Affine surface area and convex bodies of elliptic type

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Abstract

If a convex body K in $\mathbb{R}^n$ is contained in a convex body L of elliptic type (a curvature image), then it is known that the affine surface area of K is not larger than the affine surface area of L. We prove that the affine surface areas of K and L can only be equal if $K = L$.

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1 Introduction

The notion of affine surface area, originally belonging to the realm of affine differential geometry (see the books of Blaschke [4], Salkowski [16], P. and A. Schirokow [17], Li, Simon, and Zhao [11], and Section 1.4 of Leichtweiß [10]) has in recent decades been extended to general convex bodies in Euclidean space $\mathbb{R}^n$ ($n \geq 2$). The affine surface area, denoted by Ω, is now a real functional on the space $\mathcal{K}^n_n$ of n-dimensional convex bodies in $\mathbb{R}^n$, equipped with the Hausdorff metric, which is invariant under volume preserving affine transformations of $\mathbb{R}^n$ and is an upper semi-continuous valuation. It vanishes on polytopes and is positive on convex bodies of class C^2_+ (i.e., those with twice continuously differentiable boundary and positive curvatures). We refer to the book of Leichtweiß [10] for introduction, history, and references. A brief sketch of the development is also found in the introduction to the paper of Ludwig and Reitzner [12].

Since the affine surface area vanishes on polytopes, it cannot generally be monotonic under set inclusion. In the early days of affine surface area, a special monotonicity property was proved by Winternitz [20]: if $K \subset E$, where K is a sufficiently smooth convex body and E is an ellipsoid, then $\Omega(K) \leq \Omega(E)$. This was extended in the following way. If $K, L \in \mathcal{K}^n_n$, where K is arbitrary and L is of elliptic type (as defined below), then $K \subset L$ implies $\Omega(K) \leq \Omega(L)$. Proofs were given, independently, by Leichtweiß [9] (Satz 1 (d)) and by Lutwak [13] (Theorem (8.1)).

Let $K \in \mathcal{K}^n_n$. Since the affine surface area is upper semicontinuous, among the convex bodies contained in K there is at least one, say K_a, of maximal affine surface area. The nature of K_a (and, possibly, its uniqueness) is of considerable interest. We mention that

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Sheng, Trudinger, and Wang [19] have studied regularity properties of K_a. In the plane, K_a plays an astonishing role in work of Bárány [1, 2] and of Bárány and Prodromou [3], on limit shapes of convex hulls of lattice points or random points in K. Motivated by this, Bárány has asked the following two questions. If $K \in \mathcal{K}_n^n$ is of elliptic type, is it true that $K_a = K$? For any $K \in \mathcal{K}_n^n$, is K_a of elliptic type? For $n = 2$, both questions were answered affirmatively by Bárány and Prodromou [3]. The method of proof does not seem to extend to higher dimensions. In this note, we give a positive answer to Bárány's first question in $\mathbb{R}^n$, for $n \geq 2$.

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2 Explanations and results

First we have to explain 'elliptic type'. In affine differential geometry, a sufficiently smooth closed convex hypersurface is called of elliptic type if its affine principal curvatures are positive, equivalently, if its Blaschke affine curvature image is convex. Without differentiability assumptions, one needs the notion of a curvature function. A convex body $K \in \mathcal{K}_n^n$ has the curvature function f_K (on the unit sphere S^{n-1} of $\mathbb{R}^n$) if f_K is a density of the surface area measure $S_{n-1}(K, \cdot)$ of K with respect to spherical Lebesgue measure σ, equivalently, if

$$V(K, \dots, K, L) = \frac{1}{n} \int_{S^{n-1}} h_L f_K \, \mathrm{d}\sigma$$

for all $L \in \mathcal{K}_n^n$, where V is the mixed volume and h_L denotes the support function of L. We refer to [18] for these standard notions (and others used below) from the theory of convex bodies. Following Petty [15], we consider two subclasses

$$\mathcal{V}^n \subset \mathcal{F}^n \subset \mathcal{K}_n^n.$$

Here, $\mathcal{F}^n$ is the set of convex bodies $K \in \mathcal{K}_n^n$ with a positive, continuous curvature function f_K. The set $\mathcal{V}^n$ is defined as the set of all convex bodies $K \in \mathcal{F}^n$ for which the function

$$f_K^{-1/(n+1)}$$

is the restriction to S^{n-1} of a support function. The convex bodies in $\mathcal{V}^n$ are called *of elliptic type*. See Leichtweiß [9] for a proof that this extends the classical notion. Now we can state the announced result.

Theorem 1. *If $K \in \mathcal{V}^n$, $L \in \mathcal{K}_n^n$, and $L \subset K$, then*

$$\Omega(L) \leq \Omega(K), \tag{1}$$

with equality if and only if $L = K$.

As already mentioned, inequality (1) was proved by Leichtweiß [9] and Lutwak [13], so the only new aspect here is the equality condition. Neither of the approaches in [9] or [13] leads to an identification of the equality case, although Leichtweiß obtained the equality $L = K$ under the additional assumption that also L is of elliptic type. A proof of Theorem 1 becomes possible by combining Petty's theory of geominimal surface area with the information on the general affine surface area that is nowadays available. For this, we recall some facts about curvature images and geominimal surface area.

If $K \in \mathcal{V}^n$, then by definition there exists a convex body M with support function

$$h_M = f_K^{-1/(n+1)}.$$

Since $h_M > 0$, we have $o \in \operatorname{int} M$. Hence, the polar body M° is defined. Since its radial function is given by $\rho_{M^\circ} = h_M^{-1}$, we get

$$o = \int_{S^{n-1}} u \, \mathrm{d}S_{n-1}(K, u) = \int_{S^{n-1}} f_K(u) u \, \mathrm{d}\sigma(u) = \int_{S^{n-1}} \rho_{M^\circ}(u)^{n+1} u \, \mathrm{d}\sigma(u)$$

and hence (using spherical coordinates)

$$\int_{M^\circ} x \, \mathrm{d}x = o.$$

Thus, M° has centroid o. Conversely, let $M \in \mathcal{K}_n^n$ be a convex body, containing o in the interior, such that M° has centroid o. Then, as above,

$$\int_{S^{n-1}} h_M(u)^{-(n+1)} u \, \mathrm{d}\sigma(u) = o.$$

By Minkowski's existence theorem ([18], Theorem 7.1.2), there exists a convex body $\mathrm{C}M \in \mathcal{K}_n^n$ with curvature function

$$f_{\mathrm{C}M} = h_M^{-(n+1)}.$$

It is uniquely determined up to a translation. Every translate of $\mathrm{C}M$ is called a *curvature image* of M (this notion of curvature image is not to be confused with Blaschke's affine curvature image). Thus, $\mathcal{V}^n$ is precisely the set of curvature images of convex bodies.

For $K \in \mathcal{K}_n^n$, there exists a unique point $s(K)$ in the interior of K, the *Santaló point* of K, such that

$$V_n(K^s) := V_n((K - s(K))^\circ) = \min_{p \in \operatorname{int} K} V_n((K - p)^\circ),$$

where V_n denotes the volume. We have $s(K) = o$ if and only if K° has centroid o. Let κ_n denote the volume of the n-dimensional unit ball and let

$$\mathcal{T}^n := \{T \in \mathcal{K}_n^n : s(T) = o,\ V_n(T^\circ) = \kappa_n\}.$$

Every convex body from $\mathcal{K}_n^n$ has a unique homothet in $\mathcal{T}^n$. The relative surface area $A(K, T)$ of a convex body K with respect to $T \in \mathcal{T}^n$ is defined by

$$A(K, T) := nV(K, \dots, K, T).$$

Petty [15] has defined the *geominimal surface area* of $K \in \mathcal{K}_n^n$ by

$$G(K) := \inf\{A(K, T) : T \in \mathcal{T}^n\}.$$

We need some of Petty's results.

Lemma 1 and Definition (Petty [15], Theorems (2.5) and (2.8)). *To each $K \in \mathcal{K}_n^n$, there exists a unique $T \in \mathcal{T}^n$ with $G(K) = A(K,T)$. It is denoted by $T = T(K)$.*

Lemma 2 (Petty [15], Corollary (3.13)). *If $K \in \mathcal{V}^n$ and $T \in \mathcal{T}^n$, then K is a curvature image of T if and only if $T = T(K)$.*

Now we are ready for the following result.

Theorem 2. *If $K \in \mathcal{K}_n^n$, then*

$$\Omega(K)^{n+1} \leq n\kappa_n G(K)^n, \tag{2}$$

with equality if and only if $K \in \mathcal{V}^n$.

The inequality (2) was proved by Petty [15] for $K \in \mathcal{F}^n$ and was extended by Lutwak [13] to $K \in \mathcal{K}_n^n$. Up to now, the equality condition for (2) was only known under the additional assumption that $K \in \mathcal{F}^n$; see Lutwak [14], Theorem 4.13. We will show in the next section that this is a consequence of the equality.

Theorem 2 implies Theorem 1, essentially by Petty's [15] proof of his Theorem (3.21). Observe that the latter requires that $K_1 \in \mathcal{F}^n$, since in [15] the affine surface area is only defined for bodies from $\mathcal{F}^n$.

Proof of Theorem 1. Let $K \in \mathcal{V}^n$, $L \in \mathcal{K}_n^n$, and $L \subset K$. From Theorem 2, and since the geominimal surface area is monotonic under set inclusion, we get

$$\Omega(L)^{n+1} \leq n\kappa_n G(L)^n \leq n\kappa_n G(K)^n = \Omega(K)^{n+1}.$$

This proves inequality (1). If equality holds here, then the equality condition of Theorem 2 shows that also L is of elliptic type; moreover,

$$G(K) = G(L) \leq A(L, T(K)) \leq A(K, T(K)) = G(K).$$

The first inequality follows from the inf-definition of the geominimal surface area, the second from the monotonicity of mixed volumes. It follows that $A(L, T(L)) = G(L) = A(L, T(K))$. By the uniqueness result of Lemma 1, we have $T(L) = T(K)$. By Lemma 2, L and K are curvature images of the same set, thus they are translates and hence identical. □

3 Proof of Theorem 2

Of the many equivalent representations of the affine surface area of general convex bodies, we need here the one by

$$\Omega(K) = \int_{S^{n-1}} [D_{n-1}(h_K)]^{n/(n+1)} \, \mathrm{d}\sigma. \tag{3}$$

Here $D_{n-1}(h_K)$ denotes the sum of the principal minors of the Hessian matrix of the homogeneous support function of K. The function $D_{n-1}(h_K)$ exists σ-almost everywhere on S^{n-1}, it is measurable and nonnegative. The representation (3) was first established by Leichtweiß [8], for his definition of extended affine surface area. Since later all the differently defined extensions of affine surface area, by Leichtweiß, Lutwak, Schütt and Werner, were shown to be equivalent (see Leichtweiss [10]), formula (3) can be used for any of these definitions. We refer also to Hug [6] for an extension of this representation.

Let $K \in \mathcal{K}_n^n$. By Lemma 1, there exists a unique convex body $T \in \mathcal{T}^n$ with

$$G(K) = A(K,T) = \int_{S^{n-1}} h_T \,\mathrm{d}S_{n-1}(K,\cdot). \tag{4}$$

With respect to spherical Lebesgue measure σ, the measure $S_{n-1}(K,\cdot)$ has a Lebesgue decomposition into the sum of an absolutely continuous measure S_K^a and a singular measure S_K^s. It is known that

$$S_K^a(\omega) = \int_\omega D_{n-1}(h_K)\,\mathrm{d}\sigma \qquad \text{for Borel sets } \omega \subset S^{n-1}. \tag{5}$$

A proof is given, for example, in Hug [7], Section 3. With (4) this gives

$$G(K) = \int_{S^{n-1}} h_T \,\mathrm{d}S_K^a + \int_{S^{n-1}} h_T \,\mathrm{d}S_K^s \geq \int_{S^{n-1}} h_T D_{n-1}(h_K)\,\mathrm{d}\sigma. \tag{6}$$

Hölder's inequality with a negative exponent reads

$$\int g\bar{g} \geq \left(\int g^k\right)^{1/k} \left(\int \bar{g}^{k'}\right)^{1/k'},$$

where $k < 0$, $k' = k/(k-1)$ and $g, \bar{g}$ are nonnegative measurable functions. We apply this with $k = -n$, $g = h_T$, $\bar{g} = D_{n-1}(h_K)$ and obtain (integrations are over the unit sphere)

$$\begin{aligned}
\int h_T D_{n-1}(h_K)\,\mathrm{d}\sigma &\geq \left(\int h_T^{-n}\,\mathrm{d}\sigma\right)^{-1/n} \left(\int [D_{n-1}(h_K)]^{n/(n+1)}\,\mathrm{d}\sigma\right)^{(n+1)/n} \\
&= [nV_n(T^\circ)]^{-1/n}\Omega(K)^{(n+1)/n} \\
&= (n\kappa_n)^{-1/n}\Omega(K)^{(n+1)/n}.
\end{aligned}$$

The inequality (2) follows. Suppose that equality holds here. Then equality holds in Hölder's inequality. This implies (see [5], p. 140) that g^k and $\bar{g}^{k'}$ are proportional outside a set of measure zero, hence there is a (necessarily positive) constant A with

$$Ah_T^{-(n+1)} = D_{n-1}(h_K) \quad \sigma\text{-almost everywhere on } S^{n-1}. \tag{7}$$

But since equality holds also in (6), and $h_T > 0$ everywhere on S^{n-1}, the singular part S_K^s is the zero measure. Now it follows from (5) and (7) that $Ah_T^{-(n+1)}$ is a density for $S_{n-1}(K,\cdot)$. Thus, K has the curvature function $f_K = Ah_T^{-(n+1)}$, and $f_K^{-1/(n+1)}$ is a support function. Therefore, $K \in \mathcal{V}^n$, that is, K is of elliptic type, which completes the proof. $\square$

Remark 1. In retrospect, the use of the geominimal surface area for establishing the equality case in Theorem 1 seems quite natural, due to the lemmas in Section 2: if K is a curvature image, then it is the curvature image of $T(K)$, but the body $T(K)$ exists also if K is not a curvature image.

Remark 2. The extension of the previous approach to p-affine surface area for $p \geq 1$ fails, since the p-mixed volume is no longer monotonic under set inclusion.

References

[1] I. Bárány, Affine perimeter and limit shape, *J. reine angew. Math.*, **484** (1997), 71–84.

[2] I. Bárány, Sylvester's question: the probability that n points are in convex position, *Ann. Probab.*, **27** (1999), 2020–2034.

[3] I. Bárány and M. Prodromou, On maximal convex lattice polygons inscribed in a plane convex set, *Israel J. Math.*, **154** (2006), 337–360.

[4] W. Blaschke, W., *Vorlesungen über Differentialgeometrie, II. Affine Differentialgeometrie*, Springer, Berlin 1923.

[5] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press., Cambridge 1934.

[6] D. Hug, Contributions to affine surface area, *Manuscripta Math.*, **91** (1996), 283–301.

[7] D. Hug, Absolute continuity for curvature measures of convex sets I, *Math. Nachr.*, **195** (1998), 139–158.

[8] K. Leichtweiss, Zur Affinoberfläche konvexer Körper, *Manuscripta Math.*, **56** (1986), 429–464.

[9] K. Leichtweiss, Über einige Eigenschaften der Affinoberfläche konvexer Körper, *Results Math.*, **13** (1988), 255–282.

[10] K. Leichtweiss, *Affine Geometry of Convex Bodies*, Johann Ambrosius Barth, Heidelberg 1998.

[11] A.–M. Li, U. Simon and G. Zhao, *Global Affine Differential Geometry of Hypersurfaces*, W. de Gruyter, Berlin 1993.

[12] M. Ludwig and M. Reitzner, A characterization of affine surface area, *Adv. Math.*, **147** (1999), 138–172.

[13] E. Lutwak, Extended affine surface area, *Adv. Math.*, **85** (1991), 39–68.

[14] E. Lutwak, The Brunn–Minlowski–Firey theory II: Affine and geominimal surface areas, *Adv. Math.*, **118** (1996), 244–294.

[15] C. M. Petty, Geominimal surface area. *Geom. Dedicata*, **3** (1974), 77–97.

[16] E. Salkowski, *Affine Differentialgeometrie*, W. de Gruyter, Berlin 1934.

[17] P. A. Schirokow and P. A. Schirokow, *Affine Differentialgeometrie*, B. G. Teubner Verlagsgesellschaft, Leipzig 1962.

[18] R. Schneider, *Convex Bodies – the Brunn–Minkowski Theory*, Cambridge Univ. Press, Cambridge 1993.

[19] W. Sheng, N. Trudinger and X.–J. Wang, Enclosed convex hypersurfaces with maximal affine surface area, *Math. Z.*, **252** (2006), 497–510.

[20] A. Winternitz, Neuer Beweis für Blaschkes isoperimetrische Sätze der Affingeometrie, *Abh. Math. Sem. Univ. Hamburg*, **1** (1922), 99–101.

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