A Brunn–Minkowski theory for coconvex sets of finite volume

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Abstract

Let C be a closed convex cone in $\mathbb{R}^n$, pointed and with interior points. We consider sets of the form $A = C \setminus A^\bullet$, where $A^\bullet \subset C$ is a closed convex set. If A has finite volume (Lebesgue measure), then A is called a C-coconvex set. The family of C-coconvex sets is closed under the addition $\oplus$ defined by $C \setminus (A_1 \oplus A_2) = (C \setminus A_1) + (C \setminus A_2)$. We develop first steps of a Brunn–Minkowski theory for C-coconvex sets, which relates this addition to the notion of volume. In particular, we establish the equality conditions for a Brunn–Minkowski type inequality (with reversed inequality sign), introduce mixed volumes and their integral representations, and prove a Minkowski-type uniqueness theorem for C-coconvex sets with equal surface area measures.

1 Introduction

Let C be a pointed closed convex cone with apex o and with interior points in Euclidean space $\mathbb{R}^n$. This cone will be fixed throughout the following. Let $\Delta \subset C$ be a closed convex set such that $C \setminus \Delta$ is bounded and nonempty. Khovanskiĭ and Timorin [1] call the set $C \setminus (\Delta \cup \{o\})$ a *coconvex body*. (The non-inclusion of certain boundary points is relevant for some of their aims, but not if volumes are considered.) The authors of [1] extend various results of the classical Brunn–Minkowski theory of convex bodies to the coconvex setting. These include the Aleksandrov–Fenchel inequalities and the Brunn–Minkowski inequality, with reversed inequality signs. The derivation of the Aleksandrov–Fenchel inequalities for coconvex bodies from those for convex bodies is brief and particularly elegant.

In the following, we extend the concept of coconvex bodies, by weakening the requirement of boundedness to that of finite volume. By a *C-close set* we understand a closed convex set $A^\bullet \subset C$ such that $C \setminus A^\bullet$ has positive finite Lebesgue measure. (Note that the boundaries of $A^\bullet$ and C may have empty or nonempty intersection.) The set $A = C \setminus A^\bullet$ is then called a *C-coconvex set*, and its Lebesgue measure is denoted by $V_n(A)$ and is called its *volume*. The C-coconvex A set determines, conversely, the C-close set $A^\bullet = C \setminus A$.

For C-coconvex sets, we develop in this paper the first steps of a Brunn–Minkowski theory, that is, a study of the relations between the notion of volume and a notion of addition, based on vector addition.

Let A_0, A_1 be C-coconvex sets. Their *co-sum* is defined by

$$A_0 \oplus A_1 = C \setminus (A_0^\bullet + A_1^\bullet),$$

where + denotes the usual Minkowski addition. Note that $A_0^\bullet + A_1^\bullet \subset C + C = C$. Whereas the Minkowski sum of two unbounded closed convex sets need not be closed in general, it is easy to see that $A_0^\bullet + A_1^\bullet$ is closed, because $A_0^\bullet, A_1^\bullet$ are subsets of a pointed cone. That $A_0 \oplus A_1$ has finite volume, is a consequence of the following theorem. Here, $\lambda A := \{\lambda a : a \in A\}$ for $\lambda \geq 0$ and a C-coconvex set A.

Theorem 1. *Let A_0, A_1 be C-coconvex sets, and let $\lambda \in (0,1)$. Then*

$$V_n((1-\lambda)A_0 \oplus \lambda A_1)^{\frac{1}{n}} \leq (1-\lambda)V_n(A_0)^{\frac{1}{n}} + \lambda V_n(A_1)^{\frac{1}{n}}. \tag{1}$$

Equality holds if and only if $A_0 = \alpha A_1$ with some $\alpha > 0$.

The essential point here is the equality condition, which will be needed below. While the inequality (1) itself could be obtained by approximation from the results in [1], and is only a special case of much more general inequalities due to Milman and Rotem [2], we don't see an easy way to get the equality condition in either case. Our proof of (1), which adapts the classical Kneser–Süss approach to the Brunn–Minkowski inequality for convex bodies, yields the equality condition for (1) as a consequence of that for the latter inequality.

In the development of the classical Brunn–Minkowski theory for convex bodies, some of the first steps are the introduction of mixed volumes, their integral representation, and consequences of the Brunn–Minkowski theorem, such as Minkowski's first and second inequality for mixed volumes. A first application is the uniqueness result in the Minkowski problem concerning convex bodies with given surface area measures. We follow a similar line for C-coconvex sets. In particular, we prove a counterpart to Minkowski's uniqueness theorem. Let A be a C-coconvex set. Its area measure is defined as follows. Let

$$C^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 0 \text{ for all } y \in C\}$$

be the polar cone of C; here $\langle \cdot\,, \cdot \rangle$ denotes the scalar product of $\mathbb{R}^n$. Denoting by $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$, we define

$$\Omega_C := \mathbb{S}^{n-1} \cap \operatorname{int} C^\circ.$$

The spherical image $\sigma(A^\bullet, \beta)$ of the closed convex set $A^\bullet$ at the set β is the set of all outer unit normal vectors of $A^\bullet$ at points of $A^\bullet \cap \beta$. For the C-close set $A^\bullet$, we have $\sigma(A^\bullet, \operatorname{int} C) \subseteq \Omega_C$, since a supporting hyperplane of $A^\bullet$ at a point of $\operatorname{int} C \cap \operatorname{bd} A^\bullet$ (where bd denotes the boundary) separates $A^\bullet$ and the origin o. For $\omega \subseteq \Omega_C$, the reverse spherical image $\tau(A^\bullet, \omega)$ is defined as the set of all points in $\operatorname{bd} A^\bullet$ at which there exists an outer unit normal vector belonging to ω. For Borel sets $\omega \subseteq \Omega_C$ one then defines

$$\overline{S}_{n-1}(A, \omega) := S_{n-1}(A^\bullet, \omega) = \mathcal{H}^{n-1}(\tau(A^\bullet, \omega)),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (so that $S_{n-1}(A^\bullet, \cdot)$ is the usual surface area measure, extended to closed convex sets). Using the theory of surface area measures of convex bodies (see [3, Sect. 4.2]), it is easily seen that this defines a Borel measure on Ω_C, the *surface area measure* $\overline{S}_{n-1}(A, \cdot)$ of A. In contrast to the case of convex bodies, the surface area measure of a C-coconvex set is only defined on the subset Ω_C of $\mathbb{S}^{n-1}$, and the total measure may be infinite.

Now we can state a counterpart to Minkowski's uniqueness theorem.

Theorem 2. *If A_0, A_1 are C-coconvex sets with $\overline{S}_{n-1}(A_0, \cdot) = \overline{S}_{n-1}(A_1, \cdot)$, then $A_0 = A_1$.*

Some interesting open questions remain. For example, which are the necessary and sufficient conditions on a Borel measure on Ω_C to be the surface area measure of a C-coconvex set? And does the uniqueness still hold if the condition that $C \setminus A^\bullet$ has finite volume is replaced by the condition that $A^\bullet$ is only 'asymptotic' to C, in the sense that the distance of the boundaries of C and $A^\bullet$ outside $B(o,r)$ (ball with center o and radius r) tends to zero, as $r \to \infty$?

2 Notation and Preliminaries

We fix some notation, and collect what has already been introduced. We work in the n-dimensional Euclidean space $\mathbb{R}^n$, with scalar product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$. The unit sphere of $\mathbb{R}^n$ is the subspace $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. We use the k-dimensional Hausdorff measure $\mathcal{H}^k$ in $\mathbb{R}^n$, for $k = n$, which on Lebesgue measurable sets coincides with Lebesgue measure, and for $k = n-1$. We use V_n to denote the Lebesgue measure, then called the volume, of convex bodies or C-coconvex sets. $V_{n-1}(K)$ will be used to denote the $(n-1)$-dimensional volume of a convex body of dimension $n-1$. The definitions of the spherical image $\sigma(K, \cdot)$ and the reverse spherical image $\tau(K, \cdot)$, as given in [3, p. 88], do not require the boundedness and make sense for nonempty closed convex sets K.

A C-coconvex set A and the C-close set $A^\bullet := C \setminus A$ determine each other uniquely, and we have $(\lambda A)^\bullet = \lambda A^\bullet$, since $\lambda C = C$. Clearly, the volume of C-coconvex sets is homogeneous of degree n, that is,

$$V_n(\lambda A) = \lambda^n V_n(A) \quad \text{for } \lambda \geq 0.$$

Since the cone C, fixed throughout this paper, is pointed, we can choose a unit vector w such that $\langle x, w \rangle > 0$ for all $x \in C \setminus \{o\}$. The vector w will be fixed; therefore it does not appear in the notation used below. We define the hyperplanes

$$H_t := \{x \in \mathbb{R}^n : \langle x, w \rangle = t\}$$

and the closed halfspaces

$$H_t^- := \{x \in \mathbb{R}^n : \langle x, w \rangle \leq t\},$$

for $t \geq 0$. For a subset $M \subseteq C$, we define

$$M_t := M \cap H_t^-$$

for $t > 0$; thus, M_t is always bounded.

We remark that a C-coconvex set A has the property that its boundary inside $\operatorname{int} C$ 'can be seen' from o. In other words, every ray with endpoint o and passing through an interior point of C meets the boundary of A precisely once. This follows easily from the finiteness of the volume of A.

3 Proof of Theorem 1

The following proof of Theorem 1 has elements from the Kneser–Süss proof of the classical Brunn–Minkowski inequality (see, e.g., [3, pp. 370–371]).

Let A_0, A_1 be C-coconvex sets. First we assume that

$$V_n(A_0) = V_n(A_1) = 1. \tag{2}$$

Let $0 < \lambda < 1$ and define

$$A_\lambda^\bullet := (1-\lambda)A_0^\bullet + \lambda A_1^\bullet, \qquad A_\lambda := C \setminus A_\lambda^\bullet = (1-\lambda)A_0 \oplus \lambda A_1.$$

In the following, $\nu \in \{0,1\}$. We write

$$v_\nu(\zeta) := V_{n-1}(A_\nu^\bullet \cap H_\zeta), \qquad w_\nu(\zeta) := V_n(\mathcal{A}_\nu^\bullet \cap H_\zeta^-)$$

for $\zeta \geq 0$, thus

$$w_\nu(\zeta) = \int_{\alpha_\nu}^{\zeta} v_\nu(s)\,\mathrm{d}s,$$

where α_ν is the number for which H_{α_ν} supports $A_\nu^\bullet$. On (α_ν, ∞), the function v_ν is continuous, hence w_ν is differentiable and

$$w_\nu'(\zeta) = v_\nu(\zeta) > 0 \quad \text{for } \alpha_\nu < \zeta < \infty.$$

Let z_ν be the inverse function of w_ν, then

$$z_\nu'(\tau) = \frac{1}{v_\nu(z_\nu(\tau))} \quad \text{for } 0 < \tau < \infty.$$

With

$$D_\nu(\tau) := A_\nu^\bullet \cap H_{z_\nu(\tau)}, \quad z_\lambda(\tau) := (1-\lambda)z_0(\tau) + \lambda z_1(\tau),$$

the inclusion

$$A_\lambda^\bullet \cap H_{z_\lambda(\tau)} \supseteq (1-\lambda)D_0(\tau) + \lambda D_1(\tau) \tag{3}$$

holds (trivially). For $\tau > 0$ we have

$$\begin{aligned} V_n(A_\nu \cap H_{z_\nu(\tau)}^-) &= V_n(C \cap H_{z_\nu(\tau)}^-) - V_n(A_\nu^\bullet \cap H_{z_\nu(\tau)}^-) \\ &= V_n(C \cap H_{z_\nu(\tau)}^-) - \tau, \\ V_n(A_\lambda \cap H_{z_\lambda(\tau)}^-) &= V_n(C \cap H_{z_\lambda(\tau)}^-) - V_n(A_\lambda^\bullet \cap H_{z_\lambda(\tau)}^-). \end{aligned} \tag{4}$$

We write

$$V_n(A_\lambda^\bullet \cap H_{z_\lambda(\tau)}^-) =: f(\tau).$$

Then, with $\alpha_\lambda = (1-\lambda)\alpha_0 + \lambda\alpha_1$,

$$\begin{aligned} f(\tau) &= \int_{\alpha_\lambda}^{z_\lambda(\tau)} V_{n-1}(A_\lambda^\bullet \cap H_\zeta)\,\mathrm{d}\zeta \\ &= \int_0^\tau V_{n-1}(A_\lambda^\bullet \cap H_{z_\lambda(t)}) z_\lambda'(t)\,\mathrm{d}t \\ &\geq \int_0^\tau V_{n-1}((1-\lambda)D_0(t) + \lambda D_1(t)) z_\lambda'(t)\,\mathrm{d}t, \end{aligned}$$

by (3). In the integrand, we use the Brunn–Minkowski inequality in dimension $n-1$ and obtain

$$\begin{aligned} f(\tau) &\geq \int_0^\tau \left[(1-\lambda)v_0(z_0(t))^{\frac{1}{n-1}} + \lambda v_1(z_1(t))^{\frac{1}{n-1}}\right]^{n-1} \left[\frac{1-\lambda}{v_0(z_0(t))} + \frac{\lambda}{v_1(z_1(t))}\right]\mathrm{d}t \\ &\geq \tau, \end{aligned} \tag{5}$$

where the last inequality follows by estimating the integrand according to [3, p. 371].

From (4) we have

$$V_n(A_\lambda \cap H^-_{z_\lambda(\tau)}) = V_n(C \cap H^-_{z_\lambda(\tau)}) - f(\tau),$$

and we intend to let $\tau \to \infty$. Since C is a cone, for $\zeta > 0$,

$$C \cap H^-_\zeta = \zeta C_1 \quad \text{with} \quad C_1 := C \cap H^-_1$$

and hence $V_n(C \cap H^-_\zeta) = \zeta^n V_n(C_1)$. Therefore,

$$V_n(C \cap H^-_{z_\lambda(\tau)}) = [(1-\lambda)z_0(\tau) + \lambda z_1(\tau)]^n V_n(C_1), \qquad V_n(C \cap H^-_{z_\nu(\tau)}) = z_\nu(\tau)^n V_n(C_1).$$

This gives

$$\begin{aligned}
&V_n(A_\lambda \cap H^-_{z_\lambda(\tau)}) \\
&= \left[(1-\lambda)V_n(C \cap H^-_{z_0(\tau)})^{\frac{1}{n}} + \lambda V_n(C \cap H^-_{z_1(\tau)})^{\frac{1}{n}}\right]^n - f(\tau) \\
&= \left[(1-\lambda)[V_n(A_0 \cap H^-_{z_0(\tau)}) + \tau]^{\frac{1}{n}} + \lambda [V_n(A_1 \cap H^-_{z_1(\tau)}) + \tau]^{\frac{1}{n}}\right]^n - f(\tau) \\
&= \left[(1-\lambda)[b_0(\tau) + \tau]^{\frac{1}{n}} + \lambda [b_1(\tau) + \tau]^{\frac{1}{n}}\right]^n - f(\tau)
\end{aligned}$$

with $b_\nu(\tau) = V_n(A_\nu \cap H^-_{z_\nu(\tau)})$ for $\nu = 0, 1$. Note that (2) implies

$$\lim_{\tau \to \infty} b_\nu(\tau) = 1.$$

Using the mean value theorem (for each fixed τ), we can write

$$(b_1(\tau) + \tau)^{\frac{1}{n}} - (b_0(\tau) + \tau)^{\frac{1}{n}} = (b_1(\tau) - b_0(\tau))\frac{1}{n}(b(\tau) + \tau)^{\frac{1}{n}-1}$$

with $b(\tau)$ between $b_0(\tau)$ and $b_1(\tau)$, and hence tending to 1 as $\tau \to \infty$. With $\frac{1}{n}(b(\tau)+\tau)^{\frac{1}{n}-1} =: h(\tau) = O\left(\tau^{\frac{1-n}{n}}\right)$ (as $\tau \to \infty$), we get

$$\begin{aligned}
&V_n(A_\lambda \cap H^-_{z_\lambda(\tau)}) \\
&= \left[(1-\lambda)(b_0(\tau) + \tau)^{\frac{1}{n}} + \lambda\left((b_0(\tau) + \tau)^{\frac{1}{n}} + (b_1(\tau) - b_0(\tau))h(\tau)\right)\right]^n - f(\tau) \\
&= \left[(b_0(\tau) + \tau)^{\frac{1}{n}} + \lambda(b_1(\tau) - b_0(\tau))h(\tau)\right]^n - f(\tau) \\
&= b_0(\tau) + \tau - f(\tau) + \sum_{r=1}^{n} \binom{n}{r} (b_0(\tau) + \tau)^{\frac{n-r}{n}} \left[\lambda(b_1(\tau) - b_0(\tau))\right]^r h(\tau)^r.
\end{aligned}$$

Since $b_0(\tau) \to 1$, $f(\tau) \geq \tau$, $(b_0(\tau) + \tau)^{\frac{n-r}{n}} h(\tau)^r = O(\tau^{1-r})$, and $b_1(\tau) - b_0(\tau) \to 0$ as $\tau \to \infty$, we conclude that

$$V_n(A_\lambda) = \lim_{\tau \to \infty} V_n(A_\lambda \cap H^-_{z_\lambda(\tau)}) \leq 1.$$

This proves that

$$V_n((1-\lambda)A_0 \oplus \lambda A_1) \leq 1. \tag{6}$$

If there exists a number $\tau_0 > 0$ for which $f(\tau_0) = \tau_0 + \varepsilon$ with $\varepsilon > 0$, then, for $\tau > \tau_0$,

$$\begin{aligned}
f(\tau) &= V_n(A_\lambda^\bullet \cap H^-_{z_\lambda(\tau)}) \\
&= \tau_0 + \varepsilon + \int_{\tau_0}^{\tau} \left[(1-\lambda)v_0(z_0(t))^{\frac{1}{n-1}} + \lambda v_1(z_1(t))^{\frac{1}{n-1}}\right]^{n-1} \left[\frac{1-\lambda}{v_0(z_0(t))} + \frac{\lambda}{v_1(z_1(t))}\right] \mathrm{d}t \\
&\geq \tau_0 + \varepsilon + (\tau - \tau_0) = \tau + \varepsilon,
\end{aligned}$$

and as above we obtain that $V_n(A_\lambda) \leq 1 - \varepsilon$.

Suppose now that (6) holds with equality. Then, as just shown, we have $f(\tau) = \tau$ for all $\tau \geq 0$. Thus, we have equality in (5) and hence equality in (3), for all $\tau \geq 0$. Explicitly, this means that

$$A_\lambda^\bullet \cap H_{z_\lambda(\tau)} = (1-\lambda)(A_0^\bullet \cap H_{z_0(\tau)}) + \lambda(A_1^\bullet \cap H_{z_1(\tau)}) \quad \text{for all } \tau \geq 0. \tag{7}$$

We claim that this implies

$$A_\lambda^\bullet \cap H^-_{z_\lambda(\tau)} = (1-\lambda)(A_0^\bullet \cap H^-_{z_0(\tau)}) + \lambda(A_1^\bullet \cap H^-_{z_1(\tau)}) \tag{8}$$

for all $\tau \geq 0$. For the proof, let $x \in A_\lambda^\bullet \cap H^-_{z_\lambda(\tau)}$. Then there is a number $\sigma \in [0, \tau]$ such that $x \in A_\lambda^\bullet \cap H_{z_\lambda(\sigma)}$. By (7),

$$\begin{aligned}
x &\in (1-\lambda)(A_0^\bullet \cap H_{z_0(\sigma)}) + \lambda(A_1^\bullet \cap H_{z_1(\sigma)}) \\
&\subset (1-\lambda)(A_0^\bullet \cap H^-_{z_0(\tau)}) + \lambda(A_1^\bullet \cap H^-_{z_1(\tau)}),
\end{aligned}$$

since $\sigma \leq \tau$ implies $H_{z_\nu(\sigma)} \subset H^-_{z_\nu(\tau)}$. This shows the inclusion $\subseteq$ in (8). The inclusion $\supseteq$ is trivial.

To (8), we can now apply the Brunn–Minkowski inequality for n-dimensional convex bodies and conclude that

$$V_n(A_\lambda^\bullet \cap H^-_{z_\lambda(\tau)}) \geq \tau.$$

But we know that equality holds here, since equality holds in (5), hence the convex bodies $A_0^\bullet \cap H^-_{z_0(\tau)}$ and $A_1^\bullet \cap H^-_{z_1(\tau)}$, which have the same volume, are translates of each other. The translation vector might depend on τ, but in fact, it does not, since for $0 < \sigma < \tau$, the body $A_\nu^\bullet \cap H_{z_\nu(\sigma)}$ is the intersection of $A_\nu^\bullet \cap H_{z_\nu(\tau)}$ with a closed halfspace. We conclude that $A_1^\bullet$ is a translate of $A_0^\bullet$, thus there is a vector v with $A_0^\bullet + v = A_1^\bullet \subset C$. Suppose that $v \neq o$. Let M be the set of all points $x \in \operatorname{int} C \cap \operatorname{bd} A_0^\bullet$ for which $x + \lambda v \notin A_0^\bullet$ for $\lambda > 0$. The set $\bigcup_{x \in M}(x, x+v]$ is contained in A_0 and has infinite Lebesgue measure, a contradiction. Thus, $v = o$ and hence $A_0^\bullet = A_1^\bullet$.

This proves Theorem 1 under the assumption (2). Now let A_0, A_1 be arbitrary C-coconvex sets. As mentioned, also the volume of C-coconvex sets is homogeneous of degree n. Therefore (as in the case of convex bodies, see [3, p. 370]), we define

$$\overline{A}_\nu := V_n(A_\nu)^{-1/n} A_\nu \ \text{ for } \nu = 0, 1, \qquad \overline{\lambda} := \frac{\lambda V_n(A_1)^{1/n}}{(1-\lambda)V_n(A_0)^{1/n} + \lambda V_n(A_1)^{1/n}}.$$

Then $V_n(\overline{A}_\nu) = 1$ for $\nu = 0, 1$, hence $V_n((1-\overline{\lambda})\overline{A}_0 \oplus \overline{\lambda}\,\overline{A}_1) \leq 1$, as just proved. This gives the assertion.

4 A volume representation

The proof of Theorem 2 requires that we develop the initial steps of a theory of mixed volumes for C-coconvex sets. First we derive an integral representation of the volume of C-coconvex sets.

Let A be a C-coconvex set, and let $u \in \Omega_C$. Since $o \notin A^\bullet$ (because $A \neq \emptyset$), there is a supporting halfspace of $A^\bullet$ with outer normal vector u and not containing o. Therefore, the support function $h(A^\bullet, \cdot)$ of $A^\bullet$, defined by $h(A^\bullet, u) = \sup\{\langle x, u\rangle : x \in A^\bullet\}$ for $u \in \Omega_C$, satisfies

$$-\infty < h(A^\bullet, u) < 0 \quad \text{for } u \in \Omega_C.$$

We set

$$\overline{h}(A, u) := -h(A^\bullet, u)$$

and call the function $\overline{h}(A, \cdot) : \Omega_C \to \mathbb{R}_+$ thus defined the *support function* of A. The *area measure* $\overline{S}_{n-1}(A, \cdot)$ of A was already defined, namely by

$$\overline{S}_{n-1}(A, \omega) := S_{n-1}(A^\bullet, \omega) = \mathcal{H}^{n-1}(\tau(A^\bullet, \omega))$$

for Borel sets $w \subseteq \Omega_C$. Recall that $\tau(A^\bullet, \omega)$ was defined as the set of boundary points of $A^\bullet$ at which there exists an outer unit normal vector falling in ω.

The volume of the C-coconvex body A has an integral representation similar to that in the case of convex bodies, as stated in the following lemma.

Lemma 1. *The volume of a C-coconvex set A can be represented by*

$$V_n(A) = \frac{1}{n} \int_{\Omega_C} \overline{h}(A, u)\, \overline{S}_{n-1}(A, \mathrm{d}u). \tag{9}$$

Proof. Recall that $M_t := M \cap H_t^-$ for $M \subseteq C$, in particular, $C_t = C \cap H_t^-$. We write $(A^\bullet)_t = A_t^\bullet$, and later also $(A_i^\bullet)_t = A_{i,t}^\bullet$.

Let $t > 0$ be such that $A_t^\bullet$ has interior points. Let

$$\omega_t := \sigma(A_t^\bullet, \operatorname{int} C_t),$$

that is, the spherical image of the set of boundary points of $A^\bullet$ in the interior of C_t. Further, let

$$\eta_t := \sigma(A_t^\bullet, \operatorname{bd} C) \cap \operatorname{bd} C^\circ.$$

By a standard representation of the volume of convex bodies (formula (5.3) in [3]), we have

$$V_n(A_t^\bullet) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(A_t^\bullet, u)\, S_{n-1}(A_t^\bullet, \mathrm{d}u).$$

Here,

$$\int_{\eta_t} h(A_t^\bullet, u)\, S_{n-1}(A_t^\bullet, \mathrm{d}u) = 0,$$

since $u \in \eta_t$ implies $h(A_t^\bullet, u) = 0$. We state that

$$S_{n-1}(A_t^\bullet, \mathbb{S}^{n-1} \setminus (\omega_t \cup \eta_t \cup \{w\})) = 0. \tag{10}$$

For the proof, let x be a boundary point of $A_t^\bullet$ where a vector $u \in \mathbb{S}^{n-1} \setminus (\omega_t \cup \eta_t \cup \{w\})$ is attained as outer normal vector. Then $x \notin \operatorname{int} C_t$ and hence $x \in H_t$ or $x \in \operatorname{bd} C$. If

$x \in H_t$, then $u \neq w$ implies that x lies in two distinct supporting hyperplanes of $A^\bullet_t$. If $x \in (\operatorname{bd} C) \setminus H_t$, then $u \notin \eta_t$ implies that x lies in two distinct supporting hyperplanes of $A^\bullet_t$. In each case, x is a singular boundary point of $A^\bullet_t$. Now the assertion (10) follows from [3, (4.32) and Thm. 2.2.5].

As a result, we have

$$V_n(A^\bullet_t) = \frac{1}{n} \int_{\omega_t \cup \{w\}} h(A^\bullet_t, u)\, S_{n-1}(A^\bullet_t, \mathrm{d}u).$$

Since

$$h(A^\bullet_t, w) = t, \qquad S_{n-1}(A^\bullet_t, \{w\}) = V_{n-1}(A^\bullet \cap H_t),$$

we obtain

$$V_n(A^\bullet_t) = -\frac{1}{n} \int_{\omega_t} \overline{h}(A, u)\, \overline{S}_{n-1}(A, \mathrm{d}u) + \frac{1}{n} t V_{n-1}(A^\bullet \cap H_t),$$

by the definition of $\overline{h}(A, \cdot)$ and $\overline{S}_{n-1}(A, \cdot)$. Writing

$$B(t) := \operatorname{conv}((A^\bullet \cap H_t) \cup \{o\}) \setminus A^\bullet_t,$$

we have

$$V_n(B(t)) = \frac{1}{n} t V_{n-1}(A^\bullet \cap H_t) - V_n(A^\bullet_t)$$

and thus

$$V_n(B(t)) = \frac{1}{n} \int_{\omega_t} \overline{h}(A, u)\, \overline{S}_{n-1}(A, \mathrm{d}u).$$

On the other hand, writing

$$q(t) := V_{n-1}(C \cap H_t) - V_{n-1}(A^\bullet \cap H_t),$$

we get

$$V_n(A_t) = V_n(B(t)) + \frac{1}{n} t q(t) = \frac{1}{n} \int_{\omega_t} \overline{h}(A, u)\, \overline{S}_{n-1}(A, \mathrm{d}u) + \frac{1}{n} t q(t).$$

Given $\varepsilon > 0$, to each $t_0 > 0$ there exists $t \geq t_0$ with $tq(t) < \varepsilon$. Otherwise, there would exist t_0 with $tq(t) \geq \varepsilon$ for $t \geq t_0$ and hence $\int_{t_0}^{\infty} q(t) \mathrm{d}t = \infty$, which yields $V_n(A) = \infty$, a contradiction. Therefore, we can choose an increasing sequence $(t_i)_{i \in \mathbb{N}}$ with $t_i \to \infty$ for $i \to \infty$ such that $t_i q(t_i) \to 0$. From

$$V_n(A_{t_i}) = \frac{1}{n} \int_{\omega_{t_i}} \overline{h}(A, u)\, \overline{S}_{n-1}(A, \mathrm{d}u) + \frac{1}{n} t_i q(t_i)$$

and $\omega_{t_i} \uparrow \Omega_C$ we then obtain

$$V_n(A) = \frac{1}{n} \int_{\Omega_C} \overline{h}(A, u)\, \overline{S}_{n-1}(A, \mathrm{d}u),$$

as stated. □

5 Mixed volumes of bounded C-coconvex sets

First we introduce, in this section, mixed volumes and their representations for bounded coconvex sets. Let A be a bounded C-coconvex set. Then $A \subset \operatorname{int} H_t^-$ for all sufficiently large t. For bounded C-coconvex sets $A_1, \dots, A_{n-1}$, we define their *mixed area measure* by

$$\overline{S}(A_1, \dots, A_{n-1}, \omega) = S(A_{1,t}^\bullet, \dots, A_{n-1,t}^\bullet, \omega)$$

for Borel sets $\omega \subseteq \Omega_C$, where t is chosen sufficiently large. Here $S(A_{1,t}^\bullet, \dots, A_{n-1,t}^\bullet, \cdot)$ is the usual mixed area measure of the convex bodies $A_{1,t}^\bullet, \dots, A_{n-1,t}^\bullet$ (see [3, Sect. 5.1]). Clearly, the definition does not depend on t. It should be noted that the mixed area measure of bounded C-coconvex sets is only defined on Ω_C, and it is finite. For bounded C-coconvex sets $A_1, \dots, A_n$, we define their *mixed volume* by

$$\overline{V}(A_1, \dots, A_n) = \frac{1}{n} \int_{\Omega_C} \overline{h}(A_1, u)\, \overline{S}(A_2, \dots, A_n, \mathrm{d}u). \tag{11}$$

Lemma 2. *The mixed volume $\overline{V}(A_1, \dots, A_n)$ is symmetric in $A_1, \dots, A_n$.*

Proof. We choose t so large that $A_i \subset H_t^-$ for $i = 1, \dots, n$. The mixed volume of the convex bodies $A_{1,t}^\bullet, \dots, A_{n,t}^\bullet$ is given by

$$V(A_{1,t}^\bullet, \dots, A_{n,t}^\bullet) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(A_{1,t}^\bullet, u)\, S(A_{2,t}^\bullet, \dots, A_{n,t}^\bullet, \mathrm{d}u).$$

The sphere $\mathbb{S}^{n-1}$ is the disjoint union of the sets

$$\Omega_C,\ \mathbb{S}^{n-1} \cap \operatorname{bd} C^\circ,\ \{w\},\ \text{and the remaining set } \omega_0.$$

For $u \in \mathbb{S}^{n-1} \cap \operatorname{bd} C^\circ$, we have $h(A_{1,t}^\bullet, u) = 0$. Since for each body $A_{i,t}^\bullet$ the support set with outer normal vector w is equal to $C \cap H_t$, we get $S_{n-1}(A_{i,t}^\bullet, \{w\}) = V_{n-1}(C \cap H_t)$ for $i = 2, \dots, n$ and thus, by [3, (5.18)],

$$S(A_{2,t}^\bullet, \dots, A_{n,t}^\bullet, \{w\}) = V_{n-1}(C \cap H_t).$$

Therefore,

$$\frac{1}{n} \int_{\{w\}} h(A_{1,t}^\bullet, u)\, S(A_{2,t}^\bullet, \dots, A_{n,t}^\bullet, \mathrm{d}u) = \frac{1}{n} t V_{n-1}(C \cap H_t) = V_n(C_t).$$

Further, we have

$$S(A_{2,t}^\bullet, \dots, A_{n,t}^\bullet, \omega_0) = 0, \tag{12}$$

since for $\lambda_2, \dots, \lambda_n \geq 0$, the convex body $\lambda_2 A_{2,t}^\bullet + \dots + \lambda_n A_{n,t}^\bullet$ has the property that any of its points at which some $u \in \omega_0$ is an outer normal vector, is a singular point. Equation (12) then follows from [3, (5.21) and Thm. 2.2.5]. As a result, we obtain

$$\begin{aligned} V(A_{1,t}^\bullet, \dots, A_{n,t}^\bullet) &= V_n(C_t) + \frac{1}{n} \int_{\Omega_C} h(A_{1,t}^\bullet, u)\, S(A_{2,t}^\bullet, \dots, A_{n,t}^\bullet, \mathrm{d}u) \\ &= V_n(C_t) - \frac{1}{n} \int_{\Omega_C} \overline{h}(A_1, u)\, \overline{S}(A_2, \dots, A_n, \mathrm{d}u) \\ &= V_n(C_t) - \overline{V}(A_1, \dots, A_n). \end{aligned}$$

Since $V(A_{1,t}^\bullet, \dots, A_{n,t}^\bullet)$ is symmetric in its arguments, also $\overline{V}(A_1, \dots, A_n)$ is symmetric in its arguments. $\square$

Now let $A_1, \ldots, A_m$, with $m \in \mathbb{N}$, be bounded C-coconvex sets, and choose $t > 0$ with $A_i \subset C_t$ for $i = 1, \ldots, m$. By (9), for $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\begin{aligned} & V_n(\lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m) \\ & = \frac{1}{n} \int_{\Omega_C} \overline{h}(\lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m, u)\, \overline{S}_{n-1}(\lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m, \mathrm{d}u). \end{aligned}$$

Here, for $u \in \Omega_C$,

$$\begin{aligned} \overline{h}(\lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m, u) &= -h((\lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m)^\bullet, u) \\ &= -h(\lambda_1 A_1^\bullet + \cdots + \lambda_m A_m^\bullet, u) \\ &= -[\lambda_1 h(A_1^\bullet, u) + \cdots + \lambda_m (A_m^\bullet, u)] \end{aligned} \tag{13}$$

and, for t sufficiently large and Borel sets $\omega \subseteq \Omega_C$,

$$\begin{aligned} & \overline{S}_{n-1}(\lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m, \omega) \\ & = S_{n-1}(\lambda_1 A_{t,1}^\bullet + \cdots + \lambda_m A_{t,m}^\bullet, \omega) \qquad (14) \\ & = \sum_{i_1, \ldots, i_{n-1}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n-1}} S(A_{t,i_1}^\bullet, \ldots, A_{t,i_{n-1}}^\bullet, \omega), \\ & = \sum_{i_1, \ldots, i_{n-1}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \overline{S}(A_{i_1}, \ldots, A_{i_{n-1}}, \omega), \end{aligned}$$

by [3, (5.18)]. Using Lemma 2, we conclude that

$$V_n(\lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m) = \sum_{i_1, \ldots, i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \overline{V}(A_{i_1}, \ldots, A_{i_n}), \tag{15}$$

in analogy to [3, (5.17)].

6 Mixed volumes of general C-coconvex sets

We extend the mixed volumes to not necessarily bounded C-coconvex sets. For this, we use approximation by mixed volumes of bounded C-coconvex sets.

Let $\omega \subset \Omega_C$ be an open subset whose closure (in $\mathbb{S}^{n-1}$) is contained in Ω_C. Let A be a C-coconvex set, so that $A^\bullet = C \setminus A$ is closed and convex. We define

$$A_{(\omega)}^\bullet := C \cap \bigcap_{u \in \omega} H^-(A^\bullet, u), \qquad A_{(\omega)} := C \setminus A_{(\omega)}^\bullet,$$

where $H^-(A^\bullet, u)$ denotes the supporting halfspace of the closed convex set $A^\bullet$ with outer normal vector u. We claim that $A_{(\omega)}$ is bounded. For the proof, we note that the set ω, whose closure, $\operatorname{clos} \omega$, is contained in Ω_C, has a positive distance from the boundary of Ω_C (relative to $\mathbb{S}^{n-1}$). Therefore, there is a number $a_0 > 0$ such that

$$\langle x, u \rangle \leq -a_0 \quad \text{for } x \in C \text{ with } \|x\| = 1 \text{ and } u \in \omega. \tag{16}$$

Let $x \in A_{(\omega)}$. Then there is some $u \in \omega$ with $x \notin H^-(A^\bullet, u)$, hence with $\langle x, u\rangle > h(A^\bullet, u)$. Since $\langle x, u\rangle \le -a_0\|x\|$ by (16), we obtain

$$\|x\| \le \frac{1}{a_0} \max\{-h(A^\bullet, u) : u \in \operatorname{clos}\omega\}.$$

Thus, $A_{(\omega)}$ is a bounded C-coconvex set.

With A and ω as above, we associate another set, namely

$$A[\omega] := \bigcup_{x \in \tau(A^\bullet, \omega) \cap \operatorname{int} C} (o, x),$$

where (o, x) denotes the open line segment with endpoints o and x. We choose an increasing sequence $(\omega_j)_{j\in\mathbb{N}}$ of open subsets of Ω_C with closures in Ω_C and with $\bigcup_{j\in\mathbb{N}} \omega_j = \Omega_C$. Then

$$A[\omega_j] \uparrow \operatorname{int} A \qquad \text{as } j \to \infty. \tag{17}$$

In fact, that the set sequence is increasing, follows from the definition. Let $y \in \operatorname{int} A$. Then there is a boundary point x of $A^\bullet$ with $y \in (o, x)$. Let u be an outer unit normal vector of $A^\bullet$ at x. Then $u \in \Omega_C$, hence $u \in \omega_j$ for some j. For this j, we have $y \in A[\omega_j]$.

Lemma 3. *If $A_1, \dots, A_n$ are C-coconvex sets and $\lambda_1, \dots, \lambda_n \ge 0$, then*

$$\lim_{j\to\infty} V_n(\lambda_1 A_{1(\omega_j)} \oplus \cdots \oplus \lambda_n A_{n(\omega_j)}) = V_n(\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n). \tag{18}$$

Proof. We state that

$$(\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n)[\omega_j] \subseteq \lambda_1 A_{1(\omega_j)} \oplus \cdots \oplus \lambda_n A_{n(\omega_j)} \subseteq \lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n. \tag{19}$$

For the proof of the first inclusion, let $y \in (\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n)[\omega_j]$. Then there exists a point $x \in \tau(\lambda_1 A_1^\bullet + \cdots + \lambda_n A_n^\bullet, \omega_j) \cap \operatorname{int} C$ with $y \in (o, x)$. Let $u \in \omega_j$ be an outer unit normal vector of $\lambda_1 A_1^\bullet + \cdots + \lambda_n A_n^\bullet$ at x. Denoting by $F(K, u)$ the support set of a closed convex set K with outer normal vector u, we have (by [3, Thm. 1.7.5])

$$F(\lambda_1 A_1^\bullet + \cdots + \lambda_n A_n^\bullet, u) = \lambda_1 F(A_1^\bullet, u) + \cdots + \lambda_n F(A_n^\bullet, u),$$

hence there are points $x_i \in F(A_i^\bullet, u)$ $(i = 1, \dots, n)$ with $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$. We have $x_i \in A^\bullet_{i(\omega_j)}$, hence $x \in \lambda_1 A_{1(\omega_j)} \oplus \cdots \oplus \lambda_n A_{n(\omega_j)}$. This proves the first inclusion of (19). The second inclusion follows immediately from the definitions. From (19) and (17) we obtain

$$\lambda_1 A_{1(\omega_j)} \oplus \cdots \oplus \lambda_n A_{n(\omega_j)} \uparrow \operatorname{int}(\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n) \qquad \text{as } j \to \infty,$$

from which the assertion (18) follows. □

For the bounded C-coconvex sets $A_{1(\omega_j)}, \dots, A_{n(\omega_j)}$ we have from (15) that

$$V_n(\lambda_1 A_{1(\omega_j)} \oplus \cdots \oplus \lambda_n A_{n(\omega_j)}) = \sum_{i_1,\dots,i_n=1}^{n} \lambda_{i_1} \cdots \lambda_{i_n} \overline{V}(A_{i_1(\omega_j)}, \cdots, A_{i_n(\omega_j)}).$$

By Lemma 3, the left side converges, for $j \to \infty$, to $V_n(\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n)$. Since this holds for all $\lambda_1, \dots, \lambda_n \ge 0$, we can conclude that the limit

$$\lim_{j\to\infty} \overline{V}(A_{i_1(\omega_j)}, \dots, A_{i_n(\omega_j)}) =: \overline{V}(A_{i_1}, \dots, A_{i_n})$$

exists and that

$$V_n(\lambda_1 A_1 \oplus \cdots \oplus \lambda_n A_n) = \sum_{i_1,\dots,i_n=1}^{n} \lambda_{i_1}\cdots\lambda_{i_n} \overline{V}(A_{i_1},\dots,A_{i_n}). \tag{20}$$

We call $\overline{V}(A_1,\dots,A_n)$ the *mixed volume* of the C-coconvex sets $A_1,\dots,A_n$.

For this mixed volume, we shall now establish an integral representation. To that end, we note first that the support functions of $A^\bullet$ and $A^\bullet_{(\omega_j)}$ satisfy

$$h(A^\bullet, u) = h(A^\bullet_{(\omega_j)}, u) \quad \text{for } u \in \omega_j. \tag{21}$$

Since ω_j is open, then for $u \in \omega_j$ the support functions of $A^\bullet$ and $A^\bullet_{(\omega_j)}$ coincide in a neighborhood of u. By [3, Thm. 1.7.2], the support sets of $A^\bullet$ and $A^\bullet_{(\omega_j)}$ with outer normal vector u are the same. It follows that $\tau(A^\bullet,\omega_j) = \tau(A^\bullet_{(\omega_j)},\omega_j)$ and, therefore, that also

$$S_{n-1}(A^\bullet,\cdot) = S_{n-1}(A^\bullet_{(\omega_j)},\cdot) \quad \text{on } \omega_j. \tag{22}$$

More generally, if $A_1,\dots,A_{n-1}$ are C-coconvex sets, we can define their mixed area measure by

$$S(A^\bullet_1,\dots,A^\bullet_{n-1},\cdot) = S(A^\bullet_{1(\omega_j)},\dots,A^\bullet_{n-1(\omega_j)},\cdot) \quad \text{on } \omega_j, \tag{23}$$

for $j \in \mathbb{N}$. Since $\omega_j \uparrow \Omega_C$, this yields a Borel measure on all of Ω_C. It need not be finite. Then we define

$$\overline{S}(A_1,\dots,A_{n-1},\cdot) := S(A^\bullet_1,\dots,A^\bullet_{n-1},\cdot).$$

By Lemma 1, (21) and (22) we have

$$\begin{aligned} & V_n(A_{(\omega_j)}) \\ &= \frac{1}{n}\int_{\omega_j} \overline{h}(A_{(\omega_j)},u)\,\overline{S}_{n-1}(A_{(\omega_j)},\mathrm{d}u) + \frac{1}{n}\int_{\Omega_C\setminus\omega_j} \overline{h}(A_{(\omega_j)},u)\,\overline{S}_{n-1}(A_{(\omega_j)},\mathrm{d}u) \\ &= \frac{1}{n}\int_{\omega_j} \overline{h}(A,u)\,\overline{S}_{n-1}(A,\mathrm{d}u) + \frac{1}{n}\int_{\Omega_C\setminus\omega_j} \overline{h}(A_{(\omega_j)},u)\,\overline{S}_{n-1}(A_{(\omega_j)},\mathrm{d}u). \end{aligned}$$

From $A_{(\omega_j)} \uparrow A$ we get

$$\lim_{j\to\infty} V_n(A_{(\omega_j)}) = V_n(A), \tag{24}$$

and $\omega_j \uparrow \Omega_C$ gives

$$\lim_{j\to\infty} \frac{1}{n}\int_{\omega_j} \overline{h}(A,u)\,\overline{S}_{n-1}(A,\mathrm{d}u) = \frac{1}{n}\int_{\Omega_C} \overline{h}(A,u)\,\overline{S}_{n-1}(A,\mathrm{d}u) = V_n(A).$$

It follows that

$$\lim_{j\to\infty} \int_{\Omega_C\setminus\omega_j} \overline{h}(A_{(\omega_j)},u)\,\overline{S}_{n-1}(A_{(\omega_j)},\mathrm{d}u) = 0. \tag{25}$$

From (11) and using (22) and (23), we get

$$
\begin{aligned}
&\overline{V}(A_{1(\omega_j)},\ldots,A_{n(\omega_j)})\\
&=\frac{1}{n}\int_{\omega_j}\overline{h}(A_{1(\omega_j)},u)\,\overline{S}(A_{2(\omega_j)},\ldots,A_{n(\omega_j)},\mathrm{d}u)\\
&\quad+\frac{1}{n}\int_{\Omega_C\setminus\omega_j}\overline{h}(A_{1(\omega_j)},u)\,\overline{S}(A_{2(\omega_j)},\ldots,A_{n(\omega_j)},\mathrm{d}u)\\
&=\frac{1}{n}\int_{\omega_j}\overline{h}(A_1,u)\,\overline{S}(A_2,\ldots,A_n,\mathrm{d}u)\\
&\quad+\frac{1}{n}\int_{\Omega_C\setminus\omega_j}\overline{h}(A_{1(\omega_j)},u)\,\overline{S}(A_{2(\omega_j)},\ldots,A_{n(\omega_j)},\mathrm{d}u)
\end{aligned}
\tag{26}
$$

Writing $A := A_1 \oplus \cdots \oplus A_n$ we have the trivial estimates

$$
\overline{h}(A_{1(\omega_j)},u)\le\overline{h}(A_{(\omega_j)},u),\qquad \overline{S}(A_{2(\omega_j)},\ldots,A_{n(\omega_j)},\cdot)\le\overline{S}_{n-1}(A_{(\omega_j)},\cdot).
$$

Hence, the term (26) can be estimated by

$$
\begin{aligned}
&\frac{1}{n}\int_{\Omega_C\setminus\omega_j}\overline{h}(A_{1(\omega_j)},u)\,\overline{S}(A_{2(\omega_j)},\ldots,A_{n(\omega_j)},\mathrm{d}u)\\
&\le\frac{1}{n}\int_{\Omega_C\setminus\omega_j}\overline{h}(A_{(\omega_j)},u)\,\overline{S}_{n-1}(A_{(\omega_j)},\mathrm{d}u),
\end{aligned}
$$

and by (25) this tends to zero for $j \to \infty$. We conclude that

$$
\overline{V}(A_1,\ldots,A_n)=\frac{1}{n}\int_{\Omega_C}\overline{h}(A_1,u)\,\overline{S}(A_2,\ldots,A_n,\mathrm{d}u). \tag{27}
$$

7 Proof of Theorem 2

Theorem 1 together with the polynomial expansion (20) now allows similar conclusions as in the case of convex bodies. Let A_0, A_1 be C-coconvex sets, and write $A_\lambda = (1-\lambda)A_0 \oplus \lambda A_1$ for $0 \le \lambda \le 1$. A special case of (20) reads

$$
V_n(A_\lambda)=\sum_{i=0}^{n}\binom{n}{i}(1-\lambda)^{n-i}\lambda^i\overline{V}(\underbrace{A_0,\ldots,A_0}_{n-i},\underbrace{A_1,\ldots,A_1}_{i}).
$$

The function f defined by $f(\lambda) = V_n(A_\lambda)^{1/n} - (1-\lambda)V_n(A_0)^{1/n} - \lambda V_n(A_1)^{1/n}$ for $0 \le \lambda \le 1$ is convex, as follows from Theorem 1 and a similar argument as in the case of convex bodies (see [3, pp. 369–370]). Also as in the convex body case (see [3, p. 382]), one obtains the counterpart to Minkowski's first inequality, namely

$$
\overline{V}(A_0,\ldots,A_0,A_1)^n \le V_n(A_0)^{n-1}V_n(A_1), \tag{28}
$$

with equality if and only if $A_0 = \alpha A_1$ with some $\alpha > 0$.

Now we assume, as in Theorem 2, that A_0, A_1 are C-coconvex sets with $\overline{S}_{n-1}(A_0,\cdot) = \overline{S}_{n-1}(A_1,\cdot)$. By (27),

$$\overline{V}(A_0,\dots,A_0,A_1) = \frac{1}{n}\int_{\Omega_C} \overline{h}(A_1,u)\,\overline{S}_{n-1}(A_0,\mathrm{d}u).$$

Therefore, the assumption gives $\overline{V}(A_0,\dots,A_0,A_1) = V_n(A_1)$. Similarly, $\overline{V}(A_1,\dots,A_1,A_0) = V_n(A_0)$, hence multiplication gives $\overline{V}(A_0,\dots,A_0,A_1)\overline{V}(A_1,\dots,A_1,A_0) = V_n(A_0)V_n(A_1)$. On the other hand, from (28) we get $\overline{V}(A_0,\dots,A_0,A_1)\overline{V}(A_1,\dots,A_1,A_0) \le V_n(A_0)V_n(A_1)$. Thus, equality holds here, and hence in (28), which implies that $A_0 = \alpha A_1$ with $\alpha > 0$. Since $\overline{S}_{n-1}(A_0,\cdot) = \overline{S}_{n-1}(A_1,\cdot)$, we have $\alpha = 1$.

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