemma will later (in the proof of Lemma 5.2) allow us to investigate the zero cell by taking into account only a bounded number of hyperplanes, with controllable error.

LEMMA 4.2. *Let $\alpha > 0$ be given. There is a number $\nu \in \mathbb{N}$ depending only on B and α such that the following is true. For each $P \in \mathcal{P}_0^d$, there exists a polytope $L = L(P) \in \mathcal{P}_0^d$ satisfying $\operatorname{ext} L \subset \operatorname{ext} P$, $f_0(L) \le \nu$ and $V_1(B, L) \ge (1-\alpha)V_1(B, P)$. Moreover, there exists a measurable selection $P \mapsto L(P)$.*

PROOF. We need the following approximation result, which follows from a result of Bronshtein and Ivanov [1]. There exist numbers $k_0 = k_0(d)$ and $b_0 = b_0(d)$ such that the following is true. If $K \subset B^d$ is a d-dimensional convex body and $k \ge k_0$ is an integer, then there exists a polytope $Q \in \mathcal{P}_0^d$ with k vertices, without loss of generality on the boundary of K, such that

$$\delta(K, Q) \le b_0 k^{-2/(d-1)},$$

where δ is the Hausdorff distance.

Now let $P \in \mathcal{P}_0^d$ be given. We can assume that the circumball of P has center 0; let R be its radius. Then there is a vector $u \in S^{d-1}$ such that $[0, Ru] \subset P$, which implies that

$$V_1(B, P) \ge V_1(B, [0, Ru]) \ge R V_1(B, U_0). \tag{15}$$

For $k \in \mathbb{N}$ with $k \ge k_0$, there exists a polytope $Q \in \mathcal{P}_0^d$ with k vertices, all on the boundary of P, such that

$$\delta(R^{-1}P, R^{-1}Q) \le b_0 k^{-2/(d-1)} =: \kappa.$$

From this we infer that $P \subset Q + \kappa R B^d$ and hence

$$V_1(B, P) \le V_1(B, Q) + \kappa R V_1(B, B^d).$$

Together with (15), this gives

$$\begin{aligned} V_1(B, Q) &\ge V_1(B, P) - \kappa R V_1(B, B^d) \\ &= V_1(B, P)\left(1 - \frac{\kappa R V_1(B, B^d)}{V_1(B, P)}\right) \\ &\ge V_1(B, P)\left(1 - \kappa \frac{V_1(B, B^d)}{V_1(B, U_0)}\right). \end{aligned}$$

It is now clear that, for given $\alpha > 0$, there is a constant $k_1 = k_1(B, \alpha)$ such that the choice $k \ge k_1$ implies $V_1(B, Q) \ge (1-\alpha)V_1(B, P)$.

Each vertex of Q lies in a facet of P and hence, by Carathéodory's theorem, in the convex hull of some d vertices of P. Thus, the convex hull L of a suitable set of at most $\nu = dk$ vertices of P satisfies the conditions of the lemma.

By [10], page 236, the map $\psi_0 : P \mapsto \operatorname{ext} P$ is measurable as a map from $\mathcal{P}^d$ to $\mathcal{F}(\mathcal{F}')$ (see [10] for the notation). Arguing as in the proof of Lemma 3.1.7 in [10], one can show that there exists a measurable map

$$\psi : \mathcal{P}^d \to (\mathbb{R}^d)^{\mathbb{N}}, \qquad \psi = (\psi_1, \psi_2, \dots),$$

such that, for every $P \in \mathcal{P}^d$,

$$\operatorname{ext} P = \{\psi_i(P) : 1 \le i \le f_0(P)\}.$$

Now that we have a measurable enumeration of the vertices of all polytopes in $\mathcal{P}^d$, it is easy to construct a measurable choice $P \mapsto L(P)$, $P \in \mathcal{P}^d$. $\square$

The next lemma will be used later, roughly, to show that very elongated shapes of zero cells appear only with small probability. The "elongation" of a convex body is measured by the quotient of diameter and width, and for a body for which this quotient and the volume are in given intervals, we need lower and upper inclusion estimates.

For $K \in \mathcal{K}^d$, we denote by

$$\Delta(K) := \min\{h(K, u) + h(K, -u) : u \in S^{d-1}\}$$

the width of K. For $m \in \mathbb{N}$, $a > 0$ and $\varepsilon \in (0, 1)$, we consider

$$\mathcal{K}^d_{a,\varepsilon}(m) := \{K \in \mathcal{K}^d : 0 \in K,\ V(K) \in a[1, 2],$$
$$D(K)/\Delta(K) \in [m^d, (m+1)^d),\ r_B(K) \ge \varepsilon\}.$$

The condition $r_B(K) \ge \varepsilon$ will not be needed until Lemma 5.2. We write $C^d := [-1, 1]^d$.

LEMMA 4.3. *Let $m \in \mathbb{N}$ and $K \in \mathcal{K}^d_{a,\varepsilon}(m)$. Then:*

(a) *K contains a segment $S(K)$ of length at least $ma^{1/d}$;*
(b) *$K \subset c_9(d) m^{d-1} a^{1/d} C^d =: C$;*
(c) *there exists a measurable selection $\mathcal{K}^d_{a,\varepsilon}(m) \cap \mathcal{P}^d \ni P \mapsto S(P)$ such that the endpoints of $S(P)$ are vertices of P.*

PROOF. Let $m \in \mathbb{N}$ and $K \in \mathcal{K}^d_{a,\varepsilon}(m)$ be fixed.

(a) We can enclose K in a rectangular parallelepiped with one edge length equal to $\Delta(K)$. This shows that

$$a \le V(K) \le \Delta(K) D(K)^{d-1}.$$

Since $D(K) \ge m^d \Delta(K)$, this gives $m^d a \le D(K)^d$, which implies that K contains a segment of length $ma^{1/d}$.

(b) First, we note that

$$V(K) \geq \frac{1}{d!}\Delta(K)^{d-1}D(K). \tag{16}$$

For the proof, let $x, y \in K$ and $u \in S^{d-1}$ be such that $x - y = D(K)u$. The hyperplanes through x and y orthogonal to u support K. Hence, $V(K) \geq d^{-1}D(K)V_{d-1}(K|u^{\perp})$. Now the inequality (16) can be proved by induction, using that $D(K|u^{\perp}), \Delta(K|u^{\perp}) \geq \Delta(K)$. We continue with

$$2a \geq V(K) \geq \Delta(K)^{d-1}D(K)/d! \geq (m+1)^{-d(d-1)}D(K)^d/d!$$

and

$$D(K) \leq (2d!a)^{1/d}(m+1)^{d-1} \leq c_9(d)m^{d-1}a^{1/d}.$$

(c) A longest segment contained in a polytope P has its endpoints at vertices of P. The existence of a measurable selection now follows along the lines of the corresponding argument in the proof of Lemma 4.2. □

5. Two upper bounds. Let $a > 0$, $\varepsilon > 0$ be given. For $m \in \mathbb{N}$, we define

$$q_{a,\varepsilon}(m) := \mathbb{P}(Z_0 \in \mathcal{K}^d_{a,\varepsilon}(m)).$$

We prove two estimates concerning the decay of $q_{a,\varepsilon}(m)$ as $a^{1/d}\lambda \to \infty$. The first one will be used to estimate $q_{a,\varepsilon}(m)$ for almost all $m \in \mathbb{N}$; the second is more subtle and will be used to estimate $q_{a,\varepsilon}(m)$ for the initial values $m = 1, \ldots, m_0$. It is this second case that requires the stability estimate for Minkowski's inequality.

LEMMA 5.1. *For $m \in \mathbb{N}$ and $a^{1/d}\lambda \geq \sigma_0 > 0$,*

$$q_{a,\varepsilon}(m) \leq c_{13}(B, \sigma_0)\exp\{-c_{14}(B)ma^{1/d}\lambda\}. \tag{17}$$

PROOF. Let C be the cube defined in Lemma 4.3(b). Then

$$q_{a,\varepsilon}(m) = \sum_{N=d+1}^{\infty} \mathbb{P}(Z_0 \in \mathcal{K}^d_{a,\varepsilon}(m) | X(\mathcal{H}_C) = N)\mathbb{P}(X(\mathcal{H}_C) = N), \tag{18}$$

where

$$\mathbb{P}(X(\mathcal{H}_C) = N) = \frac{[2dV_1(B,C)\lambda]^N}{N!}\exp\{-2dV_1(B,C)\lambda\}.$$

Next we estimate the conditional probability:

$$\begin{aligned} p_N &:= \mathbb{P}(Z_0 \in \mathcal{K}^d_{a,\varepsilon}(m) | X(\mathcal{H}_C) = N) \\ &= [dV_1(B,C)]^{-N} \int_{E^N} \mathbf{1}\{P(u_{(N)}, t_{(N)}) \in \mathcal{K}^d_{a,\varepsilon}(m)\} \\ &\qquad \times \prod_{i=1}^{N} \mathbf{1}\{H(u_i, t_i) \cap C \neq \varnothing\} \\ &\qquad \times d\mu_N(u_1, \ldots, u_N, t_1, \ldots, t_N). \end{aligned} \tag{19}$$

Suppose that $u_1, \dots, u_N, t_1, \dots, t_N$ are such that the indicator functions occurring in the integral are all equal to 1. We may assume (excluding a set of measure 0 in the integration domain) that $P(u_{(N)}, t_{(N)})$ is a simple polytope. The segment $S(P(u_{(N)}, t_{(N)}))$, according to Lemma 4.3(c), connects two vertices of $P(u_{(N)}, t_{(N)})$, say x and y. Each of these is the intersection of precisely d facets of $P(u_{(N)}, t_{(N)})$. Hence, there exist mutually disjoint index sets $J_1, J_2, J_3 \subset \{1, \dots, N\}$ such that $x, y \in H(u_i, t_i)$ for $i \in J_1$, $x \in H(u_i, t_i)$ for $i \in J_2$, and $y \in H(u_i, t_i)$ for $i \in J_3$. The cardinalities $j_i := |J_i|$, $i = 1, 2, 3$, satisfy $0 \le j_1 \le d-1$, $1 \le j_2, j_3 \le d$, $j_1 + j_2 = j_1 + j_3 = d$. Thus, we have

$$S\big(P\big(u_{(N)}, t_{(N)}\big)\big) \cap H(u_l, t_l) = \varnothing \qquad \text{for } l \in \{1, \dots, N\} \setminus (J_1 \cup J_2 \cup J_3)$$

and

$$S\big(P\big(u_{(N)}, t_{(N)}\big)\big) = \mathrm{CH}_{J_1, J_2, J_3}(u_1, \dots, u_N; t_1, \dots, t_N),$$

where

$$\mathrm{CH}_{J_1, J_2, J_3}(u_1, \dots, u_N; t_1, \dots, t_N) := \mathrm{conv}\Bigg(\bigcap_{i \in J_1 \cup J_2} H(u_i, t_i) \cup \bigcap_{i \in J_1 \cup J_3} H(u_i, t_i)\Bigg).$$

We extend the latter definition by defining the left-hand side as the empty set if the vectors $(u_i : i \in J_1 \cup J_2)$ or $(u_i : i \in J_1 \cup J_3)$ are linearly dependent. Subsequently, we put $j := j_1 + j_2 + j_3$ whenever j_1, j_2, j_3 are given. The summation $\sum_*$ below extends over all choices of pairwise disjoint subsets $J_1, J_2, J_3 \subset \{1, \dots, N\}$ whose cardinalities satisfy $0 \le j_1 \le d-1$, $j_1 + j_2 = j_1 + j_3 = d$, $j \le N$. Then we obtain, using Lemma 4.3(a) and denoting the length of a segment S by $|S|$,

$$\begin{aligned} p_N \le {} & [d V_1(B, C)]^{-N} \\ & \times \sum_* \int_{E^j} \int_{E^{N-j}} \mathbf{1}\{H(u_i, t_i) \cap C \ne \varnothing \text{ for } i = 1, \dots, N\} \\ & \qquad \times \mathbf{1}\big\{\big|\mathrm{CH}_{J_1, J_2, J_3}(u_1, \dots, u_N; t_1, \dots, t_N)\big| \ge m a^{1/d}\big\} \\ & \qquad \times \mathbf{1}\big\{\mathrm{CH}_{J_1, J_2, J_3}(u_1, \dots, u_N; t_1, \dots, t_N) \cap H(u_l, t_l) = \varnothing \\ & \qquad\qquad \text{for } l \notin J_1 \cup J_2 \cup J_3\big\} \\ & \qquad \times d\mu_{N-j}(u_{j+1}, \dots, u_N, t_{j+1}, \dots, t_N) \\ & \qquad \times d\mu_j(u_1, \dots, u_j, t_1, \dots, t_j). \end{aligned}$$

Observe that, for a segment $S \subset C$,

$$\begin{aligned} & \int_{E^1} \mathbf{1}\{H(u, t) \cap C \ne \varnothing,\ S \cap H(u, t) = \varnothing\}\, d\mu_1(u, t) \\ & \quad = d V_1(B, C) - d V_1(B, S) \le d V_1(B, C) - d|S| V_1(B, U_0). \end{aligned}$$

Writing $\sum_{**}$ for the sum over all $j_1, j_2, j_3 \in \{0, \dots, N\}$ such that $0 \le j_1 \le d-1$, $1 \le j_2, j_3 \le d$, $j_1 + j_2 = j_1 + j_3 = d$, $j \le N$, we can estimate further

$$\begin{aligned}
p_N &\le \sum_{**} \binom{N}{j_1, j_2, j_3, N-j} [dV_1(B,C)]^{-N} \\
&\qquad \times \int_{E^j} \mathbf{1}\{H(u_i,t_i) \cap C \neq \varnothing \text{ for } i = 1, \dots, j\} \\
&\qquad\qquad \times [dV_1(B,C) - dma^{1/d} V_1(B,U_0)]^{N-j} \, d\mu_j(u_1, \dots, u_j, t_1, \dots, t_j) \\
&= \sum_{**} \binom{N}{j_1, j_2, j_3, N-j} \left(1 - \frac{ma^{1/d} V_1(B,U_0)}{V_1(B,C)}\right)^{N-j}.
\end{aligned}$$

This finally leads to the estimate

$$\begin{aligned}
q_{a,\varepsilon}(m) &\le \sum_{N=d+1}^{\infty} \sum_{**} \frac{[2dV_1(B,C)\lambda]^N}{N!} \exp\{-2dV_1(B,C)\lambda\} \\
&\qquad\qquad \times \frac{N!}{j_1! j_2! j_3! (N-j)!} \left(1 - \frac{2dma^{1/d} V_1(B,U_0)\lambda}{2dV_1(B,C)\lambda}\right)^{N-j} \\
&\le \sum_{\substack{j_1,j_2,j_3=0 \\ j_1+j_2=d=j_1+j_3}}^{\infty} \frac{1}{j_1! j_2! j_3!} [2dV_1(B,C)\lambda]^j \exp\{-2dV_1(B,C)\lambda\} \\
&\qquad \times \sum_{N \ge j} \frac{1}{(N-j)!} [2dV_1(B,C)\lambda]^{N-j} \left(1 - \frac{2dma^{1/d} V_1(B,U_0)\lambda}{2dV_1(B,C)\lambda}\right)^{N-j} \\
&= \sum_{\substack{j_1,j_2,j_3=0 \\ j_1+j_2=d=j_1+j_3}}^{\infty} \frac{1}{j_1! j_2! j_3!} [2dV_1(B,C)\lambda]^j \exp\{-2dma^{1/d} V_1(B,U_0)\lambda\}.
\end{aligned}$$

By the definition of C [cf. Lemma 4.3(b)],

$$2dV_1(B,C) = 2dc_9(d) m^{d-1} a^{1/d} V_1(B, C^d) = c_{10}(B) m^{d-1} a^{1/d}.$$

In the summation, we have $j \le 2d$ and hence $(a^{1/d}\lambda)^j = (a^{1/d}\lambda/\sigma_0)^j \sigma_0^j \le (a^{1/d}\lambda/\sigma_0)^{2d} \sigma_0^j$. This gives

$$\begin{aligned}
q_{a,\varepsilon}(m) &\le \sum_{\substack{j_1,j_2,j_3=0 \\ j_1+j_2=d=j_1+j_3}}^{\infty} \frac{1}{j_1! j_2! j_3!} (\sigma_0)^{j-2d} c_{10}(B)^j m^{2d(d-1)} (a^{1/d}\lambda)^{2d} \\
&\qquad\qquad \times \exp\{-c_{11}(B) ma^{1/d}\lambda\} \\
&\le c_{12}(B,\sigma_0) m^{2d(d-1)} (a^{1/d}\lambda)^{2d} \exp\{-c_{11}(B) ma^{1/d}\lambda\} \\
&\le c_{13}(B,\sigma_0) \exp\{-c_{14}(B) ma^{1/d}\lambda\},
\end{aligned}$$

where the latter estimate follows by splitting the exponential factor into a product (by splitting c_{11} into a sum of smaller positive constants) and using that $m \geq 1$ and $a^{1/d}\lambda \geq \sigma_0$. This proves the lemma. □

LEMMA 5.2. *For $m \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $a^{1/d}\lambda \geq \sigma_0 > 0$,*

$$q_{a,\varepsilon}(m) \leq c_{17}(B,\varepsilon,\sigma_0) m^{d^2\nu} \exp\{-2d(1+c_{15}\varepsilon^{d+1}) V(B)^{(d-1)/d} a^{1/d}\lambda\},$$

where ν depends only on B and ε.

PROOF. Let $\rho > 0$ be defined by $V(\rho B) = a$. We define C as in Lemma 4.3(b) and use (18) and (19). Assume that $u_1, \dots, u_N, t_1, \dots, t_N$ are such that the indicator functions under the integrals in (19) are all equal to 1. Then, by Lemma 4.1,

$$V_1(\rho B, P(u_{(N)}, t_{(N)})) \geq (1 + c_8\varepsilon^{d+1}) V(\rho B). \tag{20}$$

Let $\alpha := c_8\varepsilon^{d+1}/(2 + c_8\varepsilon^{d+1})$; then $(1-\alpha)(1 + c_8\varepsilon^{d+1}) = 1 + \alpha$. Set $c_{15} := c_8/(2 + c_8)$; then $\alpha > c_{15}\varepsilon^{d+1}$ (which will be needed at the end of the proof). By Lemma 4.2, there are $\nu = \nu(B,\varepsilon)$ vertices of $P(u_{(N)}, t_{(N)})$ such that the convex hull $L = L(P(u_{(N)}, t_{(N)}))$ of these vertices satisfies

$$V_1(\rho B, L) \geq (1-\alpha) V_1(\rho B, P(u_{(N)}, t_{(N)})). \tag{21}$$

The inequalities (20) and (21) imply that

$$V_1(B, L) \geq (1+\alpha)\rho V(B).$$

Excluding a set of measure 0 in the domain of integration, we can assume that each of the vertices of L lies in precisely d of the hyperplanes $H(u_1, t_1), \dots, H(u_N, t_N)$, and the remaining hyperplanes are disjoint from L. Hence, at most $d\nu$ of the hyperplanes $H(u_1, t_1), \dots, H(u_N, t_N)$ meet L; let $j \in \{d+1, \dots, d\nu\}$ denote their precise number. Then there are subsets $J_1, \dots, J_\nu \subset \{1, \dots, j\}$, each of cardinality d, such that the intersections

$$\bigcap_{i \in J_r} H(u_i, t_i), \qquad r = 1, \dots, \nu,$$

yield the vertices of L. This leads to

$$\begin{aligned}
&\mathbb{P}(Z_0 \in \mathcal{K}^d_{a,\varepsilon}(m) \mid X(\mathcal{H}_C) = N)[dV_1(B,C)]^N \\
&\quad \leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \int_{E^N} \mathbf{1}\{P(u_{(N)}, t_{(N)}) \in \mathcal{K}^d_{a,\varepsilon}(m)\} \\
&\qquad\qquad \times \mathbf{1}\{H(u_i, t_i) \cap C \neq \varnothing \text{ for } i = 1, \dots, N\} \\
&\qquad\qquad \times \mathbf{1}\{H(u_i, t_i) \cap L(P(u_{(N)}, t_{(N)})) \neq \varnothing \text{ for } i = 1, \dots, j\} \\
&\qquad\qquad \times \mathbf{1}\{H(u_i, t_i) \cap L(P(u_{(N)}, t_{(N)})) = \varnothing \text{ for } i = j+1, \dots, N\} \\
&\qquad\qquad \times d\mu_N(u_1, \dots, u_N, t_1, \dots, t_N)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=d+1}^{d\nu} \binom{N}{j} \binom{j}{d}^{\nu} \\
&\qquad \times \int_{E^j} \int_{E^{N-j}} \mathbf{1}\{H(u_i,t_i) \cap C \neq \varnothing \text{ for } i = 1, \dots, N\} \\
&\qquad\qquad \times \mathbf{1}\left\{\operatorname{conv}\left\{\bigcap_{i \in J_r} H(u_i,t_i) : r = 1, \dots, \nu\right\} \subset C\right\} \\
&\qquad\qquad \times \mathbf{1}\left\{V_1\left(B, \operatorname{conv}\left\{\bigcap_{i \in J_r} H(u_i,t_i) : r = 1, \dots, \nu\right\}\right)\right. \\
&\qquad\qquad\qquad \left. \ge (1+\alpha)\rho V(B)\right\} \\
&\qquad\qquad \times \mathbf{1}\left\{H(u_i,t_i) \cap \operatorname{conv}\left\{\bigcap_{i \in J_r} H(u_i,t_i) : r = 1, \dots, \nu\right\} = \varnothing\right. \\
&\qquad\qquad\qquad \left. \text{for } i = j+1, \dots, N\right\} \\
&\qquad\qquad \times d\mu_{N-j}(u_{j+1}, \dots, u_N, t_{j+1}, \dots, t_N) \\
&\qquad\qquad \times d\mu_j(u_1, \dots, u_j, t_1, \dots, t_j) \\
&\le \sum_{j=d+1}^{d\nu} \binom{N}{j} \binom{j}{d}^{\nu} [dV_1(B,C) - d(1+\alpha)\rho V(B)]^{N-j} [dV_1(B,C)]^j.
\end{aligned}
$$

Summation over N yields

$$
\begin{aligned}
q_{a,\varepsilon}(m) &\le \sum_{N=d+1}^{\infty} \frac{1}{N!} [2dV_1(B,C)\lambda]^N \exp\{-2dV_1(B,C)\lambda\} \\
&\qquad \times \sum_{j=d+1}^{d\nu} \binom{N}{j} \binom{j}{d}^{\nu} \frac{[dV_1(B,C) - d(1+\alpha)\rho V(B)]^{N-j}}{[dV_1(B,C)]^{N-j}} \\
&= \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{1}{j!} [2dV_1(B,C)\lambda]^j \exp\{-2dV_1(B,C)\lambda\} \\
&\qquad \times \sum_{N=j}^{\infty} \frac{1}{(N-j)!} [2dV_1(B,C)\lambda - 2d(1+\alpha)V(B)\rho\lambda]^{N-j} \\
&= \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{1}{j!} [2dV_1(B,C)\lambda]^j \exp\{-2d(1+\alpha)V(B)\rho\lambda\}.
\end{aligned}
$$

Estimating $[2dV_1(B,C)\lambda]^j$ as in the proof of Lemma 5.1 [and recalling that $\rho^d V(B) = a$], we get

$$\begin{aligned} q_{a,\varepsilon}(m) &\le c_{16}(B,\varepsilon,\sigma_0)(a^{1/d}\lambda)^{d\nu} m^{d^2\nu} \exp\{-2d(1+\alpha)V(B)^{(d-1)/d}a^{1/d}\lambda\} \\ &\le c_{17}(B,\varepsilon,\sigma_0) m^{d^2\nu} \exp\{-2d(1+c_{15}\varepsilon^{d+1})V(B)^{(d-1)/d}a^{1/d}\lambda\}, \end{aligned}$$

as stated. □

6. A transformation. Let $a > 0$ and $\varepsilon \in (0,1)$ be given. For $h \in (0,1]$ and $m \in \mathbb{N}$, we extend the definition of $\mathcal{K}^d_{a,\varepsilon}(m)$ by

$$\begin{aligned} \mathcal{K}^d_{a,\varepsilon,h}(m) := \{K \in \mathcal{K}^d : 0 \in K,\ V(K) \in a[1, 1+h], \\ D(K)/\Delta(K) \in [m^d, (m+1)^d), r_B(K) \ge \varepsilon\}. \end{aligned}$$

Thus,

$$\mathbb{P}(V(Z_0) \in a[1,1+h],\ r_B(K) \ge \varepsilon) = \sum_{m=1}^{\infty} q^{(h)}_{a,\varepsilon}(m),$$

where

$$q^{(h)}_{a,\varepsilon}(m) := \mathbb{P}(Z_0 \in \mathcal{K}^d_{a,\varepsilon,h}(m)).$$

The probability $q^{(h)}_{a,\varepsilon}(m)$ is split further into

$$q^{(h)}_{a,\varepsilon}(m,n) := \mathbb{P}(Z_0 \in \mathcal{K}^d_{a,\varepsilon,h}(m),\ f_{d-1}(Z_0) = n)$$

for $n \in \mathbb{N}$; here $f_{d-1}(P)$ denotes the number of facets of a polytope P. Then we have

$$\mathbb{P}(V(Z_0) \in a[1,1+h],\ r_B(K) \ge \varepsilon) = \sum_{m=1}^{\infty} \sum_{n=d+1}^{\infty} q^{(h)}_{a,\varepsilon}(m,n).$$

Finally, we define

$$\begin{aligned} &R^{(h)}_{a,\varepsilon}(m,n) \\ &\quad := \{(u_1,\dots,u_n,t_1,\dots,t_n) \in E^n : P(u_{(n)},t_{(n)}) \in \mathcal{K}^d_{a,\varepsilon,h}(m), \\ &\qquad f_{d-1}(P(u_{(n)},t_{(n)})) = n, H(u_i,t_i) \cap C \ne \varnothing \text{ for } i = 1,\dots,n\}, \end{aligned}$$

where the cube C is again defined as in Lemma 4.3(b) for the given a, ε, m.

LEMMA 6.1. *For $m, n \in \mathbb{N}$, $n \ge d+1$ and $h \in (0,1]$,*

$$\begin{aligned} q^{(h)}_{a,\varepsilon}(m,n) = \frac{(2\lambda)^n}{n!} \int_{R^{(h)}_{a,\varepsilon}(m,n)} \exp\{-2dV_1(B, P(u_{(n)},t_{(n)}))\lambda\} \\ \times d\mu_n(u_1,\dots,u_n,t_1,\dots,t_n). \end{aligned}$$

PROOF. Conditioning on $X(\mathcal{H}_C) = N$, $N \in \mathbb{N}$, we get, for $N \geq n$,

$$\begin{aligned}
&\mathbb{P}\big(Z_0 \in \mathcal{K}^d_{a,\varepsilon,h}(m),\ f_{d-1}(Z_0) = n \big| X(\mathcal{H}_C) = N\big) \\
&\quad = \frac{1}{[dV_1(B,C)]^N} \int_{E^N} \mathbf{1}\{H(u_i,t_i) \cap C \neq \varnothing \text{ for } i = 1,\dots,N\} \\
&\qquad\qquad \times \mathbf{1}\big\{P\big(u_{(N)},t_{(N)}\big) \in \mathcal{K}^d_{a,\varepsilon,h}(m),\ f_{d-1}\big(P\big(u_{(N)},t_{(N)}\big)\big) = n\big\} \\
&\qquad\qquad \times d\mu_N(u_1,\dots,u_N,t_1,\dots,t_N) \\
&\quad = \frac{\binom{N}{n}}{[dV_1(B,C)]^N} \int_{R^{(h)}_{a,\varepsilon}(m,n)} \big[dV_1(B,C) - dV_1\big(B, P\big(u_{(n)},t_{(n)}\big)\big)\big]^{N-n} \\
&\qquad\qquad \times d\mu_n(u_1,\dots,u_n,t_1,\dots,t_n).
\end{aligned}$$

Hence,

$$\begin{aligned}
q^{(h)}_{a,\varepsilon}(m,n) &= \sum_{N=n}^{\infty} \mathbb{P}\big(Z_0 \in \mathcal{K}^d_{a,\varepsilon,h}(m),\ f_{d-1}(Z_0) = n \big| X(\mathcal{H}_C) = N\big) \\
&\qquad \times \frac{[2dV_1(B,C)\lambda]^N}{N!} \exp\{-2dV_1(B,C)\lambda\} \\
&= \exp\{-2dV_1(B,C)\lambda\} \frac{1}{n!} (2\lambda)^n \\
&\quad \times \int_{R^{(h)}_{a,\varepsilon}(m,n)} \sum_{N=n}^{\infty} \frac{1}{(N-n)!} \\
&\qquad\qquad \times \big(2dV_1(B,C)\lambda - 2dV_1\big(B, P\big(u_{(n)},t_{(n)}\big)\big)\lambda\big)^{N-n} \\
&\qquad\qquad \times d\mu_n(u_1,\dots,u_n,t_1,\dots,t_n).
\end{aligned}$$

Writing the sum under the integral as

$$\exp\{2dV_1(B,C)\lambda\} \exp\big\{-2dV_1\big(B, P\big(u_{(n)},t_{(n)}\big)\big)\lambda\big\},$$

we obtain the stated result. □

Next, we will bound $q^{(h)}_{a,\varepsilon}(m)$ from above by $q_{a,\varepsilon}(m)$ $[= q^{(1)}_{a,\varepsilon}(m)]$. To prepare this, we need the following lemma.

LEMMA 6.2. *Let $w > 0$ and $h \in (0, 1/2)$. Then, for $n \in \mathbb{N}$,*

$$\begin{aligned}
&\int_1^{\sqrt[d]{1+h}} x^{n-1} \exp\{-wx\}\, dx \\
&\qquad \leq \tfrac{1}{2} hw\big[1 + \big(\exp\{w/(4d)\} - 1\big)^{-1}\big] \int_1^{\sqrt[d]{2}} x^{n-1} \exp\{-wx\}\, dx.
\end{aligned}$$

PROOF. By the mean value theorem, there is some $\eta \in (1, \sqrt[d]{1+h}\,)$ such that

$$\int_1^{\sqrt[d]{1+h}} x^{n-1} \exp\{-wx\}\, dx \le \frac{h}{2}\eta^{n-1} \exp\{-w\eta\}, \tag{22}$$

since $\sqrt[d]{1+h} \le 1 + h/d \le 1 + h/2$. Furthermore, we can estimate

$$\begin{aligned}
&\int_1^{\sqrt[d]{2}} x^{n-1} \exp\{-wx\}\, dx \\
&\qquad \ge \int_\eta^{\sqrt[d]{2}} x^{n-1} \exp\{-wx\}\, dx \\
&\qquad \ge \eta^{n-1} \int_\eta^{\sqrt[d]{2}} \exp\{-wx\}\, dx \\
&\qquad = \eta^{n-1}(-w^{-1})\big[\exp\{-w\sqrt[d]{2}\,\} - \exp\{-w\eta\}\big] \\
&\qquad = \eta^{n-1}(w^{-1}) \exp\{-w\eta\}\big[1 - \exp\{-w(\sqrt[d]{2} - \eta)\}\big] \\
&\qquad \ge \eta^{n-1}(w^{-1}) \exp\{-w\eta\}\big[1 - \exp\{-w(\sqrt[d]{2} - \sqrt[d]{3/2})\}\big].
\end{aligned} \tag{23}$$

Using the estimate $\sqrt[d]{2} - \sqrt[d]{3/2} \ge 1/(4d)$, we obtain the assertion by combining (22) and (23). $\square$

LEMMA 6.3. *For $m \in \mathbb{N}$, $h \in (0, 1/2)$ and $a^{1/d}\lambda \ge \sigma_0 > 0$,*

$$q_{a,\varepsilon}^{(h)}(m) \le c_{20}(B, \sigma_0) h a^{1/d} \lambda m^{d-1} q_{a,\varepsilon}^{(1)}(m).$$

PROOF. Let m, h be fixed according to the assumptions and let $n \in \mathbb{N}$ with $n \ge d + 1$. We define

$$T_h : R_{a,\varepsilon}^{(h)}(m, n) \to (S^{d-1})^n \times (0, \infty)^{n-1} \times (0, \infty)$$
$$(u_1, \dots, u_n, t_1, \dots, t_n) \mapsto (u_1, \dots, u_n, t_1/t_n, \dots, t_{n-1}/t_n, t_n)$$

and set

$$U(m, n) := \big\{\zeta \in (S^{d-1})^n \times (0, \infty)^{n-1} : (\zeta, t) \in T_h\big(R_{a,\varepsilon}^{(h)}(m, n)\big)$$
$$\text{for some } t \in (0, \infty)\big\}.$$

Clearly, the map T_h is injective, and $U(m, n)$ is independent of h. For the following, we note that $(\zeta, t) = T_h(u_1, \dots, u_n, t_1, \dots, t_n)$ is equivalent to

$$\zeta = (u_1, \dots, u_n, t_1/t_n, \dots, t_{n-1}/t_n) \quad \text{and} \quad t = t_n.$$

For $(\zeta, t) = T_h(u_1, \dots, u_n, t_1, \dots, t_n)$, we set

$$K(\zeta, t) := P\big(u_{(n)}, t_{(n)}\big).$$

Since, for each $\zeta \in U(m,n)$, $V(K(\zeta,\cdot))$ is continuous and increasing from 0 to ∞, there is a unique $t(\zeta) > 0$ such that $V(K(\zeta, t(\zeta))) = a$; consequently,

$$V\big(K\big(\zeta, \sqrt[d]{1+h}\, t(\zeta)\big)\big) = (1+h)a.$$

We apply Lemma 6.1, the transformation formula for integrals (in $\mathbb{R}^n$, with fixed vectors $u_1, \dots, u_n$), and Fubini's theorem, to obtain

$$\begin{aligned} q_{a,\varepsilon}^{(h)}(m,n) &= \frac{(2\lambda)^n}{n!} \int_{R_{a,\varepsilon}^{(h)}(m,n)} \exp\{-2d V_1\big(B, P\big(u_{(n)}, t_{(n)}\big)\big)\lambda\} \\ &\qquad\qquad \times d\mu_n(u_1,\dots,u_n,t_1,\dots,t_n) \\ &= \frac{(2\lambda)^n}{n!} \int_{U(m,n)} \int_{t(\zeta)}^{\sqrt[d]{1+h}\,t(\zeta)} \exp\{-2d V_1\big(B, K(\zeta,t)\big)\lambda\} t^{n-1}\, dt \\ &\qquad\qquad \times dt_1 \cdots dt_{n-1}\, S_{d-1}(B, du_1) \cdots S_{d-1}(B, du_n). \end{aligned}$$

Here we substitute $s = t/t(\zeta)$ and get

$$\begin{aligned} q_{a,\varepsilon}^{(h)}(m,n) &= \frac{(2\lambda)^n}{n!} \int_{U(m,n)} t(\zeta)^n \int_1^{\sqrt[d]{1+h}} \exp\{-2d V_1\big(B, K\big(\zeta, t(\zeta)\big)\big)s\lambda\} s^{n-1}\, ds \\ &\qquad\qquad \times dt_1 \cdots dt_{n-1}\, S_{d-1}(B, du_1) \cdots S_{d-1}(B, du_n). \end{aligned}$$

Lemma 6.2 now yields

$$\begin{aligned} q_{a,\varepsilon}^{(h)}(m,n) &\le \frac{(2\lambda)^n}{n!} \int_{U(m,n)} t(\zeta)^n h d V_1\big(B, K\big(\zeta, t(\zeta)\big)\big)\lambda \\ &\qquad \times \left(1 + \left(\exp\left\{\frac{1}{2} V_1\big(B, K\big(\zeta, t(\zeta)\big)\big)\lambda\right\} - 1\right)^{-1}\right) \\ &\qquad \times \int_1^{\sqrt[d]{2}} s^{n-1} \exp\{-2d V_1\big(B, K\big(\zeta, t(\zeta)\big)\big)\lambda s\}\, ds \\ &\qquad \times dt_1 \cdots dt_{n-1}\, S_{d-1}(B, du_1) \cdots S_{d-1}(B, du_n). \end{aligned}$$

Since $K(\zeta, t(\zeta)) \in \mathcal{K}_{a,\varepsilon,h}^d(m) \subset \mathcal{K}_{a,\varepsilon}^d(m)$, we have

$$m a^{1/d} S \subset K\big(\zeta, t(\zeta)\big) \subset c_9(d) m^{d-1} a^{1/d} C^d$$

by Lemma 4.3, where S is a suitable segment of unit length. Hence,

$$V_1\big(B, K\big(\zeta, t(\zeta)\big)\big) \le c_{18}(B) m^{d-1} a^{1/d}$$

and

$$\exp\{\tfrac{1}{2}V_1(B, K(\zeta, t(\zeta)))\lambda\} \geq \exp\{c_{19}(B)ma^{1/d}\lambda\}.$$

Therefore,

$$\begin{aligned} &q_{a,\varepsilon}^{(h)}(m,n) \\ &\quad \leq c_{18}(B)m^{d-1}ha^{1/d}\lambda(1 + (\exp\{c_{19}(B)ma^{1/d}\lambda\} - 1)^{-1}) \\ &\qquad \times \frac{(2\lambda)^n}{n!}\int_{U(m,n)}\int_{t(\zeta)}^{\sqrt[d]{2}t(\zeta)} t^{n-1}\exp\{-2dV_1(B, K(\zeta, t))\lambda\}\,dt \\ &\qquad\qquad \times dt_1\cdots dt_{n-1}\,S_{d-1}(B, du_1)\cdots S_{d-1}(B, du_n) \\ &\quad \leq c_{20}(B,\sigma_0)m^{d-1}ha^{1/d}\lambda q_{a,\varepsilon}^{(1)}(m,n). \end{aligned}$$

Summation over $n \in \mathbb{N}$, $n \geq d+1$, yields the required estimate. □

7. Proofs of the theorems. We prepare the proofs of Theorems 1 and 2 by establishing an upper estimate for an unconditional probability.

PROPOSITION 7.1. *Let* $\varepsilon \in (0,1)$, $h \in (0,1/2)$ *and* $a^{1/d}\lambda \geq \sigma_0 > 0$. *Then*

$$\begin{aligned} &\mathbb{P}(V(Z_0) \in a[1,1+h],\ r_B(Z_0) \geq \varepsilon) \\ &\quad \leq c_{25}(B,\varepsilon,\sigma_0)h\exp\{-2d(1+(c_{15}/2)\varepsilon^{d+1})V(B)^{(d-1)/d}a^{1/d}\lambda\}. \end{aligned}$$

PROOF. Using Lemma 6.3, we get

$$\begin{aligned} &\mathbb{P}(V(Z_0) \in a[1,1+h],\ r_B(Z_0) \geq \varepsilon) \\ &\quad = \sum_{m\in\mathbb{N}} \mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon,h}^d(m)) = \sum_{m\in\mathbb{N}} q_{a,\varepsilon}^{(h)}(m) \\ &\quad \leq c_{20}(B,\sigma_0)ha^{1/d}\lambda\sum_{m\in\mathbb{N}} m^{d-1}q_{a,\varepsilon}^{(1)}(m). \end{aligned}$$

We define $c_{21}(B) := 4(1+c_{15})\,dV(B)^{(d-1)/d}c_{14}^{-1}$, where $c_{14} = c_{14}(B)$ is the constant appearing in (17) in the argument of the exponential, and we set $m_0 := \lfloor c_{21} \rfloor$. Then

$$c_{14}m \geq 4(1+c_{15})\,dV(B)^{(d-1)/d} \tag{24}$$

for $m > m_0$. Using Lemma 5.2 for $m \leq m_0$ and Lemma 5.1 for $m > m_0$, we get

$$\begin{aligned} &\mathbb{P}(V(Z_0) \in a[1,1+h],\ r_B(Z_0) \geq \varepsilon) \\ &\quad \leq c_{20}(B,\sigma_0)ha^{1/d}\lambda\left(\sum_{m=1}^{m_0} m^{d-1}q_{a,\varepsilon}(m) + \sum_{m>m_0} m^{d-1}q_{a,\varepsilon}(m)\right) \end{aligned}$$

$$\begin{aligned}
&\le c_{22}(B,\varepsilon,\sigma_0)ha^{1/d}\lambda \\
&\quad\times \sum_{m=1}^{m_0} m^{d-1}m^{d^2\nu}\exp\{-2d(1+c_{15}\varepsilon^{d+1})V(B)^{(d-1)/d}a^{1/d}\lambda\} \\
&\quad + c_{23}(B,\sigma_0)ha^{1/d}\lambda\sum_{m>m_0} m^{d-1}\exp\{-c_{14}ma^{1/d}\lambda\} \\
&\le c_{22}(B,\varepsilon,\sigma_0)ha^{1/d}\lambda m_0^{d+d^2\nu} \\
&\quad\times \exp\{-2d(1+c_{15}\varepsilon^{d+1})V(B)^{(d-1)/d}a^{1/d}\lambda\} \\
&\quad + c_{23}(B,\sigma_0)ha^{1/d}\lambda\sum_{m>m_0} m^{d-1}\exp\{-c_{14}ma^{1/d}\lambda\}.
\end{aligned}$$

Here we estimate

$$\begin{aligned}
&\exp\{-c_{14}ma^{1/d}\lambda\} \\
&\qquad\le \exp\{-(c_{14}/2)ma^{1/d}\lambda\}\exp\{-2d(1+c_{15}\varepsilon^{d+1})V(B)^{(d-1)/d}a^{1/d}\lambda\},
\end{aligned}$$

which follows from (24) and $\varepsilon < 1$. Since m_0 is bounded from above by $c_{21}(B)$, and since $\sum m^{d-1}\exp\{-(c_{14}/2)ma^{1/d}\lambda\}$ converges, we obtain

$$\begin{aligned}
&\mathbb{P}\big(V(Z_0)\in a[1,1+h],\ r_B(Z_0)\ge\varepsilon\big) \\
&\quad\le c_{24}(B,\varepsilon,\sigma_0)ha^{1/d}\lambda\exp\{-2d(c_{15}/2)\varepsilon^{d+1}V(B)^{(d-1)/d}a^{1/d}\lambda\} \\
&\qquad\times\exp\{-2d\big(1+(c_{15}/2)\varepsilon^{d+1}\big)V(B)^{(d-1)/d}a^{1/d}\lambda\} \\
&\quad\le c_{25}(B,\varepsilon,\sigma_0)h\exp\{-2d\big(1+(c_{15}/2)\varepsilon^{d+1}\big)V(B)^{(d-1)/d}a^{1/d}\lambda\},
\end{aligned}$$

as asserted. □

PROOF OF THEOREM 1. Let $\varepsilon\in(0,1)$, $a>0$ and $\lambda>0$ with $a^{1/d}\lambda\ge\sigma_0>0$ be given. We define $\beta := (c_{15}/4)\varepsilon^{d+1}$ and choose h_0 and N (depending on B and ε) as in Lemma 3.2. We set $h_1 := \min\{h_0, 1/2\}$ and fix an interval $I=[a,b)$, where $b=\infty$ is allowed. Then we distinguish two cases.

CASE 1: $h_1 > (b-a)/a$. Then we set $h_2 := (b-a)/a$, so that $a[1,1+h_2) = [a,b)$. Now Lemma 3.2 (which obviously also holds if $a[1,1+h]$ is replaced by $a[1,1+h)$) implies that

$$\begin{aligned}
&\mathbb{P}\big(V(Z_0)\in I\big) \\
&\qquad\ge c_{26}(B,\varepsilon,\sigma_0)h_2\exp\{-2d\big(1+(c_{15}/4)\varepsilon^{d+1}\big)V(B)^{(d-1)/d}a^{1/d}\lambda\}.
\end{aligned} \tag{25}$$

By Proposition 7.1, we have

$$
\begin{aligned}
&\mathbb{P}\big(V(Z_0) \in I,\ r_B(Z_0) \geq \varepsilon\big) \\
&\quad \leq c_{25}(B, \varepsilon, \sigma_0) h_2 \exp\big\{-2d\big(1 + (c_{15}/2)\varepsilon^{d+1}\big) V(B)^{(d-1)/d} a^{1/d} \lambda\big\}.
\end{aligned} \tag{26}
$$

Therefore, (25) and (26) imply that

$$
\begin{aligned}
&\mathbb{P}\big(r_B(Z_0) \geq \varepsilon | V(Z_0) \in I\big) \\
&\quad \leq c_{27}(B, \varepsilon, \sigma_0) \exp\big\{-d(c_{15}/2)\varepsilon^{d+1} V(B)^{(d-1)/d} a^{1/d} \lambda\big\}.
\end{aligned}
$$

CASE 2: $h_1 \leq (b-a)/a$. Then $1 + h_1 \leq b/a$, $a[1, 1+h_1) \subset [a, b)$, and hence

$$
\begin{aligned}
&\mathbb{P}\big(V(Z_0) \in I\big) \\
&\quad \geq c_{26}(B, \varepsilon, \sigma_0) h_1 \exp\big\{-2d\big(1 + (c_{15}/4)\varepsilon^{d+1}\big) V(B)^{(d-1)/d} a^{1/d} \lambda\big\}.
\end{aligned} \tag{27}
$$

On the other hand, Proposition 7.1 [together with $a^{1/d}(1 + h_1)^{i/d}\lambda \geq \sigma_0$] implies that, for $i \in \mathbb{N}_0$,

$$
\begin{aligned}
&\mathbb{P}\big(V(Z_0) \in a(1+h_1)^i[1, 1+h_1],\ r_B(Z_0) \geq \varepsilon\big) \\
&\quad \leq c_{25}(B, \varepsilon, \sigma_0) h_1 \\
&\qquad \times \exp\big\{-2d\big(1 + (c_{15}/2)\varepsilon^{d+1}\big) V(B)^{(d-1)/d} a^{1/d} (1+h_1)^{i/d} \lambda\big\} \\
&\quad = c_{25}(B, \varepsilon, \sigma_0) h_1 \\
&\qquad \times \exp\big\{-2d\big(1 + (c_{15}/4)\varepsilon^{d+1}\big) V(B)^{(d-1)/d} a^{1/d} (1+h_1)^{i/d} \lambda\big\} \\
&\qquad \times \exp\big\{-2d(c_{15}/4)\varepsilon^{d+1} V(B)^{(d-1)/d} a^{1/d} (1+h_1)^{i/d} \lambda\big\} \\
&\quad \leq c_{25}(B, \varepsilon, \sigma_0) h_1 \exp\big\{-2d\big(1 + (c_{15}/4)\varepsilon^{d+1}\big) V(B)^{(d-1)/d} a^{1/d} \lambda\big\} \\
&\qquad \times \exp\big\{-d(c_{15}/4)\varepsilon^{d+1} V(B)^{(d-1)/d} a^{1/d} \lambda\big\} \\
&\qquad \times \exp\big\{-d(c_{15}/4)\varepsilon^{d+1} V(B)^{(d-1)/d} \sigma_0 (1+h_1)^{i/d}\big\}.
\end{aligned}
$$

Since

$$
[a, b) \subset \bigcup_{i=0}^{\infty} a(1+h_1)^i[1, 1+h_1],
$$

we obtain that

$$
\begin{aligned}
&\mathbb{P}\big(V(Z_0) \in I,\ r_B(Z_0) \geq \varepsilon\big) \\
&\quad \leq c_{25}(B, \varepsilon, \sigma_0) h_1 \exp\big\{-2d\big(1 + (c_{15}/4)\varepsilon^{d+1}\big) V(B)^{(d-1)/d} a^{1/d} \lambda\big\} \\
&\qquad \times \exp\big\{-d(c_{15}/4)\varepsilon^{d+1} V(B)^{(d-1)/d} a^{1/d} \lambda\big\} \\
&\qquad \times \sum_{i=0}^{\infty} \exp\big\{-d(c_{15}/4)\varepsilon^{d+1} V(B)^{(d-1)/d} \sigma_0 (1+h_1)^{i/d}\big\}
\end{aligned} \tag{28}
$$

$$\leq c_{28}(B,\varepsilon,\sigma_0)h_1 \exp\{-2d(1+(c_{15}/4)\varepsilon^{d+1})V(B)^{(d-1)/d}a^{1/d}\lambda\}$$
$$\times \exp\{-d(c_{15}/4)\varepsilon^{d+1}V(B)^{(d-1)/d}a^{1/d}\lambda\}.$$

Combining (27) and (28), we find that

$$\mathbb{P}(r_B(Z_0)\geq \varepsilon | V(Z_0)\in I)$$
$$\leq c_{29}(B,\varepsilon,\sigma_0)\exp\{-d(c_{15}/4)\varepsilon^{d+1}V(B)^{(d-1)/d}a^{1/d}\lambda\}.$$

Hence, in any case the required estimate has been established. □

PROOF OF THEOREM 2. We adopt the same notation as in the proof of Theorem 1. We set $\beta_1 := \beta/2$.

If $f : \mathcal{K}^d \to [0,\infty)$ is any translation-invariant and measurable function, then

$$\mathbb{E}[f(Z_0)] = \lambda^{(d)}\mathbb{E}[f(Z)V(Z)], \tag{29}$$

where $\lambda^{(d)}$ is the intensity of the particle process $X^{(d)}$ of the tessellation $T(X)$ generated by X; cf. Theorem 6.1.11 in [10]. From Theorem 6.3.3, (6.46), in [10], we see that $\lambda^{(d)} = V(\Pi_X)$, where Π_X is the zonoid associated with X. Its support function is given by

$$h(\Pi_X,\cdot) = \frac{\lambda}{2}\int_{S^{d-1}} |\langle v,\cdot\rangle|\,\varphi(dv);$$

hence, $\lambda^{(d)} = c_{30}(B)\lambda^d$.

We apply (29) with

$$f(K) := \mathbf{1}\{V(K)\in a[1,1+h]\}V(K)^{-1}$$

to get

$$\begin{aligned}
&\mathbb{P}(V(Z)\in a[1,1+h])\\
&\quad = c_{31}(B)\lambda^{-d}\mathbb{E}[\mathbf{1}\{V(Z_0)\in a[1,1+h]\}V(Z_0)^{-1}]\\
&\quad \geq c_{31}(B)\lambda^{-d}\mathbb{E}[\mathbf{1}\{V(Z_0)\in a[1,1+h]\}(a(1+h))^{-1}]\\
&\quad \geq c_{32}(B,\beta)(a\lambda^d)^{-1}\mathbb{E}[\mathbf{1}\{V(Z_0)\in a[1,1+h]\}],
\end{aligned}$$

where we assumed that $h \leq h_0$, where h_0 is chosen according to Lemma 3.2 (applied to β_1). Hence, Lemma 3.2 shows that, for $h\in(0,h_0]$,

$$\begin{aligned}
&\mathbb{P}(V(Z)\in a[1,1+h])\\
&\quad \geq c_{33}(B,\beta)h\sigma_0^N(a^{1/d}\lambda)^{-d}\exp\{-2d(1+\beta_1)V(B)^{(d-1)/d}a^{1/d}\lambda\}\\
&\quad \geq c_{34}(B,\varepsilon,\sigma_0)h\exp\{-2d(1+\beta)V(B)^{(d-1)/d}a^{1/d}\lambda\}.
\end{aligned}$$

Similarly, applying (29) with

$$f(K) := \mathbf{1}\{V(K) \in a[1, 1+h],\ r_B(K) \geq \varepsilon\} V(K)^{-1}$$

(and assuming $h \leq 1/2$), we deduce for $\varepsilon \in (0, 1)$ that

$$\begin{aligned}
&\mathbb{P}(V(Z) \in a[1, 1+h],\ r_B(Z) \geq \varepsilon) \\
&\quad = c_{31}(B)\lambda^{-d}\mathbb{E}[\mathbf{1}\{V(Z_0) \in a[1, 1+h],\ r_B(Z_0) \geq \varepsilon\} V(Z_0)^{-1}] \\
&\quad \leq c_{31}(B)\sigma_0^{-d}\mathbb{E}[\mathbf{1}\{V(Z_0) \in a[1, 1+h],\ r_B(Z_0) \geq \varepsilon\}] \\
&\quad \leq c_{35}(B, \varepsilon, \sigma_0)h \\
&\qquad \times \exp\{-2d(1 + (c_{15}/2)\varepsilon^{d+1}) V(B)^{(d-1)/d} a^{1/d} \lambda\},
\end{aligned}$$

where Proposition 7.1 was used for the last estimate. With these two estimates instead of Lemma 3.2 and Proposition 7.1, respectively, we can continue as in the proof of Theorem 1. □

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