Integral geometry – measure theoretic approach and stochastic applications

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Preface

Integral geometry, as it is understood here, deals with the computation and application of geometric mean values with respect to invariant measures. In the following, I want to give an introduction to the integral geometry of polyconvex sets (i.e., finite unions of compact convex sets) in Euclidean spaces. The invariant or Haar measures that occur will therefore be those on the groups of translations, rotations, or rigid motions of Euclidean space, and on the affine Grassmannians of k-dimensional affine subspaces. However, it is also in a different sense that invariant measures will play a central role, namely in the form of finitely additive functions on polyconvex sets. Such functions have been called additive functionals or valuations in the literature, and their motion invariant specializations, now called intrinsic volumes, have played an essential role in Hadwiger's [2] and later work (e.g., [8]) on integral geometry. More recently, the importance of these functionals for integral geometry has been rediscovered by Rota [5] and Klain-Rota [4], who called them 'measures' and emphasized their role in certain parts of geometric probability. We will, more generally, deal with local versions of the intrinsic volumes, the curvature measures, and derive integral-geometric results for them. This is the third aspect of the measure theoretic approach mentioned in the title. A particular feature of this approach is the essential role that uniqueness results for invariant measures play in the proofs.

As prerequisites, we assume some familiarity with basic facts from measure and integration theory. We will also have to use some notions and results from the geometry of convex bodies. These are intuitive and easy to grasp, and we will apply them without proof. In order to understand the applications to stochastic geometry that we intend to explain, the knowledge of fundamental notions from probability theory will be sufficient.

The material is taken from different sources, essentially from the lecture notes on "Integralgeometrie" [8] and "Stochastische Geometrie" [9], both written together with Wolfgang Weil. Another source is the fourth chapter of the book [7] on convex bodies.

Contents

1 Introduction

It will be one aim of the following lectures to develop some integral geometric formulae for sets in Euclidean space and to show how they can be applied in parts of stochastic geometry. In particular, I want to emphasize the role that integral geometry can play in the theoretical foundations of stereology. By stereology one understands a collection of procedures which are used to estimate certain parameters of real materials by means of measurements in small probes and plane sections. Stereology is applied in biology and medicine as well as in material sciences (e.g., metallography, mineralogy).

Since much of the motivation for the later theoretical investigations comes from these practical procedures, let me first explain the underlying ideas by two typical examples.

In geology, one may be interested in determining the volume proportion of some mineral in a rock. Thus one assumes that for the material in question there is a well-defined parameter, traditionally denoted by V_V, that specifies the volume of the investigated mineral per unit volume of the total material. In order to determine this specific volume V_V, one will first take a probe of the material "at random". As a second step, Delesse (1847) proposed to produce a (polished) plane section of the probe, possibly again "at random", and to determine the specific area A_A of the investigated mineral in that section. On the basis of heuristic arguments, Delesse asserted that

$$V_V = A_A,$$

or rather that the measured value A_A is a good estimate for the unknown parameter V_V.

A second example is taken from medicine. One may be interested in the gas exchange of a mammal lung, and this depends on the alveolar surface of the lung. To measure this specific area, denoted by S_V, only a small probe of the lung tissue will be available, and usually only a thin slice can be observed under the microscope. Tomkeieff (1945) proposed to determine the specific boundary length L_A of the tissue in the section and then to estimate the unknown specific area S_V by means of the formula

$$S_V = \frac{4}{\pi} L_A,$$

again supported by heuristic arguments.

Scientists working in practice have developed similar formulae. The so-called 'fundamental equations of stereology' are

$$V_V = A_A, \quad S_V = \frac{4}{\pi} L_A, \quad M_V = 2\pi \chi_A.$$

Here M denotes the integral of the mean curvature, and χ is the Euler characteristic.

It is evident that such heuristic procedures are implicitly based on many tacit assumptions. A theoretical justification has to begin by analyzing these assumptions, it has to provide suitable models and must finally lead to exactly proven formulae of the type used in practice. The first assumption is that the parameter of the material to be determined, like volume or surface area per unit volume, exists and can be estimated with sufficient accuracy from taking randomly placed probes and averaging. A solid foundation and justification can be achieved if the material under investigation is modelled as the realization of a random set. Taking a probe at random can then be modelled as follows. We fix a shape for the probe or 'observation window', say a compact convex set K with positive volume. Inside K we observe a realization $Z(\omega)$ of our random set Z. We assume that for the intersection $Z(\omega) \cap K$ we are able to measure a geometric functional φ of interest, like volume or surface area. Instead of placing K in a random position, one assumes that the random set Z has a suitable invariance property, meaning that Z and its image under any translation or rigid motion are stochastically equivalent. Under suitable model assumptions, the mathematical expectation $\mathbb{E}\,\varphi(Z \cap K)$ will exist, and the measured value $\varphi(Z(\omega) \cap K)$ can be considered as an unbiased estimator. If the model is such that the random set Z has a well-defined φ-density, the next question is then how this is related to the local expectation $\mathbb{E}\,\varphi(Z \cap K)$, depending on the test body K. Similar considerations will be necessary to justify the determination of parameters from randomly placed lower-dimensional sections.

This program, of which we have merely given a rough sketch, will obviously require the development of

- a theory of random sets with suitable invariance properties, admitting densities of geometric functionals, like volume, surface area, Euler characteristic,
- a theory of mean values of geometric functionals, evaluated at intersections of fixed and moving geometric objects.

2 Elementary mean value formulae

We begin with the second part of the program, the development of mean value formulae for fixed and moving geometric objects. By "moving" we mean here that the geometric objects, which are in Euclidean space, undergo translations or rigid motions. The mean values will be taken with respect to invariant measures on the groups of translations or rigid motions. The present section is still part of the introduction and will discuss a few elementary examples of such mean value formulae.

We work in n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. The subsets of $\mathbb{R}^n$ which will later (in dimensions two and three) be used to model real material, should not be too complicated, in order that functionals like surface area or Euler characteristic are defined (locally). It is sufficient for practical applications to consider only sets which can locally be represented as finite unions of convex bodies (non-empty, compact convex sets). We begin by considering only convex bodies; it will later be easy to extend the results to more general sets of the type just described. By $\mathcal{K}^n$ we denote the set of convex bodies in $\mathbb{R}^n$.

The following is a basic example of the type of questions that we will have to answer. Let $K, M \in \mathcal{K}^n$ be two convex bodies. Let M undergo translations, that is, we consider $M + t$ for $t \in \mathbb{R}^n$. What is the mean value of the volume of $K \cap (M + t)$, taken over all t with $K \cap (M + t) \neq \emptyset$? The mean value here refers to the invariant measure on the translation group, which can be identified with the Lebesgue measure λ on $\mathbb{R}^n$. For convex bodies K, we write $V_n(K) = \lambda(K)$ for the volume. Thus we are asking for the mean value

$$\frac{\int_{\mathbb{R}^n} V_n(K \cap (M + t))\, d\lambda(t)}{\int_{\mathbb{R}^n} \chi(K \cap (M + t))\, d\lambda(t)}. \tag{1}$$

Note that $\chi(K') = 1$ for a non-empty convex body K' and $\chi(\emptyset) = 0$, so that the denominator is indeed the total measure of all translation vectors t for which $K \cap (M + t) \neq \emptyset$. Thus we have to determine integrals of the type

$$\int_{\mathbb{R}^n} \varphi(K \cap (M + t))\, d\lambda(t)$$

for different functionals φ. Extensions of this problem will be our main concern in these lectures.

It is not difficult to determine the numerator in (1). Denoting the indicator function of a set $A \subset \mathbb{R}^n$ by $\mathbf{1}_A$, we have

$$V_n(K \cap (M + t)) = \int_{\mathbb{R}^n} \mathbf{1}_{K \cap (M+t)}(x)\, d\lambda(x)$$

and

$$\mathbf{1}_{K \cap (M+t)}(x) = \mathbf{1}_K(x) \mathbf{1}_{M+t}(x)$$

with

$$\mathbf{1}_{M+t}(x) = 1 \quad \Leftrightarrow \quad x \in M + t \quad \Leftrightarrow \quad t \in M^* + x \quad \Leftrightarrow \quad \mathbf{1}_{M^*+x}(t) = 1.$$

Here we have denoted by

$$M^* := \{y \in \mathbb{R}^n : -y \in M\}$$

the set obtained from M by reflection in the origin. Now Fubini's theorem gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} V_n(K \cap (M+t))\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{1}_{K\cap(M+t)}(x)\, d\lambda(x)\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{1}_K(x) \mathbf{1}_{M^*+x}(t)\, d\lambda(t)\, d\lambda(x) \\
&\quad = \int_{\mathbb{R}^n} \mathbf{1}_K(x) V_n(M^* + x)\, d\lambda(x) \\
&\quad = V_n(M^*) \int_{\mathbb{R}^n} \mathbf{1}_K(x)\, d\lambda(x)
\end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} V_n(K \cap (M+t))\, d\lambda(t) = V_n(K) V_n(M). \tag{2}$$

Note that we have used the invariance of the volume under translations and reflections.

The denominator in (1) is of a different type. We have

$$\begin{aligned}
\chi(K \cap (M+t)) = 1 \quad &\Leftrightarrow \quad K \cap (M+t) \neq \emptyset \\
&\Leftrightarrow \quad \exists\, k \in K\, \exists\, m \in M : k = m + t \\
&\Leftrightarrow \quad t = k - m \text{ with } k \in K, m \in M \\
&\Leftrightarrow \quad t \in K + M^* \\
&\Leftrightarrow \quad \mathbf{1}_{K+M^*}(t) = 1
\end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} \chi(K \cap (M+t))\, d\lambda(t) = V_n(K + M^*). \tag{3}$$

Convex geometry tells us that

$$V_n(K + M^*) = \sum_{i=0}^{n} \binom{n}{i} V(\underbrace{K, \ldots, K}_{i}, \underbrace{M^*, \ldots, M^*}_{n-i}),$$

where the function $V : (\mathcal{K}^n)^n \to \mathbb{R}$ is the so-called *mixed volume*. The essential observation for us is here that the obtained expression cannot be simplified further. In particular, there is no separation of the roles of K and M on the right-hand side, as it occurred in (2). Such a separation is only achieved if we integrate, not only over the translations of M as in (3), but over all rigid motions of M. This will be one of the fundamental results of integral geometry to be obtained later.

For the moment, however, we stay with the translation group alone. The idea leading to (2) can be extended, to give a first general formula of translative integral geometry.

When we talk of a *measure* on a locally compact space E, we always mean a non-negative, countably additive, extended real-valued function on the σ-algebra $\mathcal{B}(E)$ of Borel sets of E. Such a measure is called *locally finite* if it is finite on compact sets.

2.1 Theorem. *Let α be a locally finite measure on $\mathbb{R}^n$, and let $A, B \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \alpha(A \cap (B + t))\, d\lambda(t) = \alpha(A)\lambda(B). \tag{4}$$

Proof. Using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(A \cap (B + t))\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_{A\cap(B+t)}(x)\, d\alpha(x)\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_{A}(x)\mathbf{1}_{B+t}(x)\, d\lambda(t)\, d\alpha(x) \\
&\quad = \int_{\mathbb{R}^n} \mathbf{1}_{A}(x)\int_{\mathbb{R}^n} \mathbf{1}_{B^*+x}(t)\, d\lambda(t)\, d\alpha(x) \\
&\quad = \int_{\mathbb{R}^n} \mathbf{1}_{A}(x)\lambda(B^* + x)\, d\alpha(x) \\
&\quad = \alpha(A)\lambda(B).
\end{aligned}$$

■

This can be used to obtain a counterpart to the translative integral formula (2), with volume replaced by surface area. First we have to explain what we mean by the surface area of a general convex body, which need not satisfy any smoothness assumptions. For that purpose, let us first recall the notion of the p-dimensional Hausdorff measure, for $p \geq 0$.

We equip $\mathbb{R}^n$ with the usual scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. For a subset $G \subset \mathbb{R}^n$, the *diameter* is defined by

$$D(G) := \sup\{\|x - y\| : x, y \in G\}.$$

Now for an arbitrary subset M and for $\delta > 0$ one defines

$$\mathcal{H}^p_\delta(M) := \frac{\pi^{p/2}}{2^p \Gamma(1 + \frac{p}{2})} \inf \left\{ \sum_{i=1}^{\infty} D(G_i)^p : (G_i)_{i \in \mathbb{N}} \text{ sequence of open sets with } D(G_i) \leq \delta \text{ and } M \subset \bigcup_{i=1}^{\infty} G_i \right\}.$$

The limit

$$\mathcal{H}^p(M) := \lim_{\delta \to 0+} \mathcal{H}^p_\delta(M) = \sup_{\delta > 0} \mathcal{H}^p_\delta(M)$$

exists in $\mathbb{R} \cup \{\infty\}$ and is called the p-dimensional (outer) Hausdorff measure of M. The restriction of $\mathcal{H}^p$ to the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ of Borel sets is a measure. One can show that $\mathcal{H}^n(A) = \lambda(A)$ for $A \in \mathcal{B}(\mathbb{R}^n)$.

Now the *surface area* of a convex body $K \in \mathcal{K}^n$ with interior points is defined by

$$\mathcal{H}^{n-1}(\partial K) =: 2V_{n-1}(K),$$

where ∂ denotes the boundary. The notation $2V_{n-1}$ is chosen with respect to later developments. For $K \in \mathcal{K}^n$ without interior points, we define $V_{n-1}(K) := \mathcal{H}^{n-1}(K)$. This is zero if K is of dimension less than $n - 1$.

2.2 Theorem. *Let $K, M \in \mathcal{K}^n$ be convex bodies with interior points. Then*

$$\int_{\mathbb{R}^n} V_{n-1}(K \cap (M + t))\, d\lambda(t) = V_{n-1}(K)V_n(M) + V_n(K)V_{n-1}(M). \tag{5}$$

Proof. The boundary of the intersection $K \cap (M + t)$ consists of two parts:

$$\partial(K \cap (M + t)) = [\partial K \cap (M + t)] \cup [K \cap (\partial M + t)].$$

The intersection of the two sets on the right satisfies

$$[\partial K \cap (M + t)] \cap [K \cap (\partial M + t)] \subset \partial K \cap (\partial M + t).$$

We define

$$\alpha(A) := \mathcal{H}^{n-1}(\partial K \cap A) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^n).$$

Then α is a finite measure on $\mathbb{R}^n$. From (4) (with $A = \partial K$ and $B = \partial M$) we get

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial K \cap (\partial M + t))\, d\lambda(t) = \mathcal{H}^{n-1}(\partial K)\lambda(\partial M) = 0.$$

Since the integrand is nonnegative, it follows that

$$\mathcal{H}^{n-1}(\partial K \cap (\partial M + t)) = 0 \qquad \text{for } \lambda\text{-almost all } t,$$

that is, for all $t \in \mathbb{R}^n \setminus N$, with some set N satisfying $\lambda(N) = 0$. Hence, for all $t \in \mathbb{R}^n \setminus N$ we have

$$\mathcal{H}^{n-1}(\partial(K \cap (M+t)) = \mathcal{H}^{n-1}(\partial K \cap (M+t)) + \mathcal{H}^{n-1}(K \cap (\partial M + t)). \quad (6)$$

Using (4) with $A = \partial K$ and $B = M$, we further obtain

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial K \cap (M + t))\, d\lambda(t) = \mathcal{H}^{n-1}(\partial K)\lambda(M).$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(K \cap (\partial M + t))\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n} \mathcal{H}^{n-1}((K - t) \cap \partial M)\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial M \cap (K + t))\, d\lambda(t) \\
&\quad = \mathcal{H}^{n-1}(\partial M)\lambda(K).
\end{aligned}$$

Here we have used the facts that $\mathcal{H}^{n-1}$ is translation invariant and that the Lebesgue measure is invariant under the inversion $t \mapsto -t$. Finally we have used (4) again.

Since equation (6) holds for all $t \in \mathbb{R}^n \setminus N$ and since the null set N can be neglected in the integration, we deduce that

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial(K \cap (M + t))\, d\lambda(t) = \mathcal{H}^{n-1}(\partial K)\lambda(M) + \mathcal{H}^{n-1}(\partial M)\lambda(K).$$

This is precisely the assertion (5). ■

Instead of intersecting a fixed convex body with a translated one, we now briefly consider the intersections with a translated hyperplane. We parameterize hyperplanes in the form

$$H(u,\tau) := \{x \in \mathbb{R}^n : \langle x, u\rangle = \tau\}$$

with a unit vector $u \in \mathbb{R}^n$ and a real number $\tau \in \mathbb{R}$. Thus u is one of the two unit normal vectors of the (unoriented) hyperplane $H(u,\tau)$.

For a convex body $K \in \mathcal{K}^n$, Fubini's theorem immediately gives

$$\int_{\mathbb{R}} V_{n-1}(K \cap H(u,\tau))\, d\tau = V_n(K).$$

Can we obtain the surface area of a convex body $K \in \mathcal{K}^n$ with interior points in a similar way, that is, by a formula of type

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-2}(\partial K \cap H(u,\tau))\, d\tau = c_n V_{n-1}(K)$$

with some constant c_n? Simple examples (balls and cubes in $\mathbb{R}^3$) show that such a formula does not hold with a constant independent of K. However, we shall later see that

$$\int_{S^{n-1}} \int_{\mathbb{R}^n} \mathcal{H}^{n-2}(\partial K \cap H(u,\tau))\, d\tau\, d\sigma(u) = c_n V_{n-1}(K) \tag{7}$$

does hold with a constant c_n. Here the outer integration is over the unit sphere S^{n-1} with respect to the rotation invariant measure σ.

Both integrations in (7) together can be interpreted as one integration over the space of hyperplanes, with respect to a rigid motion invariant measure on that space. Thus we have now two examples for the simplifying effect in obtaining mean values when the integrations are performed with respect to motion invariant measures. This observation will be considerably elaborated in the following.

3 Invariant measures of Euclidean geometry

Integral geometry is based on the notion of invariant measure. Here invariance refers to a group operation and thus to a homogeneous space. Invariant measures on homogeneous spaces are also known as Haar measures. We do not presuppose here any knowledge of the theory of Haar measure. In the present section, we give an elementary introduction to the invariant measures on the groups and homogeneous spaces that are used in the integral geometry of Euclidean space.

A *topological group* is a group G together with a topology on G such that the map from $G \times G$ to G defined by $(x, y) \mapsto xy$ and the map from G to G defined by $x \mapsto x^{-1}$ are continuous. Let G be a group and X a non-empty set. An *operation* of G on X is a map $\varphi : G \times X \to X$ satisfying

$$\varphi(g, \varphi(g', x)) = \varphi(gg', x), \qquad \varphi(e, x) = x$$

for all $g, g' \in G$, the unit element e of G and all $x \in X$. One also says that G *operates on* X, by means of φ. For $\varphi(g, x)$ one usually writes gx, provided that the operation is clear from the context. The group G *operates transitively* on X if for any $x, y \in X$ there exists $g \in G$ so that $y = gx$. If G is a topological group, X is a topological space, and the operation φ is continuous, one says that G *operates continuously* on X.

The following situation often occurs: X is a nonempty set and G is a group of transformations (bijective mappings onto itself) of X, with the composition as group multiplication; the operation of G on X is given by $(g, x) \mapsto gx :=$ image of x under g. When transformation groups occur in the following, multiplication and operation are always understood in this sense.

We consider three groups of bijective affine maps of $\mathbb{R}^n$ onto itself, the *translation group* T_n, the *rotation group* SO_n, and the *rigid motion group* G_n. The *translations* $t \in T_n$ are the maps of the form $t = t_x$ with $x \in \mathbb{R}^n$, where $t_x(y) := y + x$ for $y \in \mathbb{R}^n$. The mapping $\tau : x \mapsto t_x$ is an isomorphism of the additive group $\mathbb{R}^n$ onto T_n. Hence, we can identify T_n with $\mathbb{R}^n$, which we shall often do tacitly. In particular, T_n carries the topology inherited from $\mathbb{R}^n$ via τ. Since $t_x \circ t_y = t_{x+y}$ and $t_x^{-1} = t_{-x}$, composition and inversion are continuous, hence T_n is a topological group. In view of the topological properties of $\mathbb{R}^n$ we can thus state the following.

3.1 Theorem. *The translation group T_n is an abelian, locally compact topological group with countable base. The operation of T_n on $\mathbb{R}^n$ is continuous.*

The elements of the rotation group SO_n are the linear mappings $\vartheta : \mathbb{R}^n \to \mathbb{R}^n$ that preserve scalar product and orientation; they are called *(proper) rotations*. With respect to the standard (orthonormal) basis of $\mathbb{R}^n$, every rotation ϑ is represented by an orthogonal matrix $M(\vartheta)$ with determinant 1.

The mapping $\mu : \vartheta \mapsto M(\vartheta)$ is an isomorphism of the group SO_n onto the group $\mathcal{SO}(n)$ of orthogonal (n,n)-matrices with determinant 1 under matrix multiplication. If we identify an (n,n)-matrix with the n^2-tuple of its entries (in lexicographic order, say), we can consider $\mathcal{SO}(n)$ as a subset of $\mathbb{R}^{n^2}$. This set is bounded, since the rows of an orthogonal matrix are normalized, and it is closed in $\mathbb{R}^{n^2}$, hence compact. The mappings $(M,N) \mapsto MN$ and $M \mapsto M^{-1}$ are continuous, and so is the mapping $(M,x) \mapsto Mx$ (where x is considered as an $(n,1)$-matrix) from $\mathcal{SO}(n) \times \mathbb{R}^n$ into $\mathbb{R}^n$. Using the mapping μ^{-1} to transfer the topology from $\mathcal{SO}(n)$ to SO_n, we thus obtain the following.

3.2 Theorem. *The rotation group SO_n is a compact topological group with countable base. The operation of SO_n on $\mathbb{R}^n$ is continuous.*

The elements of the motion group G_n are the affine maps $g : \mathbb{R}^n \to \mathbb{R}^n$ that preserve distances between points and the orientation; they are called *(rigid) motions*. Every rigid motion $g \in G_n$ can be represented uniquely as the composition of a rotation ϑ and a translation t_x, that is, $g = t_x \circ \vartheta$, or $gy = \vartheta y + x$ for $y \in \mathbb{R}^n$. The mapping

$$\begin{array}{rccc} \gamma : & \mathbb{R}^n \times SO_n & \to & G_n \\ & (x,\vartheta) & \mapsto & t_x \circ \vartheta \end{array}$$

is bijective. We use it to transfer the topology from $\mathbb{R}^n \times SO_n$ to G_n. Using Theorems 3.1 and 3.2, it is then easy to show the following.

3.3 Theorem. *G_n is a locally compact topological group with countable base. Its operation on $\mathbb{R}^n$ is continuous.*

After these topological groups, we now consider the homogeneous spaces that will play a role in the following. Let $q \in \{0,\dots,n\}$, let $\mathcal{L}_q^n$ be the set of all q-dimensional linear subspaces of $\mathbb{R}^n$, and let $\mathcal{E}_q^n$ be the set of all q-dimensional affine subspaces of $\mathbb{R}^n$. The natural operation of SO_n on $\mathcal{L}_q^n$ is given by $(\vartheta, L) \mapsto \vartheta L :=$ image of L under ϑ. Similarly, the natural operation of G_n on $\mathcal{E}_q^n$ is given by $(g,E) \mapsto gE :=$ image of E under g. We introduce suitable topologies on $\mathcal{L}_q^n$ and $\mathcal{E}_q^n$. For this, let $L_q \in \mathcal{L}_q^n$ be fixed and let $L_q^\perp$ be its orthogonal complement. The mappings

$$\begin{array}{rccc} \beta_q : & SO_n & \to & \mathcal{L}_q^n \\ & \vartheta & & \vartheta L_q \end{array}$$

and

$$\begin{array}{rccc} \gamma_q: & L_q^\perp \times SO_n & \to & \mathcal{E}_q^n \\ & (x, \vartheta) & \mapsto & \vartheta(L_q + x) \end{array}$$

are surjective (but not injective). We endow $\mathcal{L}_q^n$ with the finest topology for which β_q is continuous, and $\mathcal{E}_q^n$ with the finest topology for which γ_q is continuous. Thus a subset $A \in \mathcal{E}_q^n$, for example, is open if and only if $\gamma_q^{-1}(A)$ is open. It is an elementary task to prove the following.

3.4 Theorem. *$\mathcal{L}_q^n$ is compact and has a countable base, the map β_q is open, and the operation of SO_n on $\mathcal{L}_q^n$ is continuous and transitive.*

3.5 Theorem. *$\mathcal{E}_q^n$ is locally compact and has a countable base, the map γ_q is open, and the operation of G_n on $\mathcal{E}_q^n$ is continuous and transitive.*

It should be remarked that the topologies on $\mathcal{L}_q^n$ and $\mathcal{E}_q^n$, as well as the invariant measures on these spaces to be introduced below, do not depend on the special choice of the subspace L_q. This follows easily from the fact that SO_n operates transitively on $\mathcal{L}_q^n$, and G_n operates transitively on $\mathcal{E}_q^n$.

The topological spaces $\mathcal{L}_q^n$ are called *Grassmann manifolds*; a common notation for $\mathcal{L}_q^n$ is $G(n, q)$. The spaces $\mathcal{E}_q^n$ are also called *affine Grassmannians.*

Occasionally, we have talked of homogeneous spaces; it seems, therefore, appropriate here to give the general definition. If G is a topological group, a *homogeneous G-space* is, by definition, a pair (X, φ), where X is a topological space and φ is a transitive continuous operation of G on X with the additional property that the map $\varphi(\cdot, p)$ is open for $p \in X$. In this sense, $\mathcal{L}_q^n$ is a homogeneous SO_n-space (with respect to the standard operation), and $\mathcal{E}_q^n$ is a homogeneous G_n-space. Also with the standard operations, $\mathbb{R}^n$ is a homogeneous T_n-space and G_n-space, and the unit sphere

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$$

is a homogeneous SO_n-space.

We shall now introduce invariant measures on the groups and homogeneous spaces considered. We begin with some general definitions and remarks. All topological spaces occurring here are locally compact and second countable. By a *Borel measure* ρ on X we understand a measure on the σ-algebra $\mathcal{B}(X)$ of Borel sets of X satisfying $\rho(K) < \infty$ for every compact set $K \subset X$. Every such measure is regular. Instead of 'Borel measure' we often say 'measure' for short. The notion 'measurable', without extra specification, means 'Borel measurable'.

Let the topological group G operate continuously on the space X. A measure ρ on X is called *G-invariant* (or briefly *invariant*, if G is clear from the context) if

$$\rho(gA) = \rho(A) \qquad \text{for all } A \in \mathcal{B}(X) \text{ and all } g \in G.$$

This definition makes sense: for each $g \in G$, the mapping $x \mapsto gx$ is a homeomorphism, hence $A \in \mathcal{B}(X)$ implies $gA \in \mathcal{B}(X)$. Invariant regular Borel measures on locally compact homogeneous spaces are called *Haar measures*, if they are not identically zero.

From basic measure theory, we assume familiarity with Lebesgue measure on $\mathbb{R}^n$, in particular with the following result. Here we use the unit cube $C^n := [0,1]^n$ for normalization.

3.6 Theorem and Definition. *There is a unique translation invariant measure λ on $\mathcal{B}(\mathbb{R}^n)$ satisfying $\lambda(C^n) = 1$. It is called the Lebesgue measure.*

It is easy to see that λ is also rotation invariant (SO_n-invariant). If $\vartheta \in SO_n$ and if one defines $\rho(A) := \lambda(\vartheta A)$ for $A \in \mathcal{B}(\mathbb{R}^n)$, then ρ is a translation invariant measure on $\mathcal{B}(\mathbb{R}^n)$. By Theorem 3.6, $\rho = c\lambda$ with $c = \rho(C^n)$. The unit ball B^n satisfies $c\lambda(B^n) = \rho(B^n) = \lambda(\vartheta B^n) = \lambda(B^n)$, hence $c = 1$.

Since the Lebesgue measure λ is thus rigid motion invariant, it is the Haar measure on the homogeneous G_n-space $\mathbb{R}^n$, normalized in a special way.

We mention the special value

$$\kappa_n := \lambda(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})},$$

which will play a role in many later formulae. We put $\kappa_0 := 1$.

The Haar measure on the homogeneous SO_n-space S^{n-1}, the unit sphere, is easily derived from the Lebesgue measure. For $A \in \mathcal{B}(S^{n-1})$ we define

$$\hat{A} := \{\alpha x \in \mathbb{R}^n : x \in A,\ 0 \le \alpha \le 1\}.$$

A standard argument shows that $\hat{A} \in \mathcal{B}(\mathbb{R}^n)$, hence we can define $\sigma(A) := n\lambda(\hat{A})$. This yields a finite measure σ on $\mathcal{B}(S^{n-1})$ for which

$$\sigma(S^{n-1}) =: \omega_n = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

The rotation invariance of λ implies the rotation invariance of σ. We call σ, with the normalization specified above, the *spherical Lebesgue measure*.

Up to a constant factor, σ is the only rotation invariant Borel measure on $\mathcal{B}(S^{n-1})$. This follows from Corollary 3.12 below.

Our next aim is the introduction of an invariant measure on the rotation group SO_n. For a measure on a group, several notions of invariance are natural. A topological group G operates on itself by means of the mapping $(g,x) \mapsto gx$ (multiplication in G). The corresponding invariance on G is called *left invariance*. More generally, for $g \in G$ and $A \subset G$ we write

$$gA := \{ga : a \in A\}, \quad Ag := \{ag : a \in A\}, \quad A^{-1} := \{a^{-1} : a \in A\}.$$

If $A \in \mathcal{B}(G)$, then also gA, Ag, A^{-1} are Borel sets. A measure ρ on G is called *left invariant* if $\rho(gA) = \rho(A)$, and *right invariant* if $\rho(Ag) = \rho(A)$, for all $A \in \mathcal{B}(G)$ and all $g \in G$. The measure ρ is *inversion invariant* if $\rho(A^{-1}) = \rho(A)$ for all $A \in \mathcal{B}(G)$. If ρ has all three invariance properties, it is just called *invariant*.

With these definitions we connect two general remarks. Let ρ be a left invariant measure on the topological group G. Then each measurable function $f \geq 0$ on G satisfies

$$\int_G f(ag)\, d\rho(g) = \int_G f(g)\, d\rho(g) \tag{8}$$

for all $a \in G$. This follows immediately from the definition of the integral. Vice versa, if (8) holds for all measurable functions $f \geq 0$, then the left invariance of ρ is obtained by applying (8) to indicator functions. Similarly, the right invariance of ρ is equivalent to

$$\int_G f(ga)\, d\rho(g) = \int_G f(g)\, d\rho(g) \tag{9}$$

for $a \in G$, and the inversion invariance of ρ is equivalent to

$$\int_G f(g^{-1})\, d\rho(g) = \int_G f(g)\, d\rho(g), \tag{10}$$

in each case for all measurable functions $f \geq 0$.

The following theorem on invariant measures on compact groups will be needed for the rotation group only, but can be proved without additional effort in a more general setting.

3.7 Theorem. *Every left invariant Borel measure on a compact group with countable base is invariant.*

Proof. Let ν be a left invariant Borel measure on the group G satisfying the assumptions. Since it is finite on compact sets, we may assume $\nu(G) = 1$, without loss of generality. For measurable functions $f \geq 0$ on G and for $x \in G$ we have

$$\int f(y^{-1}x)\, d\nu(y) = \int f((x^{-1}y)^{-1})\, d\nu(y) = \int f(y^{-1})\, d\nu(y). \tag{11}$$

Here the integrations extend over all of G; similar conventions will be adopted in the following. Fubini's theorem gives

$$\begin{aligned}\int f(y^{-1})\, d\nu(y) &= \int \int f(y^{-1}x)\, d\nu(y)\, d\nu(x) \\ &= \int \int f(y^{-1}x)\, d\nu(x)\, d\nu(y) = \int f(x)\, d\nu(x).\end{aligned}$$

Hence, the measure ν is inversion invariant. Using this fact and (11), we get for $x \in G$ that

$$\begin{aligned}\int f(yx)\, d\nu(y) &= \int f(y^{-1}x)\, d\nu(y) \\ &= \int f(y^{-1})\, d\nu(y) = \int f(y)\, d\nu(y),\end{aligned}$$

which shows that ν is also right invariant. ■

Concerning the application of Fubini's theorem here and later, we remark the following. All topological spaces occurring in our considerations are locally compact and second countable, thus they are σ-compact. Moreover, all the measures that occur are finite on compact sets. Therefore, all measure spaces under consideration are σ-finite, so that Fubini's theorem can be applied in its usual form.

The following uniqueness result for invariant measures makes special assumptions, but in this form it is sufficient for our purposes and is easy to prove.

3.8 Theorem. *Let G be a locally compact group with a countable base, let $\nu \neq 0$ be an invariant and μ a left invariant Borel measure on G. Then $\mu = c\nu$ with a constant $c \geq 0$.*

Proof. For measurable functions $f, g \geq 0$ on G we have

$$\int f\, d\nu \int g\, d\mu = \int \int f(xy)g(y)\, d\nu(x)\, d\mu(y)$$

$$= \int\int f(xy)g(y)\,d\mu(y)\,d\nu(x) = \int\int f(y)g(x^{-1}y)\,d\mu(y)\,d\nu(x)$$
$$= \int f(y)\int g(x^{-1}y)\,d\nu(x)\,d\mu(y) = \int g\,d\nu\int f\,d\mu.$$

Here we have used, besides Fubini's theorem, the right and inversion invariance of ν and the left invariance of μ.

Since $\nu \neq 0$, there is a compact set $A_0 \subset G$ with $\nu(A_0) > 0$. For arbitrary $A \in \mathcal{B}(G)$ we put $f := \mathbf{1}_{A_0}$ and $g := \mathbf{1}_A$ and obtain $\nu(A_0)\mu(A) = \nu(A)\mu(A_0)$, hence $\mu = c\nu$ with $c := \mu(A_0)/\nu(A_0)$. ■

The notation $\mathbf{1}_A$ used here for the indicator function of a set A will also be employed in the following.

Now we turn to the existence of some invariant measures. First we describe a direct construction of the invariant measure on the rotation group, without recourse to the general theory of Haar measure.

3.9 Theorem. *On the rotation group SO_n, there is an invariant measure ν with $\nu(SO_n) = 1$.*

Proof. By LI_n we denote the set of linearly independent n-tuples of vectors from the unit sphere S^{n-1}. We define a map $\psi : LI_n \to SO_n$ in the following way. Let $(x_1, \dots, x_n) \in LI_n$. By Gram-Schmidt orthonormalization, we transform $(x_1, \dots, x_n)$ into the n-tuple $(z_1, \dots, z_n)$; then we denote by $(\overline{z}_1, \dots, \overline{z}_n)$ the positively oriented n-tuple for which $\overline{z}_i := z_i$ for $i = 1, \dots, n-1$ and $\overline{z}_n := \pm z_n$. If $(e_1, \dots, e_n)$ denotes the standard basis of $\mathbb{R}^n$, there is a unique rotation $\vartheta \in SO_n$ satisfying $\vartheta e_i = \overline{z}_i$ for $i = 1, \dots, n$. We define $\psi(x_1, \dots, x_n) := \vartheta$.

Explicitly, we have $z_i = y_i/\|y_i\|$ with $y_1 = x_1$ and

$$y_k = x_k - \sum_{j=1}^{k-1} \langle x_k, y_j\rangle \frac{y_j}{\|y_j\|^2}, \qquad k = 2, \dots, n.$$

From this representation, the following is evident. If $\rho \in SO_n$ is a rotation and if the n-tuple $(x_1, \dots, x_n) \in LI_n$ is transformed into $(z_1, \dots, z_n)$ and then into $(\overline{z}_1, \dots, \overline{z}_n)$, then the n-tuple $(\rho x_1, \dots, \rho x_n)$ is transformed into $(\rho z_1, \dots, \rho z_n)$ and subsequently into $(\rho\overline{z}_1, \dots, \rho\overline{z}_n)$. Thus we have $\psi(\rho x_1, \dots, \rho x_n) = \rho\psi(x_1, \dots, x_n)$.

For $(x_1, \dots, x_n) \in (S^{n-1})^n \setminus LI_n$ we define $\psi(x_1, \dots, x_n) := \mathrm{id}$. For the product measure

$$\sigma^{\otimes n} := \underbrace{\sigma \otimes \cdots \otimes \sigma}_{n},$$

the set $(S^{n-1})^n \setminus LI_n$ has measure zero; hence for any $\rho \in SO_n$ the equality $\psi(\rho x_1, \dots, \rho x_n) = \rho\psi(x_1, \dots, x_n)$ holds $\sigma^{\otimes n}$-almost everywhere. The mapping $\psi : (S^{n-1})^n \to SO_n$ is measurable, since LI_n is open and ψ is continuous on LI_n and constant on $(S^{n-1})^n \setminus LI_n$.

Now we define $\overline{\nu}$ as the image measure of $\sigma^{\otimes n}$ under ψ, thus $\overline{\nu} = \psi(\sigma^{\otimes n})$. Then $\overline{\nu}$ is a finite measure on SO_n, and for $\rho \in SO_n$ and measurable $f \geq 0$ we obtain

$$\begin{aligned}
&\int_{SO_n} f(\rho\vartheta)\, d\overline{\nu}(\vartheta) \\
&= \int_{(S^{n-1})^n} f(\rho\psi(x_1, \dots, x_n))\, d\sigma^{\otimes n}(x_1, \dots, x_n) \\
&= \int_{(S^{n-1})^n} f(\psi(\rho x_1, \dots, \rho x_n))\, d\sigma^{\otimes n}(x_1, \dots, x_n) \\
&= \int_{S^{n-1}} \cdots \int_{S^{n-1}} f(\psi(\rho x_1, \dots, \rho x_n))\, d\sigma(x_1) \cdots d\sigma(x_n) \\
&= \int_{S^{n-1}} \cdots \int_{S^{n-1}} f(\psi(x_1, \dots, x_n))\, d\sigma(x_1) \cdots d\sigma(x_n) \\
&= \int_{SO_n} f(\vartheta)\, d\overline{\nu}(\vartheta).
\end{aligned}$$

Here we have used the rotation invariance of the spherical Lebesgue measure. We have proved that the measure $\overline{\nu}$ is left invariant and thus invariant, by Theorem 3.7. The measure $\nu := \overline{\nu}/\overline{\nu}(SO_n)$ is invariant and normalized. ■

From now on, ν will always denote the normalized invariant measure on SO_n.
Now we turn to the motion group G_n. Since it is not compact, an invariant measure μ on G_n cannot be finite. In order to normalize μ, we specify the compact set $A_0 := \gamma(C^n \times SO_n)$ and require that $\mu(A_0) = 1$.

3.10 Theorem. *On the motion group G_n, there is an invariant measure μ with $\mu(A_0) = 1$. Up to a constant factor, it is the only left invariant measure on G_n.*

Proof. We define μ as the image measure of the product measure $\lambda \otimes \nu$ under the homeomorphism $\gamma : \mathbb{R}^n \times SO_n \to G_n$ defined by (8). Then μ is a Borel measure on G_n with $\mu(\gamma(C^n \times SO_n)) = \lambda(C^n)\nu(SO_n) = 1$.

To show the left invariance of μ, let $f \geq 0$ be a measurable function on G_n and let $g' \in G_n$. With $g' = \gamma(t', \vartheta')$ we have

$$\begin{aligned}
&\int_{G_n} f(g'g)\, d\mu(g) \\
&\quad = \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t', \vartheta')\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) \\
&\quad = \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t' + \vartheta' t, \vartheta'\vartheta))\, d\lambda(t)\, d\nu(\vartheta) \\
&\quad = \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) \\
&\quad = \int_{G_n} f(g)\, d\mu(g),
\end{aligned}$$

where we have used the motion invariance of λ and the left invariance of ν. Hence, μ is left invariant. Similarly, the right invariance of ν implies via

$$\begin{aligned}
\int_{G_n} f(gg')\, d\mu(g) &= \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t + \vartheta t', \vartheta\vartheta'))\, d\lambda(t)\, d\nu(\vartheta) \\
&= \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) = \int_{G_n} f(g)\, d\mu(g)
\end{aligned}$$

the right invariance of μ. The inversion invariance of μ is obtained from

$$\begin{aligned}
\int_{G_n} f(g^{-1})\, d\mu(g) &= \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(-\vartheta^{-1} t, \vartheta^{-1}))\, d\lambda(t)\, d\nu(\vartheta) \\
&= \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) = \int_{G_n} f(g)\, d\mu(g),
\end{aligned}$$

where the inversion invariance of ν was used.

The uniqueness assertion is a special case of Theorem 3.8. ■

Having constructed invariant measures on the groups SO_n and G_n, we next turn to the introduction of invariant measures on the homogeneous spaces $\mathcal{L}_q^n$ and $\mathcal{E}_q^n$. First we prove a formula of integral-geometric type, extending Theorem 2.1, which will be useful for obtaining uniqueness results.

3.11 Theorem. *Suppose that the compact group G operates continuously and transitively on the Hausdorff space X, and that G and X have countable bases. Let ν be an invariant measure on G with $\nu(G) = 1$, let $\rho \neq 0$ be a G-invariant Borel measure on X and α an arbitrary Borel measure on X. Then*

$$\int_G \alpha(A \cap gB)\, d\nu(g) = \alpha(A)\rho(B)/\rho(X)$$

for all $A, B \in \mathcal{B}(X)$.

Proof. If φ denotes the operation of G on X and if $x \in X$, the mapping $\varphi(\cdot, x) : G \to X$ is continuous and surjective, hence X is compact. Therefore, the Borel measures α and ρ are finite. Let $A, B \in \mathcal{B}(X)$ and $g \in G$ be given, then

$$\alpha(A \cap gB) = \int_X \mathbf{1}_{A \cap gB}\, d\alpha(x) = \int_X \mathbf{1}_A(x)\mathbf{1}_B(g^{-1}x)\, d\alpha(x).$$

Fubini's theorem yields

$$\int_G \alpha(A \cap gB)\, d\nu(g) = \int_X \mathbf{1}_A(x) \int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g)\, d\alpha(x). \tag{12}$$

The integral $\int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g)$ does not depend on x, since for $y \in X$ there exists $\tilde{g} \in G$ with $y = \tilde{g}x$ and, therefore,

$$\int_G \mathbf{1}_B(g^{-1}y)\, d\nu(g) = \int_G \mathbf{1}_B((\tilde{g}^{-1}g)^{-1}x)\, d\nu(g) = \int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g).$$

Hence we obtain

$$\begin{aligned} \rho(X) \int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g) &= \int_X \mathbf{1}_B(g^{-1}x)\, d\nu(g)\, d\rho(x) \\ &= \int_G \int_X \mathbf{1}_B(g^{-1}x)\, d\rho(x)\, d\nu(g) = \int_G \rho(gB)\, d\nu(g) = \rho(B). \end{aligned}$$

Inserting this into (12), we complete the proof. ∎

3.12 Corollary. *Suppose that the compact group G operates continuously and transitively on the Hausdorff space X and that G and X have countable bases. Let ν be an invariant measure on G with $\nu(G) = 1$.*

Then there exists a unique G-invariant measure ρ on X with $\rho(X) = 1$. It can be defined by

$$\rho(B) := \nu(\{g \in G : gx_0 \in B\}), \qquad B \in \mathcal{B}(X),$$

with arbitrary $x_0 \in X$.

Proof. Let ρ be a G-invariant measure on X with $\rho(X) = 1$. We choose $x_0 \in X$ and let α be the Dirac measure on X concentrated in x_0. Theorem 3.11 with $A := \{x_0\}$ gives

$$\rho(B) = \nu(\{g \in G : g^{-1}x_0 \in B\})$$

for $B \in \mathcal{B}(X)$. Thus ρ is unique. Vice versa, if ρ is defined in this way, it is clear that it is a G-invariant normalized measure. ■

Now we turn to invariant measures on the space $\mathcal{L}_q^n$ of q-dimensional linear subspaces and on the space $\mathcal{E}_q^n$ of q-dimensional affine subspaces. By an *invariant measure* on $\mathcal{L}_q^n$ we understand an SO_n-invariant measure on $\mathcal{L}_q^n$, and an *invariant measure* on $\mathcal{E}_q^n$ is defined as a G_n-invariant measure on $\mathcal{E}_q^n$.

3.13 Theorem. *On $\mathcal{L}_q^n$ there is a unique invariant measure ν_q, normalized by $\nu_q(\mathcal{L}_q^n) = 1$.*

This is just a special case of Corollary 3.12. We also notice that ν_q is the image measure of ν under the map β_q defined by (8).

3.14 Theorem. *On $\mathcal{E}_q^n$ there is an invariant measure μ_q. It is unique up to a constant factor.*

Proof. We recall that we have chosen a subspace $L_q \in \mathcal{L}_q^n$ and defined the map $\gamma_q : L_q^\perp \times SO_n \to \mathcal{E}_q^n$ by (8). Let $\lambda^{(n-q)}$ be Lebesgue measure on $L_q^\perp$. We define

$$\mu_q := \gamma_q(\lambda^{(n-q)} \otimes \nu), \tag{13}$$

so that μ_q is the image measure of the product measure $\lambda^{(n-q)} \otimes \nu$ under the map γ_q. If $A \subset \mathcal{E}_q^n$ is compact, the sets

$$\gamma_q(\{x \in L_q^\perp : \|x\| < k\} \times SO_n), \qquad k \in \mathbb{N},$$

constitute an open covering of A, hence A is included in one of these sets. It follows that $\mu_q(A) < \infty$.

By the definition of μ_q, integrals with respect to μ_q can be expressed in the following way. For a nonnegative measurable function f on $\mathcal{E}_q^n$,

$$\begin{aligned}\int_{\mathcal{E}_q^n} f\, d\mu_q &= \int_{SO_n}\int_{L_q^\perp} f(\rho(L_q+x))\, d\lambda^{(n-q)}(x)\, d\nu(\rho)\\ &= \int_{SO_n}\int_{(\rho L_q)^\perp} f(\rho L_q + y)\, d\lambda^{(n-q)}(y)\, d\nu(\rho).\end{aligned}$$

Since the invariant measure ν_q on $\mathcal{L}_q^n$ is the image measure under the map β_q, this can be written as

$$\int_{\mathcal{E}_q^n} f\, d\mu_q = \int_{\mathcal{L}_q^n}\int_{L^\perp} f(L+y)\, d\lambda^{(n-q)}(y)\, d\nu_q(L). \tag{14}$$

From this representation we infer that μ_q does not depend on the choice of the subspace L_q.

To show the invariance of μ_q, let $g = \gamma(x, \vartheta) \in G$ and let $f \geq 0$ be a measurable function on $\mathcal{E}_q^n$. Denoting by Π the orthogonal projection onto $L_q^\perp$, we have

$$\begin{aligned}&\int_{\mathcal{E}_q^n} f(gE)\, d\mu_q(E)\\ &= \int_{SO_n}\int_{L_q^\perp} f(g\rho(L_q+y))\, d\lambda^{(n-q)}(y)\, d\nu(\rho)\\ &= \int_{SO_n}\int_{L_q^\perp} f(\vartheta\rho(L_q+y+\Pi(\rho^{-1}\vartheta^{-1}x)))\, d\lambda^{(n-q)}(y)\, d\nu(\rho)\\ &= \int_{SO_n}\int_{L_q^\perp} f(\vartheta\rho(L_q+y))\, d\lambda^{(n-q)}(y)\, d\nu(\rho)\\ &= \int_{SO_n}\int_{L_q^\perp} f(\rho(L_q+y))\, d\lambda^{(n-q)}(y)\, d\nu(\rho)\\ &= \int_{\mathcal{E}_q^n} f(E)\, d\mu_q(E),\end{aligned}$$

where we have used the invariance properties of $\lambda^{(n-q)}$ and ν. This shows the invariance of μ_q.

To prove the uniqueness (up to a factor), we assume that τ is another invariant Borel measure on $\mathcal{E}_q^n$. Let $\tilde{\mathcal{L}}_q^n$ (respectively $\tilde{\mathcal{E}}_q^n$) be the open set of all $L \in \mathcal{L}_q^n$ (respectively $E \in \mathcal{E}_q^n$) that intersect $L_q^\perp$ in precisely one point. The mapping

$$\begin{array}{rccc} \delta_q : & L_q^\perp \times \tilde{\mathcal{L}}_q^n & \to & \tilde{\mathcal{E}}_q^n \\ & (x, L) & \mapsto & L + x \end{array}$$

is a homeomorphism. For fixed $B \in \mathcal{B}(\tilde{\mathcal{L}}_q^n)$ and arbitrary $A \in \mathcal{B}(L_q^\perp)$ we define $\eta(A) := \tau(\delta_q(A \times B))$. Then η is a Borel measure on $L_q^\perp$, which is invariant under the translations of $L_q^\perp$ into itself. Theorem 3.6 implies that $\eta(A) = \lambda^{(n-q)}(A)\alpha(B)$ with a constant $\alpha(B) \geq 0$. Hence we have

$$\tau(\delta_q(A \times B)) = \lambda^{(n-q)}(A)\alpha(B)$$

for arbitrary $A \in \mathcal{B}(L_q^\perp)$ and $B \in \mathcal{B}(\tilde{\mathcal{L}}_q^n)$. Obviously this equation defines a finite measure α on $\mathcal{B}(\tilde{\mathcal{L}}_q^n)$, and $\delta_q^{-1}(\tau) = \lambda^{(n-q)} \otimes \alpha$. For a measurable function $f \geq 0$ on $\tilde{\mathcal{E}}_q^n$ we obtain

$$\begin{aligned} \int_{\tilde{\mathcal{E}}_q^n} f \, d\tau &= \int_{\tilde{\mathcal{L}}_q^n} \int_{L_q^\perp} f(L + x) \, d\lambda^{(n-q)}(x) \, d\alpha(L) \\ &= \int_{\tilde{\mathcal{L}}_q^n} \int_{L^\perp} f(L + y) \, d\lambda^{(n-q)}(y) \, d\varphi(L) \qquad (15) \end{aligned}$$

with a measure φ on $\tilde{\mathcal{L}}_q^n$ defined by $d\varphi(L)/d\alpha(L) = D(L_q^\perp, L^\perp)^{-1}$, where $D(L_q^\perp, L^\perp)$ is the absolute determinant of the orthogonal projection from $L_q^\perp$ onto $L^\perp$.

Now let $B \in \mathcal{B}(\mathcal{L}_q^n)$ and

$$B' := \{L + y : L \in B, \ y \in L^\perp \cap B^n\}.$$

By $\beta(B) := \tau(B')$ we define a rotation invariant finite measure β on $\mathcal{L}_q^n$. According to Theorem 3.13 it is a multiple of ν_q. On the other hand, (15) gives $\tau(B') = \kappa_{n-q}\varphi(B)$ for $B \subset \tilde{\mathcal{L}}_q^n$. Hence, there is a constant c with $\varphi(B) = c\nu_q(B)$ for all Borel sets $B \subset \tilde{\mathcal{L}}_q^n$. From (15) and (14) we deduce that $\tau(A) = c\mu_q(A)$ for all Borel sets $A \subset \tilde{\mathcal{E}}_q^n$. Since μ_q does not depend on the choice of the subspace $L_q \in \mathcal{L}_q^n$, it is easy to see that $\tau = c\mu_q$. ∎

By its definition, the measure μ_q comes with a particular normalization. We want to determine the measure of all q-flats meeting the unit ball B^n.

Since

$$\{E \in \mathcal{E}_q^n : E \cap B^n \neq \emptyset\} = \gamma_q((B^n \cap L_q^\perp) \times SO_n),$$

we get

$$\mu_q(\{E \in \mathcal{E}_q^n : E \cap B^n \neq \emptyset\}) = \kappa_{n-q}.$$

For $r > 0$ we have

$$\mu_q(\{E \in \mathcal{E}_q^n : E \cap rB^n \neq \emptyset\}) = r^{n-q}\kappa_{n-q}.$$

4 Additive functionals

Beside special Haar measures, another type of invariant measures that we will use are finitely additive measures on certain systems of subsets of Euclidean space.

We begin with some general definitions. Let φ be a function on a family $\mathcal{S}$ of sets with values in some abelian group. The function φ is called *additive* or a *valuation* if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \tag{16}$$

holds whenever $K, L \in \mathcal{S}$ are sets such that also $K \cup L \in \mathcal{S}$ and $K \cap L \in \mathcal{S}$. If $\emptyset \in \mathcal{S}$, one also assumes that $\varphi(\emptyset) = 0$. We say that the system $\mathcal{S}$ is $\cap$-stable (intersection stable) if $K, L \in \mathcal{S}$ implies $K \cap L \in \mathcal{S} \cup \{\emptyset\}$. In this case, we denote by $U(\mathcal{S})$ the system of all finite unions of sets in $\mathcal{S}$ (including the empty set). The system $U(\mathcal{S})$ is closed under finite unions and intersections and thus is a lattice.

Now let φ be an additive function on $\mathcal{S}$. One may ask whether it has an extension to an additive function on the lattice $U(\mathcal{S})$. Suppose that such an extension exists, and denote it also by φ. Then for $K_1, \ldots, K_m \in U(\mathcal{S})$ the formula

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{i_1 < \cdots < i_r} \varphi(K_{i_1} \cap \cdots \cap K_{i_r}) \tag{17}$$

holds. For $m = 2$, this is just the equation (16) defining additivity. The general case of (17) is easily obtained by induction. This formula is called the *inclusion-exclusion principle*.

Formula (17) shows that an additive extension from the $\cap$-stable system $\mathcal{S}$ to the generated lattice $U(\mathcal{S})$, if it exists, is uniquely determined. Conversely, however, one cannot just use (17) for the definition of such an extension, since the representation of an element of $U(\mathcal{S})$ in the form

$K_1 \cup \cdots \cup K_m$ with $K_i \in \mathcal{S}$ is in general not unique. Hence, the existence of an additive extension, if there is one, must be proved in a different way.

We will write (17) in a more concise form. For $m \in \mathbb{N}$, let $S(m)$ denote the set of all non-empty subsets of $\{1, \ldots, m\}$. For $v \in S(m)$, let $|v| := \operatorname{card} v$. If $K_1, \ldots, K_m$ are given, we write

$$K_v := K_{i_1} \cap \cdots \cap K_{i_m} \qquad \text{for } v = \{i_1, \ldots, i_r\} \in S(m).$$

With these conventions, the inclusion-exclusion principle (17) can be written in the form

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{v \in S(m)} (-1)^{|v|-1} \varphi(K_v). \tag{18}$$

Of considerable importance in the following is the lattice $U(\mathcal{K}^n)$ generated by the $\cap$-stable family $\mathcal{K}^n \cup \{\emptyset\}$. Thus the system $U(\mathcal{K}^n)$ consists of all subsets of $\mathbb{R}^n$ that can be represented as finite unions of convex bodies. We call such sets *polyconvex* (following Klain-Rota [4], who in turn followed E. de Giorgi). Hadwiger [2] used for $U(\mathcal{K}^n)$ the name 'Konvexring', which has been translated (perhaps not so luckily) into *convex ring.*

The simplest non-zero valuation on $\mathcal{K}^n$ is given by $\chi(K) = 1$ for all $K \in \mathcal{K}^n$. We show that it has an additive extension to $U(\mathcal{K}^n)$.

4.1 Theorem. *There is a unique valuation χ on the convex ring $U(\mathcal{K}^n)$ satisfying*

$$\chi(K) = 1 \qquad \textit{for } K \in \mathcal{K}^n.$$

Proof. The proof uses induction with respect to the dimension. For $n = 0$, the existence is trivial. Suppose that $n > 0$ and the existence has been proved in Euclidean spaces of dimension $n - 1$. We choose a unit vector $u \in \mathbb{R}^n$ and define

$$\chi(K) := \sum_{\lambda \in \mathbb{R}} \left[\chi(K \cap H(u, \lambda)) - \lim_{\mu \downarrow \lambda} \chi(K \cap H(u, \mu)) \right] \tag{19}$$

for $K \in U(\mathcal{K}^n)$. On the right-hand side, χ denotes the additive function that exists by the induction hypothesis in spaces of dimension $n - 1$. It is obvious that $\chi(K) = 1$ for $K \in \mathcal{K}^n$. If $K = K_1 \cup \cdots \cup K_m$ with $K_i \in \mathcal{K}^n$, then the inclusion-exclusion principle gives

$$\chi(K \cap H(u, \lambda)) = \sum_{v \in S(m)} (-1)^{|v|-1} \chi(K_v \cap H(u, \lambda)),$$

since χ is additive on the polyconvex sets in $H(u,\lambda)$. Now the function $\lambda \mapsto \chi(K_v \cap H(u,\lambda))$ is the indicator function of a compact interval, hence it is clear that the limit in (19) exists for every $\lambda \in \mathbb{R}$ and is non-zero only for finitely many values of λ. Thus χ is well-defined on $U(\mathcal{K}^n)$. It follows from (19) and the induction hypothesis that χ is additive on $U(\mathcal{K}^n)$. This proves the existence of χ. The uniqueness is clear from the inclusion-exclusion principle. ■

The function χ is called the *Euler characteristic*. It coincides, on $U(\mathcal{K}^n)$, with the Euler characteristic as defined in algebraic topology.

Another simple example of a valuation on $U(\mathcal{K}^n)$ is given by the indicator function. For $K \in U(\mathcal{K}^n)$, let

$$\mathbf{1}_K(x) := \begin{cases} 1 & \text{for } x \in K, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus K. \end{cases}$$

For $K, L \in U(\mathcal{K}^n)$ we trivially have

$$\mathbf{1}_{K\cup L}(x) + \mathbf{1}_{K\cap L}(x) = \mathbf{1}_K(x) + \mathbf{1}_L(x)$$

for $x \in \mathbb{R}^n$. Hence, the mapping

$$\begin{array}{rccc} \varphi: & U(\mathcal{K}^n) & \to & V \\ & K & \mapsto & \mathbf{1}_K \end{array}$$

is an additive function on $U(\mathcal{K}^n)$ with values in the vector space V of finite linear combinations of indicator functions of polyconvex sets. Since $K \mapsto \mathbf{1}_K$ is additive, for $K \in U(\mathcal{K}^n)$ with $K = K_1 \cup \cdots \cup K_m$, $K_i \in \mathcal{K}^n$, the inclusion-exclusion principle gives

$$\mathbf{1}_K = \sum_{v \in S(m)} (-1)^{|v|-1} \mathbf{1}_{K_v}.$$

Thus V consists of all linear combinations of indicator functions of convex bodies.

We will now prove a general extension theorem for valuations on $\mathcal{K}^n$, which is due to Groemer [1]. We endow the set $\mathcal{K}^n$ of convex bodies with the Hausdorff metric δ, which is defined by

$$\begin{aligned} \delta(K,L) &:= \max\{\max_{x\in K}\min_{y\in L}\|x-y\|,\ \max_{x\in L}\min_{y\in K}\|x-y\|\} \\ &= \min\{\epsilon > 0 : K \subset L + \epsilon B^n,\ L \subset K + \epsilon B^n\}, \end{aligned}$$

and with the induced topology. A general extension theorem holds for continuous valuations with values in a topological vector space. This theorem will imply Theorem 4.1, but the short proof of the latter is of independent interest.

4.2 Theorem. *Let X be a topological vector space, and let $\varphi : \mathcal{K}^n \to X$ be a continuous additive mapping. Then φ has an additive extension to the convex ring $U(\mathcal{K}^n)$.*

Proof. An essential part of the proof is the following

PROPOSITION. The equality

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i} = 0$$

with $m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $K_i \in \mathcal{K}^n$ implies

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i) = 0.$$

Assume this proposition were false. Then there is a smallest number $m \in \mathbb{N}$, necessarily $m \geq 2$, for which there exist numbers $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ such that

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i} = 0, \tag{20}$$

but

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i) =: a \neq 0. \tag{21}$$

Let $H \subset \mathbb{R}^n$ be a hyperplane with $K_1 \subset \operatorname{int} H^+$, where H^+, H^- are the two closed halfspaces bounded by H. By (20) we have

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i \cap H^-} = 0, \quad \sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i \cap H} = 0.$$

Since $K_1 \cap H^- = \emptyset$ and $K_1 \cap H = \emptyset$, each of these two sums has at most $m-1$ non-zero summands. From the mimimality of m (and from $\varphi(\emptyset) = 0$) we get

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i \cap H^-) = 0, \quad \sum_{i=1}^{m} \alpha_i \varphi(K_i \cap H) = 0.$$

The additivity of φ on $\mathcal{K}^n$ yields

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i \cap H^+) = a, \tag{22}$$

whereas (20) gives

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i \cap H^+} = 0. \tag{23}$$

A standard separation theorem for convex bodies implies the existence of a sequence $(H_j)_{j\in\mathbb{N}}$ of hyperplanes with $K_1 \subset \operatorname{int} H_j^+$ for $j \in \mathbb{N}$ and

$$K_1 = \bigcap_{j=1}^{\infty} H_j^+.$$

If the argument that has led us from (20), (21) to (23), (22) is applied k-times, we obtain

$$\sum_{i=1}^{m} \alpha_i \varphi\left(K_i \cap \bigcap_{j=1}^{k} H_j^+\right) = a.$$

For $k \to \infty$ this yields

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i \cap K_1) = a, \tag{24}$$

since

$$\lim_{k\to\infty} K_i \cap \bigcap_{j=1}^{k} H_j^+ = K_i \cap K_1$$

in the sense of the Hausdorff metric (if $K_i \cap K_1 \neq \emptyset$, otherwise use $\varphi(\emptyset) = 0$) and φ is continuous. Equality (20) implies

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i \cap K_1} = 0. \tag{25}$$

The procedure leading from (20) and (21) to (25) and (24) can be repeated, replacing the bodies K_i and K_1 by $K_i \cap K_1$ and K_2, then by $K_i \cap K_1 \cap K_2$ and K_3, and so on. Finally one obtains

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_1 \cap \cdots \cap K_m} = 0$$

and

$$\sum_{i=1}^{m} \alpha_i \varphi(K_1 \cap \cdots \cap K_m) = a$$

(because of $K_i \cap K_1 \cap \cdots \cap K_m = K_1 \cap \cdots \cap K_m$). Now $a \neq 0$ implies $\sum_{i=1}^{m} \alpha_i \neq 0$ and hence $\mathbf{1}_{K_1 \cap \cdots \cap K_m} = 0$ by the first relation, but this yields $\varphi(K_1 \cap \cdots \cap K_m) = 0$, contradicting the second relation. This completes the proof of the proposition.

Now we consider the real vector space V of all finite linear combinations of indicator functions of elements of $\mathcal{K}^n$. For $K \in U(\mathcal{K}^n)$ we have $\mathbf{1}_K \in V$, as noted earlier. For fixed $f \in V$ we choose a representation

$$f = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i}$$

with $m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $K_i \in \mathcal{K}^n$ and define

$$\tilde{\varphi}(f) := \sum_{i=1}^{m} \alpha_i \varphi(K_i).$$

The proposition proved above shows that this definition is possible, since the right-hand side does not depend on the special representation chosen for f. Evidently, $\tilde{\varphi} : V \to X$ is a linear map satisfying $\tilde{\varphi}(\mathbf{1}_K) = \varphi(K)$ for $K \in \mathcal{K}^n$. We can now extend φ from $\mathcal{K}^n$ to $U(\mathcal{K}^n)$ by defining

$$\varphi(K) := \tilde{\varphi}(\mathbf{1}_K) \qquad \text{for } K \in U(\mathcal{K}^n).$$

By the linearity of $\tilde{\varphi}$ and the additivity of the map $K \mapsto \mathbf{1}_K$ we obtain, for $K, M \in U(\mathcal{K}^n)$,

$$\begin{aligned} \varphi(K \cup M) + \varphi(K \cap M) &= \tilde{\varphi}(\mathbf{1}_{K \cup M}) + \tilde{\varphi}(\mathbf{1}_{K \cap M}) \\ &= \tilde{\varphi}(\mathbf{1}_{K \cup M} + \mathbf{1}_{K \cap M}) \\ &= \tilde{\varphi}(\mathbf{1}_K + \mathbf{1}_M) \\ &= \tilde{\varphi}(\mathbf{1}_K) + \tilde{\varphi}(\mathbf{1}_M) \\ &= \varphi(K) + \varphi(M). \end{aligned}$$

Thus φ is additive on $U(\mathcal{K}^n)$. ■

5 Local parallel sets and curvature measures

One of our aims will be to compute integrals such as

$$I(K, M) := \int_{G_n} \chi(K \cap gM)\, d\mu(g) \tag{26}$$

for convex bodies $K, M \in \mathcal{K}^n$, where μ is the invariant measure on the motion group G_n; thus $I(K, M)$ is the total Haar measure of the set of rigid motions which bring M into a hitting position with K. We get a first hint to what the result will involve if we choose for M a ball ρB^n of radius $\rho > 0$. In that case,

$$I(K, \rho B^n) = \int_{\mathbb{R}^n} \chi(K \cap (\rho B^n + t))\, d\lambda(t) = V_n(K + \rho B^n),$$

as obtained in Section 2. The set $K + \rho B^n$ is known as the *outer parallel set* of K at distance ρ. It can also be represented as

$$K + \rho B^n = \{x \in \mathbb{R}^n : d(K, x) \le \rho\},$$

where

$$d(K, x) := \min\{\|x - y\| : y \in K\}$$

is the distance of x from K. A fundamental result in the geometry of convex bodies, the *Steiner formula*, says that the volume $V_n(K + \rho B^n)$ of the parallel body, as a function of the parameter ρ, is a polynomial of degree n, thus

$$V_n(K + \rho B^n) = \sum_{i=0}^{n} \rho^{n-i} \kappa_{n-i} V_i(K). \tag{27}$$

The reason for introducing the normalizing factors κ_{n-i} will become clear later in this section. The coefficients $V_0(K), \dots, V_n(K)$ appearing in (27) define important functionals of K. We have just seen that they inevitably appear when we want to compute the integral $I(K, \rho B^n)$. As it turns out, also the general integral $I(K, M)$ given by (26) can be expressed in terms of these functionals alone, evaluated for the bodies K and M.

In the present section, a more general version of the Steiner formula (27) will be obtained. Namely, we replace the parallel body $K + \rho B^n$ by a local version of it. The polynomial expansion generalizing (27) then defines a series of measures on $\mathbb{R}^n$, the *curvature measures* of the convex body K. These measures will appear in very general versions of the kinematic formula of integral geometry.

We need a simple device from convex geometry. Let $K \in \mathcal{K}^n$ be a convex body. For $x \in \mathbb{R}^n$, there is a unique point $p(K, x)$ in K nearest to x, that is,

$$\|p(K, x) - x\| = \min\{\|y - x\| : y \in K\} = d(K, x).$$

This defines a continuous map $p(K,\cdot):\mathbb{R}^n\to K$, which is called the *nearest-point map* of K, or the *metric projection* onto K. Also the map

$$\begin{array}{rcl} p:\mathcal{K}^n\times\mathbb{R}^n & \to & \mathbb{R}^n, \\ (K,x) & \mapsto & p(K,x) \end{array}$$

is continuous.

Now for $K\in\mathcal{K}^n$, a Borel set $A\in\mathcal{B}(\mathbb{R}^n)$ and a number $\rho\ge 0$, we define the *local parallel set* of (K,A) at distance ρ by

$$M_\rho(K,A):=\{x\in\mathbb{R}^n: d(K,x)\le\rho,\ p(K,x)\in A\}.$$

This is a Borel set, since $p(K,\cdot)$ is continuous. We can, therefore, define

$$\mu_\rho(K,A):=\lambda(M_\rho(K,A))\quad \text{for } A\in\mathcal{B}(\mathbb{R}^n).$$

In other words, $\mu_\rho(K,\cdot)$ is the image measure of the Lebesgue measure, restricted to the parallel body $K_\rho=K+\rho B^n$, under the nearest point map of K. In particular, $\mu_\rho(K,\cdot)$ is a finite measure on $\mathcal{B}(\mathbb{R}^n)$. We call it the *local parallel volume* of K at distance ρ.

This measure is concentrated on K, that is, $\mu_\rho(K,A)=\mu_\rho(K,A\cap K)$.

We first prove some fundamental properties of the mapping $\mu_\rho:\mathcal{K}^n\times\mathcal{B}(\mathbb{R}^n)\to\mathbb{R}$. In the following, $\xrightarrow{w}$ denotes weak convergence of finite measures.

5.1 Theorem. *Let $(K_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{K}^n$ satisfying $K_j\to K$ for $j\to\infty$. Then*

$$\mu_\rho(K_j,\cdot)\xrightarrow{w}\mu_\rho(K,\cdot)\quad \textit{for } j\to\infty, \tag{28}$$

for every $\rho>0$.

Proof. By a well-known characterization of weak convergence, we have to show that

$$\liminf_{j\to\infty}\mu_\rho(K_j,A)\ge\mu_\rho(K,A) \tag{29}$$

for every open set A, and

$$\lim_{j\to\infty}\mu_\rho(K_j,\mathbb{R}^n)=\mu_\rho(K,\mathbb{R}^n). \tag{30}$$

Let $A\subset\mathbb{R}^n$ be open. Let $x\in M_\rho(K,A)$ be a point with $d(K,x)<\rho$. Since p is continuous, we have $p(K_j,x)\to p(K,x)$ and $d(K_j,x)\to d(K,x)$ for $j\to\infty$. Hence, for all sufficiently large j we deduce that $p(K_j,x)\in A$ and $d(K_j,x)<\rho$, hence $x\in M_\rho(K_j,A)$. Thus we have

$$M_\rho(K,A)\setminus\partial K_\rho\subset\liminf_{j\to\infty}M_\rho(K_j,A)$$

and, therefore,

$$\begin{aligned} \mu_\rho(K,A) &= \lambda(M_\rho(K,A)\setminus\partial K_\rho) \\ &\le \lambda\left(\liminf_{j\to\infty} M_\rho(K_j,A)\right) \\ &\le \liminf_{j\to\infty} \lambda(M_\rho(K_j,A)) \\ &= \liminf_{j\to\infty} \mu_\rho(K_j,A), \end{aligned}$$

which proves (29). The relation (30) follows from standard results of convex geometry. ■

5.2 Theorem. *For any Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and any $\rho > 0$, the function $\mu_\rho(\cdot,A) : \mathcal{K}^n \to \mathbb{R}$ is measurable.*

Proof. For an open set A, the preceding proof shows that the function $\mu_\rho(\cdot,A)$ is lower semicontinuous, hence it is measurable.

Denote by $\mathcal{A}$ the system of all sets $A \in \mathcal{B}(\mathbb{R}^n)$ for which $\mu_\rho(\cdot,A)$ is measurable. We show that $\mathcal{A}$ is a Dynkin system. For $A_1, A_2 \in \mathcal{A}$ with $A_2 \subset A_1$ we have $M_\rho(K,A_2) \subset M_\rho(K,A_1)$ and

$$M_\rho(K,A_1\setminus A_2) = M_\rho(K,A_1)\setminus M_\rho(K,A_2),$$

hence

$$\mu_\rho(K,A_1\setminus A_2) = \mu_\rho(K,A_1) - \mu_\rho(K,A_2)$$

for all $K \in \mathcal{K}^n$, which shows that $A_1 \setminus A_2 \in \mathcal{A}$. If $(A_j)_{j\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$, then

$$\mu_\rho\left(K, \bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_\rho(K,A_j)$$

for $K \in \mathcal{K}^n$, since $\mu_\rho(K,\cdot)$ is a measure. It follows that $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$. Thus $\mathcal{A}$ is a Dynkin system. Since it contains the open sets, it also contains the σ-algebra generated by the open sets and thus all Borel sets, as asserted. ■

5.3 Theorem. *For any Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and for $\rho > 0$, the function $\mu_\rho(\cdot,A) : \mathcal{K}^n \to \mathbb{R}$ is additive.*

Proof. Let $K, L \in \mathcal{K}^n$ be convex bodies with $K \cup L \in \mathcal{K}^n$. Let $x \in \mathbb{R}^n$, and put $y := p(K \cup L, x)$. We assume $y \in K$, without loss of generality. Then

$$p(K\cup L, x) = p(K,x). \tag{31}$$

Let $z := p(L, x)$. Since $K \cup L$ is convex, there is a point $a \in [z, y]$ (the segment with endpoints z and y) with $a \in K \cap L$. From $y = p(K \cup L, x)$ it follows that $\|y - x\| \leq \|z - x\|$ and hence $\|a - x\| \leq \|z - x\|$. From $a \in L$ and the definition of z we conclude that $a = z$ and thus $z \in K \cap L$. This shows that

$$p(K \cap L, x) = p(L, x). \tag{32}$$

For $K' \in \mathcal{K}^n$, let $\mathbf{1}_\rho(K', A, \cdot)$ be the indicator function of the local parallel set $M_\rho(K', A)$. From (31) and (32) it follows that

$$\mathbf{1}_\rho(K \cup L, A, x) = \mathbf{1}_\rho(K, A, x), \quad \mathbf{1}_\rho(K \cap L, A, x) = \mathbf{1}_\rho(L, A, x).$$

Since x was arbitrary, this yields

$$\mathbf{1}_\rho(K \cup L, A, \cdot) + \mathbf{1}_\rho(K \cap L, A, \cdot) = \mathbf{1}_\rho(K, A, \cdot) + \mathbf{1}_\rho(L, A, \cdot).$$

Integrating this equation with the Lebesgue measure, we obtain

$$\mu_\rho(K \cup L, A) + \mu_\rho(K \cap L, A) = \mu_\rho(K, A) + \mu_\rho(L, A),$$

which shows that $\mu_\rho(\cdot, A)$ is additive on $\mathcal{K}^n$. ■

We will now explicitly compute the local parallel volume in the case of a convex polytope. For this, we need some elementary facts about polytopes, which we will use without proof.

A *polyhedral set* in $\mathbb{R}^n$ is a set which can be represented as the intersection of finitely many closed halfspaces. A bounded non-empty polyhedral set is called a *convex polytope* or briefly a *polytope*. Let P be a polytope. If H is a supporting hyperplane of P, then $P \cap H$ is again a polytope. The set $F := P \cap H$ is called a *face* of P, and an *m-face* if $\dim F = m$, $m \in \{0, \ldots, n-1\}$. If $\dim P = n$, we consider P as an n-face of itself. By $\mathcal{F}_m(P)$ we denote the set of all m-faces of P. For $F \in \mathcal{F}_m(P)$ we define

$$\lambda_F(B) := \lambda^{(m)}(B \cap F) \qquad \text{for } B \in \mathcal{B}(\mathbb{R}^n),$$

where $\lambda^{(m)}$ denotes m-dimensional Lebesgue measure. For $F \in \mathcal{F}_m(P)$, $m \in \{0, \ldots, n-1\}$ and a point $x \in \operatorname{relint} F$ (the relative interior of F), let $N(P, F)$ be the *normal cone* of P at F; this is the cone of outer normal vectors of supporting hyperplanes to P at x. It does not depend upon the choice of x. The number

$$\gamma(F, P) := \frac{\lambda^{(n-m)}(N(P, F) \cap B^n)}{\kappa_{n-m}}$$

is called the *external angle* of P at its face F. We also put $\gamma(P, P) = 1$ and $\gamma(F, P) = 0$ if either $F = \emptyset$ or F is not a face of P.

Now let a polytope P, a Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and a number $\rho > 0$ be given. For $x \in \mathbb{R}^n$, the nearest point $p(P, x)$ lies in the relative interior of a unique face of P. Therefore,

$$M_\rho(P, A) = \bigcup_{m=0}^{n} \bigcup_{F \in \mathcal{F}_m(P)} \left[P_\rho \cap p(P, \cdot)^{-1}(A \cap \operatorname{relint} F)\right] \tag{33}$$

is a disjoint decomposition of the local parallel set $M_\rho(P, A)$. It follows from the properties of the nearest point map that

$$P_\rho \cap p(P, \cdot)^{-1}(A \cap \operatorname{relint} F) \tag{34}$$

$$= (A \cap \operatorname{relint} F) \oplus (N(P, F) \cap \rho B^n), \tag{35}$$

where $\oplus$ denotes direct sum. An application of Fubini's theorem gives

$$\begin{aligned} &\lambda(P_\rho \cap p(P, \cdot)^{-1}(A \cap \operatorname{relint} F)) \\ &\quad = \lambda^{(m)}(A \cap F)\lambda^{(n-m)}(N(P, F) \cap \rho B^n) \\ &\quad = \lambda^{(m)}(A \cap F)\rho^{n-m}\kappa_{n-m}\gamma(F, P). \end{aligned}$$

Together with (33), this yields

$$\mu_\rho(P, A) = \sum_{m=0}^{n} \rho^{n-m}\kappa_{n-m} \sum_{F \in \mathcal{F}_m(P)} \lambda^{(m)}(A \cap F)\gamma(F, P).$$

Hence, if we define a measure $\Phi_m(P, \cdot)$ on $\mathcal{B}(\mathbb{R}^n)$ by

$$\Phi_m(P, \cdot) := \sum_{F \in \mathcal{F}_m(P)} \gamma(F, P)\lambda_F,$$

then

$$\mu_\rho(P, A) = \sum_{m=0}^{n} \rho^{n-m}\kappa_{n-m}\Phi_m(P, A).$$

This gives the desired polynomial expansion of the local parallel volume in the case of polytopes. The following theorem extends this result to general convex bodies.

5.4 Theorem. (Local Steiner formula) *For every convex body $K \in \mathcal{K}^n$, there exist finite measures $\Phi_0(K, \cdot), \ldots, \Phi_n(K, \cdot)$ on $\mathcal{B}(\mathbb{R}^n)$ such that the local parallel volume satisfies*

$$\mu_\rho(K, A) = \sum_{j=0}^{n} \rho^{n-j}\kappa_{n-j}\Phi_j(K, A)$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$ *and all* $\rho \geq 0$.

Proof. If P is a polytope, we have seen above that

$$\mu_\rho(P, A) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} \Phi_j(P, A) \tag{36}$$

with

$$\Phi_j(P, \cdot) = \sum_{F \in \mathcal{F}_j(P)} \gamma(F, P) \lambda_F. \tag{37}$$

Now let $K \in \mathcal{K}^n$ be an arbitrary convex body. As one knows from convex geometry, there is a sequence $(P_i)_{i \in \mathbb{N}}$ of polytopes converging to K in the Hausdorff metric. In (36), we replace P by P_i and ρ by each of the numbers $1, \dots, n+1$. The resulting system of linear equations,

$$\mu_k(P_i, A) = \sum_{j=0}^{n} k^{n-j} \kappa_{n-j} \Phi_j(P_i, A), \qquad k = 1, \dots, n+1,$$

can be solved for the 'unknowns' $\kappa_{n-j} \Phi_j(P_i, A)$ (it has a Vandermonde determinant), which yields representations

$$\Phi_j(P_i, A) = \sum_{k=1}^{n+1} \alpha_{jk} \mu_k(P_i, A), \qquad j = 0, \dots, n.$$

Here the coefficients α_{jk} do not depend on P_i or A, thus we have

$$\Phi_j(P_i, \cdot) = \sum_{k=1}^{n+1} \alpha_{jk} \mu_k(P_i, \cdot) \qquad \text{for } i \in \mathbb{N}.$$

By Theorem 5.1, for each fixed $\rho \geq 0$ the measures $\mu_\rho(P_i, \cdot)$ converge weakly to $\mu_\rho(K, \cdot)$. Hence, if we define a finite signed measure by

$$\Phi_j(K, \cdot) := \sum_{k=1}^{n+1} \alpha_{jk} \mu_k(K, \cdot),$$

then the measures $\Phi_j(P_i, \cdot)$ converge, for $i \to \infty$, weakly to the signed measure $\Phi_j(K, \cdot)$ $(j = 0, \dots, n)$. It follows that the latter is nonnegative, and it also follows that

$$\mu_\rho(K, \cdot) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} \Phi_j(K, \cdot),$$

using (36) and weak convergence. ■

One calls $\Phi_j(K,\cdot)$ the jth *curvature measure* of the body $K \in \mathcal{K}^n$. The reason for this name becomes clear if one considers a convex body K whose boundary is a regular hypersurface of class C^2. In that case, the local parallel volume can be computed by differential-geometric means, and one obtains for $j = 0,\dots,n-1$ the representation

$$\Phi_j(K,A) = \frac{\binom{n}{j}}{n\kappa_{n-j}} \int\limits_{A\cap\partial K} H_{n-1-j}\, dS.$$

Here H_k denotes the kth normalized elementary symmetric function of the principal curvatures of ∂K, and dS is the volume form on ∂K. Thus the curvature measures are (up to constant factors) indefinite integrals of curvature functions, and they replace the latter in the non-smooth case.

For $j = n$, we simply have

$$\Phi_n(K,A) = \lambda(K\cap A) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^n),$$

as follows immediately from the definition of the local parallel set and the local Steiner formula. For a general convex body K it is clear that the measures $\Phi_0(K,\cdot),\dots,\Phi_{n-1}(K,\cdot)$ are concentrated on ∂K, since $\mu_\rho(K,A) - \lambda(K\cap A)$ depends only on $A\cap\partial K$.

For polytopes P, we have the explicit representation (37) of the curvature measures. The external angle appearing in it does not depend on the dimension of the surrounding space, as follows easily from Fubini's theorem. In other words, if $\dim P < n$, it makes no difference if the external angle $\gamma(F,P)$ is computed in $\mathbb{R}^n$ or in the affine hull of P. This independence of dimension extends to the curvature measures $\Phi_j(P,\cdot)$ and then, by approximation and weak convergence, to the curvature measures $\Phi_j(K,\cdot)$ of arbitrary convex bodies.

We mention without proof that for arbitrary convex bodies K the measures $\Phi_0(K,\cdot)$ and $\Phi_{n-1}(K,\cdot)$ have simple intuitive interpretations. Namely, if $\dim K \neq n-1$, then

$$\Phi_{n-1}(K,A) = \frac{1}{2}\mathcal{H}^{n-1}(A\cap\partial K).$$

For $\dim K = n-1$, one trivially has $\Phi_{n-1}(K,A) = \mathcal{H}^{n-1}(A\cap\partial K)$. The measure Φ_0 is the normalized area of the spherical image. Let $\sigma(K,A) \subset S^{n-1}$ denote the set of all outer unit normal vectors of K at points of $A\cap\partial K$, then

$$\Phi_0(K,A) = \frac{1}{n\kappa_n}\mathcal{H}^{n-1}(\sigma(K,A)).$$

We can use the relation

$$\Phi_j(K,\cdot) = \sum_{k=1}^{n+1} \alpha_{jk}\mu_k(K,\cdot), \tag{38}$$

which was obtained in the proof of Theorem 5.4, to transfer properties of the local parallel volumes $\mu_\rho(K,\cdot)$ to the curvature measures $\Phi_j(K,\cdot)$. In this way Theorems 5.1, 5.2, 5.3, together with some easily obtained additional properties of the local parallel volumes, yield a series of properties of the curvature measures, which we list in the following theorem.

5.5 Theorem. *Let $j \in \{0,\dots,n\}$.*
(a) $\Phi_j(K,\cdot)$ *depends weakly continuously on K, that is, $K_i \to K$ implies the weak convergence $\Phi_j(K_i,\cdot) \xrightarrow{w} \Phi_j(K,\cdot)$ for $i \to \infty$.*
(b) *For every $A \in \mathcal{B}(\mathbb{R}^n)$, the function $\Phi_j(\cdot,A)$ is measurable on $\mathcal{K}^n$.*
(c) Φ_j *is motion covariant, that is,*

$$\Phi_j(gK, gA) = \Phi_j(K,A)$$

for every rigid motion $g \in G_n$ and all $K \in \mathcal{K}^n$ and $A \in \mathcal{B}(\mathbb{R}^n)$.
(d) Φ_j *is homogeneous of degree j, that is,*

$$\Phi_j(\alpha K, \alpha A) = \alpha^j \Phi(K,A)$$

for every $\alpha > 0$ and all $K \in \mathcal{K}^n$ and $A \in \mathcal{B}(\mathbb{R}^n)$.
(e) Φ_j *is defined locally, that is, for every open set $A \subset \mathbb{R}^n$ and all convex bodies $K, M \in \mathcal{K}^n$ with $K \cap A = M \cap A$, one has*

$$\Phi_j(K,B) = \Phi_j(M,B)$$

for every Borel set $B \subset A$.
(f) $\Phi_j(\cdot,A)$ *is additive for every $A \in \mathcal{B}(\mathbb{R}^n)$, that is,*

$$\Phi_j(K \cup L, A) + \Phi_j(K \cap L, A) = \Phi_j(K,A) + \Phi_j(L,A)$$

holds for all convex bodies $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$.

The final property, together with Theorem 4.2, has the important consequence that the curvature measures have an additive extension to polyconvex sets. This means that one can define signed measures on the convex ring $U(\mathcal{K}^n)$ in the following way. Let $K \in U(\mathcal{K}^n)$ and choose a representation $K = \bigcup_{i=1}^m K_i$ with $m \in \mathbb{N}$ and $K_i \in \mathcal{K}^n$. Then

$$\Phi_j(K,\cdot) := \sum_{v \in S(m)} (-1)^{|v|-1} \Phi_j(K_v,\cdot)$$

does not depend on the special choice of the representation; in particular, this is consistent with the already defined value $\Phi_j(K,\cdot)$ for convex K. This follows from Theorem 4.2, since the weak convergence of curvature measures can be interpreted as convergence in the vector space of finite signed measures, on a suitable compact subset of $\mathbb{R}^n$, with respect to a suitable topology.

We have now everything at hand to formulate a central result of integral geometry. This is the *principal kinematic formula*, in a version for curvature measures on the convex ring. Let $K, M \in U(\mathcal{K}^n)$ be polyconvex sets, let $A, B \in \mathcal{B}(\mathbb{R}^n)$ be Borel sets, and let $j \in \{0, \dots, n\}$. Then

$$\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(g) = \sum_{k=j}^{n} \alpha_{njk} \Phi_j(K, A) \Phi_{n+j-k}(M, B)$$

holds, with certain explicit constants α_{njk}.

We will indicate a proof of this result in Section 9. Before that, however, we will prove a global version of this formula in a different way. The method of proof is of independent interest and leads to further results for which no other access is known.

The global result refers to the total measures

$$V_j(K) := \Phi_j(K, \mathbb{R}^n), \qquad j = 0, \dots, n.$$

The number V_j is called the jth *intrinsic volume* of K. These important functionals are defined by the classical Steiner formula

$$V_n(K + \rho B^n) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} V_j(K),$$

of which Theorem 5.4 is the local generalization. As a function on $\mathcal{K}^n$, each intrinsic volume V_j is continuous, additive and rigid motion invariant. In the next section we shall prove that the intrinsic volumes are essentially characterized by these properties.

The additive extensions of the intrinsic volumes to the convex ring $U(\mathcal{K}^n)$ will be denoted by the same symbols. In the following cases, they have simple intuitive interpretations. It is clear that

$$V_n(K) = \lambda(K) \qquad \text{for } K \in U(\mathcal{K}^n),$$

since this holds true for convex bodies K and both functions, V_n and λ, are additive on $U(\mathcal{K}^n)$. It also remains true for polyconvex sets that

$$2V_{n-1}(K) = \mathcal{H}^{n-1}(\partial K)$$

if K is the closure of its interior, but this requires an extra proof. Finally,

$$V_0(K) = \chi(K) \qquad \text{for } K \in U(\mathcal{K}^n),$$

so that V_0 is nothing but the Euler characteristic. For a convex polytope P we have

$$V_0(P) = \Phi_0(P, \mathbb{R}^n) = \sum_{F \in \mathcal{F}_0(P)} \gamma(F, P) = 1,$$

since the normal cones $N(P, F)$ of P at its vertices F cover $\mathbb{R}^n$ and have pairwise no common interior points. By additivity, the equation $V_0(K) = \chi(K)$ extends from $\mathcal{K}^n$ to $U(\mathcal{K}^n)$.

6 Hadwiger's characterization theorem

The jth intrinsic volume $V_j : \mathcal{K}^n \to \mathbb{R}$ is an additive, continuous and rigid motion invariant function. A celebrated theorem due to Hadwiger (see [2]) says that any function on $\mathcal{K}^n$ with these properties is a linear combination of the intrinsic volumes $V_0, \ldots, V_n$. This result can be used to prove some formulae of the integral geometry of convex bodies in a very elegant way. Whereas Hadwiger's original proof was quite long, one has now a shorter proof due to Klain [3]. We will present his proof here, except that at one point we take a certain analytical result for granted.

The crucial step for a proof of the characterization theorem is the following result.

6.1 Theorem. *Suppose that $\psi : \mathcal{K}^n \to \mathbb{R}$ is an additive, continuous, motion invariant function satisfying $\psi(K) = 0$ whenever either $\dim K < n$ or K is a unit cube. Then $\psi = 0$.*

Proof. The proof proceeds by induction with respect to the dimension. For $n = 0$, there is nothing to prove. If $n = 1$, ψ vanishes on (closed) segments of unit length, hence on segments of length $1/k$ for $k \in \mathbb{N}$ and therefore on segments of rational length. By continuity, ψ vanishes on all segments and thus on $\mathcal{K}^1$.

Now let $n > 1$ and suppose that the assertion has been proved in dimensions less than n. Let $H \subset \mathbb{R}^n$ be a hyperplane and I a closed segment of length 1, orthogonal to H. For convex bodies $K \subset H$ define $\varphi(K) := \psi(K + I)$. Clearly φ has, relative to H, the properties of ψ in the Theorem, hence the induction hypothesis yields $\varphi = 0$. For fixed $K \subset H$, we thus have $\psi(K + I) = 0$, and a similar argument as used above for $n = 1$

shows that $\psi(K+S)=0$ for any closed segment S orthogonal to H. Thus μ vanishes on right convex cylinders.

Let $K \subset H$ again be a convex body and let $S = \mathrm{conv}\,\{0, s\}$ be a segment not parallel to H. If $m \in \mathbb{N}$ is sufficiently large, the cylinder $Z := K + mS$ can be cut by a hyperplane H' orthogonal to S so that the two closed halfspaces H^-, H^+ bounded by H' satisfy $K \subset H^-$ and $K + ms \subset H^+$. Then $\overline{Z} := [(Z \cap H^-) + ms] \cup (Z \cap H^+)$ is a right cylinder, and we deduce that $m\mu(K+S) = \mu(Z) = \mu(\overline{Z}) = 0$. Thus ψ vanishes on arbitrary convex cylinders.

By Theorem 4.2, the continuous additive function ψ has an additive extension to the convex ring; this extension is also denoted by ψ. It follows that

$$\psi\left(\bigcup_{i=1}^{k} K_i\right) = \sum_{i=1}^{k} \psi(K_i)$$

whenever $K_1, \ldots, K_k$ are convex bodies such that $\dim(K_i \cap K_j) < n$ for $i \neq j$.

Let P be a polytope and S a segment. The sum $P+S$ has a decomposition $P + S = \bigcup_{i=1}^{k} P_i$, where $P_1 = P$, the polytope P_i is a convex cylinder for $i > 1$, and $\dim(P_i \cap P_j) < n$ for $i \neq j$. It follows that $\psi(P+S) = \psi(P)$. By induction, we obtain $\psi(P+Z) = \psi(P)$ if Z is a finite sum of segments. By continuity, $\psi(K+Z) = \psi(K)$ for arbitrary convex bodies K and zonoids Z, that is, limits of sums of segments.

Now we have to use an analytic result, for which we do not give a proof. Let K be a centrally symmetric convex body which is sufficiently smooth (say, its support function is of class C^∞). Then there exist zonoids Z_1, Z_2 so that $K + Z_1 = Z_2$ (this can be seem from Section 3.5 in [7], especially Theorem 3.5.3). We conclude that $\psi(K) = \psi(K + Z_1) = \psi(Z_2) = 0$. Since every centrally symmetric convex body K can be approximated by bodies which are centrally symmetric and sufficiently smooth in the above sense, it follows from the continuity of ψ that $\psi(K) = 0$ for all centrally symmetric convex bodies.

Now let Δ be a simplex, say $\Delta = \mathrm{conv}\,\{0, v_1, \ldots, v_n\}$, without loss of generality. Let $v := v_1 + \cdots + v_n$ and $\Delta' := \mathrm{conv}\,\{v, v - v_1, \ldots, v - v_n\}$, then $\Delta' = -\Delta + v$. The vectors $v_1, \ldots, v_n$ span a parallelotope P. It is the union of Δ, Δ' and the part of P lying between the hyperplanes spanned by $v_1, \ldots, v_n$ and $v - v_1, \ldots, v - v_n$, respectively. The latter, say Q, is a centrally symmetric polytope, and $\Delta \cap Q$, $\Delta' \cap Q$ are of dimension $n-1$. We deduce that $0 = \psi(P) = \psi(\Delta) + \psi(Q) + \psi(\Delta')$, thus $\psi(-\Delta) = -\psi(\Delta)$. If the dimension n is even, then $-\Delta$ is obtained from Δ by a proper rigid motion, and the motion invariance of ψ yields $\psi(\Delta) = 0$. If the dimension

$n > 1$ is odd, we decompose Δ as follows. Let z be the centre of the inscribed ball of Δ, and let p_i be the point where this ball touches the facet F_i of Δ ($i = 1, \dots, n+1$). For $i \neq j$, let Q_{ij} be the convex hull of the face $F_i \cap F_j$ and the points z, p_i, p_j. The polytope Q_{ij} is invariant under reflection in the hyperplane spanned by $F_i \cap F_j$ and z. If $Q_1, \dots, Q_m$ are the polytopes Q_{ij} for $1 \leq c < j \leq n+1$ in any order, then $P = \bigcup_{r=1}^{m} Q_r$ and $\dim(Q_r \cap Q_s) < n$ for $r \neq s$. Since $-Q_r$ is the image of Q_r under a proper rigid motion, we have $\psi(-\Delta) = \sum \psi(-Q_r) = \sum \psi(Q_r) = \psi(\Delta)$. Thus $\psi(\Delta) = 0$ for every simplex Δ.

Decomposing a polytope P into simplices, we obtain $\psi(P) = 0$. The continuity of ψ now implies $\psi(K) = 0$ for all convex bodies K. This finishes the induction and hence the proof of Theorem 6.1. ■

Hadwiger's characterization theorem is now an easy consequence.

6.2 Theorem. *Suppose that $\psi : \mathcal{K}^n \to \mathbb{R}$ is an additive, continuous, motion invariant function. Then there are constants $c_0, \dots, c_n$ so that*

$$\psi(K) = \sum_{i=0}^{n} c_i V_i(K)$$

for all $K \in \mathcal{K}^n$.

Proof. We use induction on the dimension. For $n = 0$ the assertion is trivial. Suppose that $n > 0$ and the assertion has been proved in dimensions less than n. Let $H \subset \mathbb{R}^n$ be a hyperplane. The restriction of ψ to the convex bodies lying in H is additive, continuous and invariant under motions of H into itself. By the induction hypothesis, there are constants $c_0, \dots, c_{n-1}$ so that $\psi(K) = \sum_{i=0}^{n-1} c_i V_i(K)$ holds for convex bodies $K \subset H$ (note that the intrinsic volumes do not depend on the dimension of the surrounding space). By the motion invariance of ψ and V_i, this holds for all $K \in \mathcal{K}^n$ of dimension less than n. It follows that the function ψ' defined by

$$\psi'(K) := \psi(K) - \sum_{i=0}^{n} c_i V_i(K)$$

for $K \in \mathcal{K}^n$, where c_n is chosen so that ψ' vanishes at a fixed unit cube, satisfies the assumptions of Theorem 6.1. Hence $\psi' = 0$, which completes the proof of Theorem 6.2. ■

The late Gian-Carlo Rota, in a Colloquium Lecture at the Annual Meeting of the AMS in 1997, called Hadwiger's characterization theorem the 'Main

Theorem of Geometric Probability'. The reason is that it can be used to derive kinematic formulae of integral geometry, which can in turn be interpreted in terms of geometric hitting probabilities. We shall see this, in more elaborate versions, in the next two sections.

7 Kinematic and Crofton formulae

Our aim in this section will be to compute the integrals

$$\int_{G_n} V_j(K \cap gM)\, d\mu(g)$$

and

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E)$$

for convex bodies K, M, where V_j is an intrinsic volume. For that, we use Hadwiger's characterization theorem. From this result, we first deduce a more general kinematic formula, involving a functional on convex bodies that need not have any invariance property.

7.1 Theorem. *If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is an additive continuous function, then*

$$\int_{G_n} \varphi(K \cap gM)\, d\mu(g) = \sum_{k=0}^{n} \varphi_{n-k}(K) V_k(M) \tag{39}$$

for $K, M \in \mathcal{K}^n$, where the coefficients $\varphi_{n-k}(K)$ are given by

$$\varphi_{n-k}(K) = \int_{\mathcal{E}_k^n} \varphi(K \cap E)\, d\mu_k(E). \tag{40}$$

Proof. In order that the integral in (39) makes sense, we first have to show that for given convex bodies K, M the function $g \mapsto \varphi(K \cap gM)$ is μ-integrable. Let $G_n(K, M)$ denote the set of all rigid motions $g \in G_n$ for which K and gM touch, that is, $K \cap gM \neq \emptyset$ but K and gM can be separated weakly by a hyperplane. Using the map γ from (8), it is easy to see that $\gamma(t, \vartheta) \in G_n(K, M)$ holds if and only if $t \in \partial(K - \vartheta M)$; hence

$$\mu(G_n(K, M)) = \int_{SO_n} \int_{\mathbb{R}^n} \mathbf{1}_{G_n(K,M)}(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta)$$

$$= \int_{SO_n} \lambda(\partial(K - \vartheta M))\, d\nu(\vartheta) = 0.$$

On $G_n \setminus G_n(K, M)$, the map $g \mapsto \varphi(K \cap gM)$ is continuous. Since the continuous function φ is bounded on the compact set $\{K' \in \mathcal{K}^n : K' \subset K\}$, it follows that the integral in (39) is well-defined and finite.

Now we fix a convex body $K \in \mathcal{K}^n$ and define

$$\psi(M) := \int_{G_n} \varphi(K \cap gM)\, d\mu(g) \quad \text{for } M \in \mathcal{K}^n.$$

Then $\psi : \mathcal{K}^n \to \mathbb{R}$ is obviously additive and motion invariant. It follows from the bounded convergence theorem that ψ is continuous. Theorem 6.2 yields the existence of constants $\varphi_0(K), \dots, \varphi_n(K)$ so that

$$\psi(M) = \sum_{k=0}^{n} \varphi_k(K) V_{n-k}(M)$$

for all $M \in \mathcal{K}^n$. The constants depend, of course, on the given body K, and we have now to determine them.

Suppose first that $1 \leq k \leq n-1$ and let $L_k \in \mathcal{L}_k^n$. We choose a k-dimensional cube $W \subset L_k$ with $0 \in W$ and $\lambda^{(k)}(W) = 1$. For $r \geq 1$ we have

$$\psi(rW) = \int_{G_n} \varphi(K \cap grW)\, d\mu(g) = \sum_{i=0}^{n} \varphi_{n-i}(K) V_i(rW).$$

The intrinsic volumes have the easily established properties

$$V_i(rW) = \begin{cases} 0 & \text{for } i > k, \\ r^k & \text{for } i = k, \\ r^i V_i(W) & \text{for } i < k. \end{cases}$$

This yields

$$\psi(rW) = \varphi_{n-k}(K) r^k + O(r^{k-1}) \tag{41}$$

for $r \to \infty$. On the other hand,

$$\begin{aligned} \psi(rW) &= \int_{G_n} \varphi(K \cap grW)\, d\mu(g) \\ &= \int_{SO_n} \int_{\mathbb{R}^n} \varphi(K \cap (\vartheta rW + x))\, d\lambda(x)\, d\nu(\vartheta) \end{aligned}$$

$$= \int_{SO_n} \int_{\vartheta L_k^\perp} \int_{\vartheta L_k} \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2)$$
$$d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta).$$

For fixed $\vartheta \in SO_n$ and $x_1 \in L_k^\perp$ we put

$$\begin{aligned} X &:= \{x_2 \in \vartheta L_k : K \cap (\vartheta r W + x_1 + x_2) = K \cap (\vartheta L_k + x_1)\}, \\ Y &:= \{x_2 \in \vartheta L_k : \emptyset \neq K \cap (\vartheta r W + x_1 + x_2) \neq K \cap (\vartheta L_k + x_1)\}. \end{aligned}$$

Then

$$\begin{aligned} &\int_{\vartheta L_k} \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2) \\ &\quad = \varphi(K \cap (\vartheta L_k + x_1)) \int_X d\lambda^{(k)}(x_2) \\ &\qquad + \int_Y \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2). \end{aligned}$$

For $r \to \infty$, we get

$$\int_X d\lambda^{(k)}(x_2) = r^k + O(r^{k-1}).$$

Since φ is bounded on compact sets,

$$\int_Y \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2) = O(r^{k-1}).$$

We deduce that

$$\begin{aligned} \psi(rW) &= r^k \int_{SO_n} \int_{\vartheta L_k^\perp} \varphi(K \cap (\vartheta L_k + x_1))\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta) + O(r^{k-1}) \\ &= r^k \int_{SO_n} \int_{L_k^\perp} \varphi(K \cap \vartheta(L_k + x_1))\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta) + O(r^{k-1}) \\ &= r^k \int_{\mathcal{E}_k^n} \varphi(K \cap E)\, d\mu_k(E) + O(r^{k-1}). \end{aligned}$$

If we compare this with (41) and let r tend to infinity, we obtain the asserted formula (40) for the coefficients.

In the cases $k = 0$ and $k = n$, simpler versions of the proof, with the obvious changes, give the same result. This completes the proof of Theorem 7.1. ■

In Theorem 7.1, we can choose for φ the intrinsic volume V_j and get

$$\int_{G_n} V_j(K \cap gM)\, d\mu(g) = \sum_{k=0}^{n} V_{j,n-k}(K) V_k(M)$$

with

$$V_{j,n-k}(K) = \int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E).$$

By

$$\psi(K) := \int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E) \qquad \text{for } K \in \mathcal{K}^n$$

we again define a functional $\psi : \mathcal{K}^n \to \mathbb{R}$ which is additive, continuous and motion invariant. This is proved similarly as above. Hadwiger's characterization theorem yields a representation

$$\psi(K) = \sum_{r=0}^{n} c_r V_r(K).$$

Here only one coefficient is non-zero. In fact, from

$$\psi(K) = \int_{\mathcal{L}_k^n} \int_{L^\perp} V_j(K \cap (L+y))\, d\lambda^{(n-k)}(y)\, d\nu_k(L)$$

one sees that ψ has the homogeneity property

$$\psi(\alpha K) = \alpha^{n-k+j} \psi(K)$$

for $\alpha > 0$. Since V_k is homogeneous of degree k, we deduce that $c_r = 0$ for $r \neq n-k+j$. Thus we have obtained

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E) = \alpha_{njk} V_{n+j-k}(K)$$

with some constant α_{njk}. In order to determine this constant, we choose for K the unit ball B^n. For $\epsilon \geq 0$, the Steiner formula gives

$$\sum_{j=0}^{n} \epsilon^{n-j} \kappa_{n-j} V_j(B^n) = V_n(B^n + \epsilon B^n) = (1+\epsilon)^n \kappa_n = \sum_{j=0}^{n} \epsilon^{n-j} \binom{n}{j} \kappa_n,$$

hence

$$V_j(B^n) = \frac{\binom{n}{j}\kappa_n}{\kappa_{n-j}} \qquad \text{for } j = 0, \dots, n.$$

Choosing $L \in \mathcal{L}_k^n$, we obtain

$$\begin{aligned}
\alpha_{njk} V_{n+j-k}(B^n) &= \int_{\mathcal{E}_k^n} V_j(B^n \cap E)\, d\mu_k(E) \\
&= \int_{SO_n} \int_{L^\perp} V_j(B^n \cap \vartheta(L+x))\, d\lambda^{(n-k)}(x)\, d\nu(\vartheta) \\
&= \int_{L^\perp \cap B^n} (1 - \|x\|^2)^{j/2} V_j(B^n \cap L)\, d\lambda^{(n-k)}(x) \\
&= \frac{\binom{k}{j}\kappa_k}{\kappa_{k-j}} \int_{L^\perp \cap B^n} (1 - \|x\|^2)^{j/2}\, d\lambda^{(n-k)}(x).
\end{aligned}$$

Introducing polar coordinates, the latter integral is transformed into a Beta integral, and one obtains

$$\begin{aligned}
&\int_{L^\perp \cap B^n} (1 - \|x\|^2)^{j/2}\, d\lambda^{(n-k)}(x) \\
&= (n-k)\kappa_{n-k} \int_0^1 (1 - r^2)^{j/2} r^{n-k-1}\, dr \\
&= \frac{1}{2}(n-k)\kappa_{n-k} \int_0^1 (1-t)^{j/2} t^{\frac{n-k-2}{2}}\, dt \\
&= \frac{1}{2}(n-k)\kappa_{n-k} \mathrm{B}\left(\frac{j+2}{2}, \frac{n-k}{2}\right) \\
&= \frac{1}{2}(n-k)\kappa_{n-k} \frac{\Gamma(\frac{j+2}{2})\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+j-k+2}{2})} = \frac{\kappa_{n+j-k}}{\kappa_j}.
\end{aligned}$$

Altogether this yields

$$\alpha_{njk} = \frac{\binom{k}{j}\kappa_k \kappa_{n+j-k}}{V_{n+j-k}(B^n)\kappa_{k-j}\kappa_k} = \frac{\binom{k}{j}\kappa_k \kappa_{n+j-k}}{\binom{n}{k-j}\kappa_n \kappa_j}.$$

This can be put in still a different form by using the identity

$$n!\kappa_n = 2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right).$$

We collect what we have obtained.

7.2 Theorem. *Let $K, M \in \mathcal{K}^n$ be convex bodies and let $j \in \{0, \dots, n\}$. Then the principal kinematic formula*

$$\int_{G_n} V_j(K \cap gM)\, d\mu(g) = \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K) V_k(M)$$

holds. For $k \in \{1, \dots, n-1\}$ and $j \le k$ the Crofton formula

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E) = \alpha_{njk} V_{n+j-k}(K)$$

holds. The coefficients are given by

$$\alpha_{njk} = \frac{\binom{k}{j}\kappa_k \kappa_{n+j-k}}{\binom{n}{k-j}\kappa_j \kappa_n} = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n+j-k+1}{2})}{\Gamma(\frac{j+1}{2})\Gamma(\frac{n+1}{2})}.$$

Finally, the results are easily extended to polyconvex sets. Let $K \in U(\mathcal{K}^n)$. We choose a representation

$$K = \bigcup_{i=1}^{m} K_i$$

with convex bodies $K_1, \dots, K_m$. Since V_{n+j-k} is additive on $U(\mathcal{K}^n)$, the inclusion-exclusion principle gives

$$V_{n+j-k}(K) = \sum_{v \in S(m)} (-1)^{|v|-1} V_{n+j-k}(K_v).$$

Now let $M \in \mathcal{K}^n$ be a convex body. Since the principal kinematic formula holds for convex bodies, we obtain

$$\begin{aligned}
&\int_{G_n} V_j(K \cap gM)\, d\mu(g) \\
&\quad = \int_{G_n} V_j\left(\bigcup_{i=1}^{m}(K_i \cap gM)\right) d\mu(g) \\
&\quad = \int_{G_n} \sum_{v \in S(m)} (-1)^{|v|-1} V_j(K_v \cap gM)\, d\mu(g)
\end{aligned}$$

$$= \sum_{v \in S(m)} (-1)^{|v|-1} \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K_v) V_k(M)$$

$$= \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K) V_k(M).$$

Hence, the kinematic formula holds for $K \in U(\mathcal{K}^n)$ and $M \in \mathcal{K}^n$. In a similar way, it can now be extended to $K \in U(\mathcal{K}^n)$ and $M \in U(\mathcal{K}^n)$. An analogous extension is possible for the Crofton formula.

8 Extension to random sets

It has been announced in the introduction that we want to use integral-geometric results to give a theoretical foundation for some formulae used in stereology. To achieve this goal, we shall now extend the kinematic and Crofton formulae to certain random sets.

First we have to explain what one understands by a closed random set in $\mathbb{R}^n$. Let $\mathcal{F}$ denote the system of all closed subsets of $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$ one writes

$$\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \emptyset\},$$
$$\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \emptyset\}.$$

The system

$$\{\mathcal{F}_G : G \subset \mathbb{R}^n \text{ open}\} \cup \{\mathcal{F}^C : C \subset \mathbb{R}^n \text{ compact}\}$$

is a subbasis of a topology on $\mathcal{F}$; this topology is called the *topology of closed convergence*. By $\mathcal{B}(\mathcal{F})$ we denote the corresponding σ-algebra of Borel sets.

Now a *random closed set* in $\mathbb{R}^n$, briefly a RACS, is defined as a random variable with values in $\mathcal{F}$. More precisely, a RACS is a measurable map $Z : \Omega \to \mathcal{F}$ from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the measurable space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. For $\omega \in \Omega$, the closed set $Z(\omega)$ is called a *realisation* of Z. The image measure $\mathbb{P}_Z := Z(\mathbb{P})$ of the probability measure $\mathbb{P}$ under the map Z is called the *distribution* of Z. Thus, this is a measure on $\mathcal{B}(\mathcal{F})$, and for $A \in \mathcal{B}(\mathcal{F})$ one has

$$\mathbb{P}_Z(A) = \mathbb{P}(Z^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega : Z(\omega) \in A\}) =: \mathbb{P}(Z \in A),$$

which is the probability that Z has a realization in the prescribed set A.

The random closed set Z is called *stationary* if for each vector $t \in \mathbb{R}^n$ the random closed sets Z and $Z + t$ have the same distribution, in other words, if the distribution of Z is invariant under translations. If the distribution of Z is invariant under rotations, then Z is called *isotropic*.

For a measurable nonnegative or $\mathbb{P}$-integrable function $f : \Omega \to \mathbb{R}$, the *expectation* is

$$\mathbb{E}f := \int_\Omega f \, d\mathbb{P}.$$

We will often have a random closed set $Z : \Omega \to \mathcal{F}$ and a measurable function $f : \mathcal{F} \to \mathbb{R}$. If the expectation of $f \circ Z$ exists, it is given by

$$\mathbb{E}f(Z) := \int_\Omega f \circ Z \, d\mathbb{P} = \int_\mathcal{F} f \, d\mathbb{P}_Z,$$

by the transformation formula for integrals.

For our envisaged applications, we have to restrict the admitted random closed sets. The *extended convex ring* is defined by

$$LU(\mathcal{K}^n) := \{F \subset \mathbb{R}^n : F \cap K \in U(\mathcal{K}^n) \ \text{ for } K \in \mathcal{K}^n\}.$$

The elements of $LU(\mathcal{K}^n)$ will also be called *locally polyconvex sets*. Thus a locally polyconvex set has the property that its intersection with any convex body is a finite union of convex bodies.

If $M \in U(\mathcal{K}^n)$ is a non-empty polyconvex set, there are a number $m \in \mathbb{N}$ and convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ such that $M = K_1 \cup \dots \cup K_m$. The smallest number m with this property will be denoted as $N(M)$. We also put $N(\emptyset) = 0$. This defines a function $N : U(\mathcal{K}^n) \to \mathbb{N}_0$, which can be shown to be measurable. Now we are in a position to define the random closed sets which will be admitted in the following.

Definition. A *standard random set* in $\mathbb{R}^n$ is a closed random set Z in $\mathbb{R}^n$ with the following properties:
(a) The realizations of Z are locally polyconvex,
(b) Z is stationary,
(c) Z satisfies the integrability condition

$$\mathbb{E}2^{N(Z \cap C^n)} < \infty.$$

Here, as before, $C^n := [0, 1]^n$ is the unit cube in $\mathbb{R}^n$.

For a standard random set, one can define a volume density, a surface area density and, more generally, the density of the jth intrinsic volume. Let Z be a standard random set. We choose a 'test body' (or 'observation window') $K \in \mathcal{K}^n$ with $V_n(K) > 0$. For a given realization $Z(\omega)$, the intersection $Z(\omega) \cap K$ is polyconvex, hence the (additively extended) jth intrinsic volume $V_j(Z(\omega) \cap K)$ is defined. One can show that the function $\omega \mapsto V_j(Z(\omega) \cap K)$ is measurable, hence it defines a real random variable. Its expectation,

$$\mathbb{E}V_j(Z \cap K),$$

depends on both, the random set Z and the test body K. However, we shall see that the limit

$$\overline{V}_j(Z) := \lim_{r\to\infty} \frac{\mathbb{E}V_j(Z \cap rK)}{V_n(rK)}$$

exists and is independent of K. This number $\overline{V}_j(Z)$ is called the *density of the jth intrinsic volume* of the random set Z.

The existence proof for the limit, which is a bit technical, is preceded by two lemmas. Recall that C^n is the unit cube given by

$$C^n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_i \le 1 \text{ for } i = 1, \dots, n\}.$$

The set

$$\partial^+ C^n := \{x = (x_1, \dots, x_n) \in C^n : \max_{1\le i\le n} x_i = 1\}$$

is called the *right upper boundary* of C^n. It is polyconvex. We need the set

$$C_0^n := C^n \setminus \partial^+ C^n$$

as a ‘fundamental domain’; the space $\mathbb{R}^n$ can be represented as a disjoint union of translates of C_0^n:

$$\mathbb{R}^n = \bigcup_{z\in\mathbb{Z}^n} (C_0^n + z).$$

We write $\mathbb{Z}^n$ as a sequence $(z_i)_{i\in\mathbb{N}}$ (in any order) and put

$$C_i := C^n + z_i, \qquad \partial^+ C_i := \partial^+ C^n + z_i.$$

The set C_0^n belongs to $U(\mathcal{P}^n_{ro})$, the class of all finite unions of relatively open convex polytopes. Below we shall use the fact that every additive functional on the class of polytopes has an additive extension to $U(\mathcal{P}^n_{ro})$. We do not give a proof here, but refer to [6].

8.1 Lemma. *If $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ is an additive function and $K \in U(\mathcal{K}^n)$ is a polyconvex set, then*

$$\varphi(K) = \sum_{i\in\mathbb{N}} [\varphi(K \cap C_i) - \varphi(K \cap \partial^+ C_i)].$$

Proof. Let $K \in U(\mathcal{K}^n)$. For a polytope $P \in \mathcal{K}^n$ we define

$$\psi(P) := \varphi(K \cap P).$$

Then ψ is an additive functional on convex polytopes and hence has a unique extension to an additive function on $U(\mathcal{P}^n_{ro})$, also denoted by ψ. Without loss of generality we may assume that

$$K \subset Q := \bigcup_{i=1}^{m} (C_0^n + z_i)$$

and that $\overline{Q}$ is convex (where $\overline{Q}$ denotes the closure of Q). Then

$$\begin{aligned}
\varphi(K) &= \varphi(K \cap \overline{Q}) = \psi(\overline{Q}) = \psi(Q) \\
&= \sum_{i \in \mathbb{N}} \psi(C_0^n + z_i) \\
&= \sum_{i \in \mathbb{N}} [\psi(C_i) - \psi(\partial^+ C_i)] \\
&= \sum_{i \in \mathbb{N}} [\varphi(K \cap C_i) - \varphi(K \cap \partial^+ C_i)].
\end{aligned}$$

Here we have used the additivity of ψ on $U(\mathcal{P}^n_{ro})$ and the fact that $\psi(P) = 0$ for all convex polytopes P with $K \cap P = \emptyset$. ■

We call a function $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ *conditionally bounded* if, for each $K' \in \mathcal{K}^n$, the function φ is bounded on the set $\{K \in \mathcal{K}^n : K \subset K'\}$. When φ is translation invariant and additive, it is sufficient for this to assume that φ is bounded on the set $\{K \in \mathcal{K}^n : K \subset C^n\}$.

8.2 Lemma. *Let the function $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ be translation invariant, additive and conditionally bounded. Then*

$$\lim_{r \to \infty} \frac{\varphi(rK)}{V_n(rK)} = \varphi(C^n) - \varphi(\partial^+ C^n)$$

for every $K \in \mathcal{K}^n$ with $V_n(K) > 0$.

Proof. Let $K \in \mathcal{K}^n$ and $0 \in \operatorname{int} K$, without loss of generality. For $z \in \mathbb{R}^n$ we put

$$\varphi(K, z) := \varphi(K \cap (C^n + z)) - \varphi(K \cap (\partial^+ C^n + z)). \tag{42}$$

Lemma 8.1 shows that

$$\varphi(rK) = \sum_{z \in \mathbb{Z}^n} \varphi(rK, z) \qquad \text{for } r > 0.$$

Define

$$Z_r^1 := \{z \in \mathbb{Z}^n : (C^n + z) \cap rK \neq \emptyset,\ C^n + z \not\subset rK\}$$

and

$$Z_r^2 := \{z \in \mathbb{Z}^n : C^n + z \subset rK\}.$$

Then

$$\lim_{r\to\infty} \frac{|Z_r^1|}{V_n(rK)} = 0, \qquad \lim_{r\to\infty} \frac{|Z_r^2|}{V_n(rK)} = 1, \tag{43}$$

where $|A|$ denotes the number of elements of a set A. The limit relations follow from the fact that one easily shows the existence of numbers $r_0 > s, t > 0$ such that

$$\begin{aligned} z \in Z_r^1 \quad \Rightarrow \quad & C^n + z \subset (r+s)K \setminus (r-s)K, \\ & (r-t)K \subset \bigcup_{z \in Z_r^2} (C^n + z) \end{aligned}$$

for $r \geq r_0$.

By assumption,

$$|\varphi(rK, z)| = |\varphi(rK - z, 0)| \leq b$$

with some constant b independent of z, K and r. This gives

$$\frac{1}{V_n(rK)} \left| \sum_{z \in Z_r^1} \varphi(rK, z) \right| \leq b \frac{|Z_r^1|}{V_n(rK)} \to 0 \qquad \text{for } r \to \infty.$$

From this we deduce

$$\begin{aligned} \lim_{r\to\infty} \frac{\varphi(rK)}{V_n(rK)} &= \lim_{r\to\infty} \frac{1}{V_n(rK)} \sum_{z \in \mathbb{Z}^n} \varphi(rK, z) \\ &= \lim_{r\to\infty} \frac{1}{V_n(rK)} \sum_{z \in Z_r^2} \varphi(rK, z) \\ &= [\varphi(C^n) - \varphi(\partial^+ C^n)] \lim_{r\to\infty} \frac{|Z_r^2|}{V_n(rK)} \\ &= \varphi(C^n) - \varphi(\partial^+ C^n). \end{aligned}$$

■

We are now in a position to prove the existence of the densities of intrinsic volumes for standard random sets.

8.3 Theorem. *For a standard random set Z and for $j \in \{0, \dots, n\}$, the limit*

$$\overline{V}_j(Z) := \lim_{r\to\infty} \frac{\mathbb{E}V_j(Z \cap rK)}{V_n(rK)}$$

exists, and it satisfies

$$\overline{V}_j(Z) = \mathbb{E}[V_j(Z \cap C^n) - V_j(Z \cap \partial^+ C^n)].$$

Hence, $\overline{V}_j(Z)$ is independent of K.

Proof. Let $K \in \mathcal{K}^n$ and $V_n(K) > 0$. Without loss of generality, we can assume that $K \subset C^n$. For given $\omega \in \Omega$, there is a representation

$$Z(\omega) \cap K = \bigcup_{i=1}^{N_K(\omega)} K_i(\omega) \qquad \text{with } K_i(\omega) \in \mathcal{K}^n,$$

where $N_K(\omega) := N(Z(\omega) \cap K)$. By the inclusion-exclusion principle,

$$\begin{aligned} &V_j(Z(\omega) \cap K) \\ &= \sum_{k=1}^{N_K(\omega)} (-1)^{k-1} \sum_{1\le i_1<\dots<i_k\le N_K(\omega)} V_j(K_{i_1}(\omega) \cap \dots \cap K_{i_k}(\omega)), \end{aligned}$$

hence, by the monotoneity of the intrinsic volumes,

$$\begin{aligned} \mathbb{E}|V_j(Z \cap K)| &\le V_j(C^n)\mathbb{E}\sum_{k=1}^{N_K} \binom{N_K}{k} \\ &\le V_j(C^n)\mathbb{E}2^{N(Z\cap K)} \\ &\le V_j(C^n)\mathbb{E}2^{N(Z\cap C^n)}, \end{aligned}$$

since $N(Z(\omega) \cap K) \le N(Z(\omega) \cap C^n)$. By assumption, the right-hand side is finite, hence $V_j(Z \cap K)$ is integrable. For a polyconvex set $M \in U(\mathcal{K}^n)$, the integrability of $V_j(Z \cap M)$ then follows from additivity, using the inclusion-exclusion principle again. This shows that all expectations appearing in the theorem exist and are finite. Therefore, we can define a functional $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ by

$$\varphi(M) := \mathbb{E}V_j(Z \cap M) \qquad \text{for } M \in U(\mathcal{K}^n).$$

Then φ is additive, translation invariant (as follows from the stationarity of Z) and conditionally bounded (as follows from the last estimate above). Now the assertion of the theorem follows from Lemma 8.2. ■

After these preliminaries, we are now able to answer questions of the following kind. Suppose that the realisations $Z(\omega)$ of a closed standard set Z can be observed in a 'window', that is, in a compact convex set K with $V_n(K) > 0$.

By 'observation' we mean that, in principle, the values $V_j(Z(\omega) \cap K)$ can be measured. We want to use the values $V_j(Z(\omega) \cap K)/V_n(K)$ to estimate the densities $\overline{V}_j(Z)$. But in general, $V_j(Z \cap K)/V_n(K)$ will depend on K and thus will not be an unbiased estimator for $\overline{V}_j(Z)$. To control the error, we would have to determine the expectation of $V_j(Z \cap K)$. If Z is an isotropic standard random set, this can be achieved by means of integral geometry. From the obtained set of expectations, one can then also derive unbiased estimators for the densities of the intrinsic volumes.

The next theorem extends the principal kinematic formula to isotropic standard random sets.

8.4 Theorem. *Let Z be an isotropic standard random set in $\mathbb{R}^n$, let $K \in \mathcal{K}^n$ and $j \in \{0, \ldots, n\}$. Then*

$$\mathbb{E} V_j(Z \cap K) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) \overline{V}_{n+j-k}(Z).$$

(*The coefficients are those of Theorem* 7.2.)

Proof. First we denote that the function

$$\begin{array}{ccc} \mathbb{R}^n \times SO_n \times \Omega & \to & \mathbb{R} \\ (x, \vartheta, \omega) & \mapsto & V_j(Z(\omega) \cap K \cap (\vartheta B^n + z)) \end{array}$$

is integrable with respect to the product measure $\lambda \otimes \nu \otimes \mathbb{P}$. Since $\mathbb{E} 2^{N(Z \cap C^n)} < \infty$, this follows as in the proof of Theorem 8.3, if we additionally assume that $K \subset C^n$. For general $K \in \mathcal{K}^n$ it then follows from

$$\begin{aligned} & \int \int \int |V_j(Z(\omega) \cap K \cap (C^n + z) \cap (\vartheta B^n + x))| \, d\lambda(x) d\nu(\vartheta) d\mathbb{P}(\omega) \\ = & \int \int \int |V_j(Z(\omega) \cap (K - z) \cap C^n \cap (\vartheta B^n + x - z))| \, d\lambda(x) d\nu(\vartheta) d\mathbb{P}(\omega) \\ = & \int \int \int |V_j(Z(\omega) \cap (K - z) \cap C^n \cap (\vartheta B^n + x))| \, d\lambda(x) d\nu(\vartheta) d\mathbb{P}(\omega) \\ < & \infty \end{aligned}$$

and the inclusion-exclusion formula.

For $\vartheta \in SO_n$, $x \in \mathbb{R}^n$ and $r > 0$ we deduce from the motion invariance of V_j and the stationarity and isotropy of Z that

$$\begin{aligned}
&\mathbb{E}V_j(Z \cap K \cap (\vartheta r B^n + x)) \\
&\quad = \mathbb{E}V_j(\vartheta^{-1}(Z - x) \cap \vartheta^{-1}(K - x) \cap rB^n) \\
&\quad = \mathbb{E}V_j(Z \cap \vartheta^{-1}(K - x) \cap rB^n).
\end{aligned}$$

From Fubini's theorem (and the invariance properties of λ and ν) we get

$$\begin{aligned}
&\mathbb{E} \int_{SO_n} \int_{\mathbb{R}^n} V_j(Z \cap K \cap (\vartheta r B^n + x))\, d\lambda(x)\, d\nu(\vartheta) \\
&\quad = \mathbb{E} \int_{SO_n} \int_{\mathbb{R}^n} V_j(Z \cap (\vartheta K + x) \cap rB^n)\, d\lambda(x)\, d\nu(\vartheta).
\end{aligned}$$

We apply the principal kinematic formula (Theorem 7.2) to both sides and obtain

$$\sum_{k=j}^{n} \alpha_{njk} \mathbb{E}V_k(Z \cap K) V_{n+j-k}(rB^n) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) \mathbb{E}V_{n+j-k}(Z \cap rB^n).$$

Now we divide both sides by $V_n(rB^n)$ and let r tend to infinity. Because of $V_m(rB^n) = r^m V_m(B^n)$ and $\alpha_{njj} = 1$, the left side tends to

$$\mathbb{E}V_j(Z \cap K)$$

and by Theorem 8.3, the right side tends to

$$\sum_{k=j}^{n} \alpha_{njk} V_k(K) \overline{V}_{n+j-k}(Z).$$

This completes the proof. ■

The special cases

$$\begin{aligned}
\mathbb{E}V_n(Z \cap K) &= V_n(K)\overline{V}_n(Z), \\
\mathbb{E}V_{n-1}(Z \cap K) &= V_{n-1}(K)\overline{V}_n(Z) + V_n(K)\overline{V}_{n-1}(Z)
\end{aligned}$$

of Theorem 8.4 can be obtained without the assumption of isotropy, since corresponding translative integral-geometric formulae are available.

Now we interpret Theorem 8.4. As one application, it describes the error which is made if the measured value $V_j(Z(\omega) \cap K)/V_n(K)$ is used as an estimator for the density $\overline{V}_j(Z)$. Writing the formula of Theorem 8.4 in the form

$$\frac{\mathbb{E}V_j(Z \cap K)}{V_n(K)} = \overline{V}_j(Z) + \frac{1}{V_n(K)} \sum_{k=j}^{n-1} \alpha_{njk} V_j(K) \overline{V}_{n+j-k}(Z),$$

we see that the mean error tends to 0 for increasing windows K, thus the estimator

$$V_j(Z(\omega) \cap K)/V_n(K)$$

is asymptotically unbiased. However, one can also obtain an unbiased estimator. The system of equations given by Theorem 8.4,

$$\mathbb{E}V_j(Z \cap K) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) \overline{V}_{n+j-k}(Z), \qquad j = 0, \dots, n,$$

can be solved for $\overline{V}_0(Z), \dots, \overline{V}_n(Z)$, since the coefficient matrix is triangular. This yields formulae of the form

$$\overline{V}_i(Z) = \mathbb{E}\left(\sum_{m=0}^{n} \beta_{nim}(K) V_m(Z \cap K) \right), \qquad i = 0, \dots, n,$$

hence

$$\sum_{m=0}^{n} \beta_{nim}(K) V_m(Z \cap K)$$

is an unbiased estimator for $\overline{V}_i(Z)$. As an example, we write down the two-dimensional case, using the notations A, L, χ for area, perimeter and Euler characteristic, respectively:

$$\begin{aligned}
\overline{A}(Z) &= \mathbb{E}\frac{A(Z \cap K)}{A(K)}, \\
\overline{L}(Z) &= \mathbb{E}\left[\frac{L(Z \cap K)}{A(K)} - \frac{L(K)A(Z \cap K)}{A(K)^2} \right], \\
\overline{\chi}(Z) &= \mathbb{E}\left[\frac{\chi(Z \cap K)}{A(K)} - \frac{1}{2\pi} \frac{L(K)L(Z \cap K)}{A(K)^2} \right. \\
&\quad \left. + \left(\frac{1}{2\pi} \frac{L(K)^2}{A(K)^3} - \frac{1}{A(K)^2} \right) A(Z \cap K) \right].
\end{aligned}$$

Theorem 8.4 also immediately yields a Crofton formula for random sets. If we talk of a standard random set Z in some affine subspace E, the stationarity and isotropy of Z refer to E, and densities of intrinsic volumes have to be computed in E.

8.5 Theorem. *Let Z be an isotropic standard random set in $\mathbb{R}^n$, let $E \in \mathcal{E}_k^n$ be a k-dimensional flat, where $k \in \{1, \dots, n-1\}$, and let $j \in \{0, \dots, k\}$. Then $Z \cap E$ is an isotropic standard random set in E, and*

$$\overline{V}_j(Z \cap E) = \alpha_{njk} \overline{V}_{n+j-k}(Z).$$

Proof. We omit the (not difficult) proof that $Z \cap E$ is, with respect to E, again an isotropic standard random set. For that reason, the density $\overline{V}_j(Z \cap E)$ exists. Now let $K \in \mathcal{K}^n$, $K \subset E$ and $V_k(K) > 0$. Theorem 8.4 yields

$$\mathbb{E} V_j(Z \cap K) = \sum_{m=j}^{k} \alpha_{njm} V_m(K) \overline{V}_{n+j-m}(Z), \tag{44}$$

where only terms with $m \leq k$ appear since $V_m(K) = 0$ for $m > k$. Since Z is stationary, we can assume that $0 \in E$ and hence $rK \subset E$ for $r > 0$. In (44), we replace K by rK and divide the equation by $V_k(rK)$. For $r \to \infty$, the left side tends to $\overline{V}_j(Z \cap E)$, since $V_j(Z \cap rK) = V_j(Z \cap E \cap rK)$ (and the intrinsic volumes do not depend on the dimension of the surrounding space). The right side tends to $\alpha_{njk} \overline{V}_{n+j-k}(Z)$. ■

The implications of this theorem are clear. After Theorem 8.4, we had seen how the densities $\overline{V}_j(Z)$ of an isotropic standard random set admit asymptotically unbiased or even unbiased estimators. If Z is observed in a k-dimensional section $Z \cap E$, then we can obtain estimators for $\overline{V}_j(Z \cap E)$. Theorem 8.5 tells us that these are at the same time (asymptotically) unbiased estimators for the densities $\alpha_{njk} \overline{V}_{n+j-k}(Z)$.

As an example, we consider the practically relevant case where $n = 3$ and $k = 2$. We deal with the three-dimensional densities $\overline{V}$ (volume), $\overline{S}$ (surface area), $\overline{M}$ (integral of mean curvature) and with the two-dimensional densities $\overline{A}$ (area), $\overline{L}$ (boundary length), $\overline{\chi}$ (Euler characteristic). The equations of Theorem 8.5 now read

$$\overline{V}(Z) = \overline{A}(Z \cap E), \tag{45}$$

$$\overline{S}(Z) = \frac{4}{\pi} \overline{L}(Z \cap E), \tag{46}$$

$$\overline{M}(Z) = 2\pi \overline{\chi}(Z \cap E). \tag{47}$$

These equations, finally, provide an exact theoretic foundation for the 'fundamental equations of stereology', which are traditionally written in the form

$$V_V = A_A,$$

$$\begin{aligned} S_V &= \frac{4}{\pi} L_A, \\ M_V &= 2\pi \chi_A. \end{aligned}$$

Concluding we can say that Theorems 8.4 and 8.5 provide theoretical justifications for some practical procedures of stereology, at least in those cases where it is reasonable to model probes of real materials by realisations of isotropic standard random sets. From the practical point of view, the consideration of only locally polyconvex sets does not seem very restrictive. Of the invariance properties, stationarity is always unrealistic, requiring unbounded sets, but it may well be satisfied approximately at close range. The most critical assumption is that of isotropy. For that reason, the applicability of motion invariant stereology is limited, and translative integral geometry is under investigation.

9 The kinematic formula for curvature measures

We shall now prove the local version of the principal kinematic formula, that is, the equation

$$\begin{aligned} &\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(g) \\ &\quad = \sum_{k=j}^{n} \alpha_{njk} \Phi_j(K, A) \Phi_{n+j-k}(M, B) \end{aligned} \tag{48}$$

for the curvature measures Φ_i. It holds for polyconvex sets $K, M \in U(\mathcal{K}^n)$ and Borel sets $A, B \in \mathcal{B}(\mathbb{R}^n)$. As for the global version, involving the intrinsic volumes V_i, it is sufficient to prove (48) for convex bodies $K, M \in \mathcal{K}^n$, since the general case of polyconvex sets is then easily obtained, using additivity and the inclusion-exclusion principle.

For the proof of (48), we first consider the case where K and M are n-dimensional convex polytopes. We also consider only translations instead of rigid motions, thus we have to investigate the integral

$$I := \int_{\mathbb{R}^n} \Phi_j(K \cap (M + x), A \cap (B + x))\, d\lambda(x).$$

By (37), the jth curvature measure of a polytope P is given by

$$\Phi_j(P, \cdot) = \sum_{F \in \mathcal{F}_j(P)} \gamma(F, P) \lambda_F.$$

It follows that

$$I = \int\limits_{\mathbb{R}^n} \sum_{F' \in \mathcal{F}_j(K \cap (M+x))} \gamma(F', K \cap (M+x)) \lambda_{F'}(A \cap (B+x))\, d\lambda(x). \quad (49)$$

The faces $F' \in \mathcal{F}_j(K \cap (M+x))$ are precisely the j-dimensional sets of the form $F' = F \cap (G+x)$ with a face $F \in \mathcal{F}_k(K)$ and a face $G \in \mathcal{F}_i(M)$, where $k, i \in \{j, \dots, n\}$. In computing the integral (49), only those translation vectors x need to be considered for which a pair F, G with $F \cap (G+x) \neq \emptyset$ also satisfies $\operatorname{relint} F \cap \operatorname{relint}(G+x) \neq \emptyset$, since the remaining vectors x make up a set of Lebesgue measure zero. Moreover, the pairs F, G for which $k + i < n$ or which are in special position, do not contribute to the integral, since for them we have

$$\lambda(\{x \in \mathbb{R}^n : F \cap (G+x) \neq \emptyset\}) = \lambda(F + G^*) = 0.$$

In the remaining cases, we have $\dim F' = \dim F + \dim G - n$ and hence $k + i = n + j$. Therefore, we obtain

$$\begin{aligned} I &= \sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{n+j-k}(M)} \\ &\quad \int\limits_{\mathbb{R}^n} \gamma(F \cap (G+x), K \cap (M+x)) \lambda_{F \cap (G+x)}(A \cap (B+x))\, d\lambda(x). \end{aligned}$$

In the integrand, we may assume that $\operatorname{relint} F \cap \operatorname{relint}(G+x) \neq \emptyset$, and in this case the external angle

$$\gamma(F \cap (G+x), K \cap (M+x)) =: \gamma(F, G, K, M)$$

does not depend on x. Putting

$$J(F,G) := \int\limits_{\mathbb{R}^n} \lambda_{F \cap (G+x)}(A \cap (B+x))\, d\lambda(x),$$

we thus have

$$I = \sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{n+j-k}(M)} \gamma(F, G, K, M) J(F, G).$$

To compute the integral $J(F,G)$ for given faces $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_{n+j-k}(M)$, we decompose the space $\mathbb{R}^n$ in a way adapted to these faces. We may assume that

$$0 \in L_1 := \operatorname{aff} F \cap \operatorname{aff} G,$$

where aff denotes the affine hull. Let

$$L_2 := L_1^\perp \cap \operatorname{aff} F, \qquad L_3 := L_1^\perp \cap \operatorname{aff} G,$$

and let $\lambda^{(j)}, \lambda^{(k-j)}, \lambda^{(n-k)}$ denote the Lebesgue measures on L_1, L_2, L_3, respectively. With respect to the direct sum decomposition $\mathbb{R}^n = L_1 \oplus L_2 \oplus L_3$, every $x \in \mathbb{R}^n$ has a unique decomposition $x = x_1 + x_2 + x_3$ with $x_i \in L_i$ for $i = 1, 2, 3$. Writing

$$A' := A \cap F, \qquad B' := B \cap G,$$

we get

$$\begin{aligned} J(F,G) &= [F,G] \int_{L_3} \int_{L_2} \int_{L_1} \lambda_{F \cap (G + x_1 + x_2 + x_3)}(A' \cap (B' + x_1 + x_2 + x_3) \\ &\quad d\lambda^{(j)}(x_1)\, d\lambda^{(k-j)}(x_2)\, d\lambda^{(n-k)}(x_3). \end{aligned}$$

Here the factor $[F,G]$ is an absolute determinant, defined by

$$d\lambda(x) = [F,G]\, d\lambda^{(j)}(x_1)\, d\lambda^{(k-j)}(x_2)\, d\lambda^{(n-k)}(x_3).$$

It can be described as follows, in a more general version. Let $L, L' \subset \mathbb{R}^n$ be two linear subspaces. We choose an orthonormal basis of $L \cap L'$ and extend it to an orthonormal basis of L and also to an orthonormal basis of L'. Let P denote the parallelepiped that is spanned by the vectors obtained in this way. We define $[L, L'] := V_n(P)$. Then $[L, L']$ depends only on the subspaces L and L'. If $L + L' \neq \mathbb{R}^n$, then $[L, L'] = 0$. We extend this definition to faces F, G of polytopes by putting $[F,G] := [L, L']$, where L and L' are the linear subspaces which are translates of the affine hulls of F and G, respectively.

To compute now the inner integral over L_1, we observe that

$$(A' \cap (B' + x_1 + x_2 + x_3)) - x_2 = (A' - x_2) \cap (B' + x_1 + x_3) \subset L_1$$

and hence

$$\begin{aligned} &\int_{L_1} \lambda_{F \cap (G + x_1 + x_2 + x_3)}(A' \cap (B' + x_1 + x_2 + x_3))\, d\lambda^{(j)}(x_1) \\ &\quad = \int_{L_1} \lambda^{(j)}((A' - x_2) \cap (B' + x_3 + x_1))\, d\lambda^{(j)}(x_1) \\ &\quad = \lambda^{(j)}((A' - x_2) \cap L_1)\lambda^{(j)}((B' + x_3) \cap L_1), \end{aligned}$$

where we have used Theorem 2.1. The integrations over L_2 and L_3 now require only Fubini's theorem, and we get

$$\begin{aligned}\int_{L_2} \lambda^{(j)}((A'-x_2)\cap L_1)\, d\lambda^{(k-j)}(x_2) &= \lambda^{(j)}\otimes\lambda^{(k-j)}(A') = \lambda_F(A),\\ \int_{L_3} \lambda^{(j)}((B'+x_3)\cap L_1)\, d\lambda^{(n-k)}(x_3) &= \lambda^{(j)}\otimes\lambda^{(n-k)}(B') = \lambda_G(B).\end{aligned}$$

Together this yields

$$J(F,G) = [F,G]\lambda_F(A)\lambda_G(B).$$

Inserting this in the integral I, we end up with the following translative integral-geometric formula for polytopes.

9.1 Theorem. *If $K, M \in \mathcal{K}^n$ are polytopes and $A, B \in \mathcal{B}(\mathbb{R}^n)$ are Borel sets, then for $j \in \{0,\dots,n\}$,*

$$\begin{aligned}&\int_{G_n} \Phi_j(K\cap(M+x), A\cap(B+x))\, d\lambda(x)\\ &\quad= \sum_{k=j}^{n}\sum_{F\in\mathcal{F}_k(K)}\sum_{G\in\mathcal{F}_{n+j-k}(M)} \gamma(F,G,K,M)[F,G]\lambda_F(A)\lambda_G(B).\end{aligned}$$

The kinematic formula at which we are aiming requires, for polytopes, the computation of

$$\begin{aligned}&\int_{G_n} \Phi_j(K\cap gM, A\cap gB)\, d\mu(g)\\ &\quad= \int_{SO_n}\int_{G_n} \Phi_j(K\cap(\vartheta M+x), A\cap(\vartheta B+x))\, d\lambda(x)\, d\nu(\vartheta)\\ &\quad= \sum_{k=j}^{n}\sum_{F\in\mathcal{F}_k(K)}\sum_{G\in\mathcal{F}_{n+j-k}(M)} \lambda_F(A)\lambda_G(B)\\ &\qquad\quad \int_{SO_n} \gamma(F,\vartheta G,K,\vartheta M)[F,\vartheta G]\, d\nu(\vartheta).\end{aligned}$$

Here we have used the fact that $\lambda_{\vartheta G}(\vartheta B) = \lambda_G(B)$. The summands with

$k = j$ or $k = n$ are easily determined, since for $k = j$ we get

$$\begin{aligned}&\sum_{F\in\mathcal{F}_k(K)}\sum_{G\in\mathcal{F}_{n+j-k}(M)}\gamma(F,G,K,M)[F,G]\lambda_F\otimes\lambda_G\\&=\sum_{F\in\mathcal{F}_j(K)}\gamma(F,M,K,M)[F,M]\lambda_F\otimes\lambda_M\\&=\sum_{F\in\mathcal{F}_j(K)}\gamma(F,K)\lambda_F\otimes\lambda_M\\&=\Phi_j(K,\cdot)\otimes\Phi_n(M,\cdot),\end{aligned}$$

and similarly for $k = n$,

$$\begin{aligned}&\sum_{F\in\mathcal{F}_k(K)}\sum_{G\in\mathcal{F}_{n+j-k}(M)}\gamma(F,G,K,M)[F,G]\lambda_F\otimes\lambda_G\\&=\Phi_n(K,\cdot)\otimes\Phi_j(M,\cdot).\end{aligned}$$

The remaining integrals over the rotation group are determined in the following theorem.

9.2 Theorem. *Let $K, M \in \mathcal{K}^n$ be polytopes, let $j \in \{0,\dots,n-2\}$, $k \in \{j+1,\dots,n-1\}$, $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_{n+j-k}(M)$. Then*

$$\int_{SO_n}\gamma(F,\vartheta G,K,\vartheta M)[F,\vartheta G]\,d\nu(\vartheta)=\alpha_{njk}\gamma(F,K)\gamma(G,M),$$

where α_{njk} is as in Theorem 7.2.

Proof. In order to avoid difficult direct computations, we will give a proof based on the uniqueness of spherical Lebesgue measures. This is possible since external angles are defined in terms of such measures.

By definition,

$$\gamma(F,\vartheta G,K,\vartheta M)=\gamma(F\cap(\vartheta G+x),K\cap(\vartheta M+x))$$

with suitable $x \in \mathbb{R}^n$. As before, let $N(P,F)$ denote the normal cone of a polytope P in a relatively interior point of its face F. From the definition of the external angle we get

$$\gamma(F,\vartheta G,K,\vartheta M)=\frac{\sigma^{(L)}(N(K\cap(\vartheta M+x),F\cap(\vartheta G+x))\cap S^{n-1})}{\sigma^{(L)}(L\cap S^{n-1})},$$

where $L \in \mathcal{L}^n_{n-j}$ is the orthogonal space of $F\cap(\vartheta G+x)$ (i.e., the orthogonal complement of the linear subspace parallel to the affine hull of $F \cap (\vartheta G +$

$x)$). For a linear subspace $L \subset \mathbb{R}^n$, we have denoted by $\sigma^{(L)}$ the spherical Lebesgue measure on $L \cap S^{n-1}$.

A general property of normal cones of convex bodies gives

$$N(K \cap (\vartheta M + x), F \cap (\vartheta G + x)) = N(K, F) + \vartheta N(M, G).$$

Therefore, we have to evaluate the integral

$$\int_{SO_n} \sigma^{(L_1+\vartheta L_2)}((N(K,F) + \vartheta N(M,G)) \cap S^{n-1})[F, \vartheta G]\, d\nu(\vartheta),$$

where L_1 is the orthogonal space of F and L_2 is the orthogonal space of G.

More generally, we define the integral

$$I(A,B) := \int_{SO_n} \sigma^{(L_1+\vartheta L_2)}(C(A) + \vartheta C(B)) \cap S^{n-1})[F, \vartheta G]\, d\nu(\vartheta)$$

for arbitrary Borel sets $A \subset L_1 \subset S^{n-1}$ and $B \subset L_2 \subset S^{n-1}$, where

$$C(A) := \{\alpha x : x \in A,\ \alpha \geq 0\}$$

denotes the cone spanned by A. Concerning the measurability of the integrand, we give the following hints for a proof. The function $\vartheta \mapsto [F, \vartheta G]$ is continuous, hence measurable. Let U denote the set of all rotations $\vartheta \in SO_n$ for which L_1 and ϑL_2 are not in special position. Then it can be shown that $\nu(SO_n \setminus U) = 0$. For $\vartheta \in U$ we have

$$\dim L_1 + \dim L_2 = (n-k) + (k-j) = n - j \leq n,$$

hence the sum $L_1 + \vartheta L_2$ is direct. From this one can deduce that $C(A) + \vartheta C(B)$ is a Borel set (in general, the sum of two Borel sets need not be a Borel set). For different $\vartheta \in U$, the sets $C(A) + \vartheta C(B)$ are connected by linear transformations. All this together is sufficient to show that the mapping

$$\vartheta \mapsto \sigma^{(L_1+\vartheta L_2)}((C(A) + \vartheta C(B)) \cap S^{n-1})$$

is measurable on U.

For fixed $B \in \mathcal{B}(L_2 \cap S^{n-1})$ we now define

$$\varphi(A) := I(A,B) \qquad \text{for } A \in \mathcal{B}(L_1 \cap S^{n-1}).$$

If $\bigcup_{i=1}^{\infty} A_i$ is a disjoint union of sets $A_i \in \mathcal{B}(L_1 \cap S^{n-1})$, then

$$\left(C\left(\bigcup_{i=1}^{\infty} A_i\right) + \vartheta C(B)\right) \cap S^{n-1} = \bigcup_{i=1}^{\infty}((C(A_i) + \vartheta C(B)) \cap S^{n-1})$$

for $\vartheta \in U$, and this union is disjoint up to a set of $\sigma^{(L_1+\vartheta L_2)}$-measure zero. We deduce that

$$\sigma^{(L_1+\vartheta L_2)}\left(\left(C\left(\bigcup_{i=1}^{\infty} A_i\right)+\vartheta C(B)\right)\cap S^{n-1}\right)$$
$$=\sum_{i=1}^{\infty}\sigma^{(L_1+\vartheta L_2)}((C(A_i)+\vartheta C(B))\cap S^{n-1})$$

for $\vartheta \in U$ and thus

$$\varphi\left(\bigcup_{i=1}^{\infty} A_i\right)=\sum_{i=1}^{\infty}\varphi(A_i),$$

by the theorem of monotone convergence. It follows that φ is a finite measure on $L_1 \cap S^{n-1}$. Let $\rho \in SO_n(L_1)$ be a rotation mapping the subspace L_1 into itself. Then

$$C(\rho A)+\vartheta C(B)=\rho(C(A)+\rho^{-1}\vartheta C(B))$$

and

$$[F,\vartheta G]=[\rho F,\vartheta G]=[F,\rho^{-1}\vartheta G],$$

hence

$$\begin{aligned}
&\varphi(\rho A)\\
&=\int_{SO_n}\sigma^{(L_1+\vartheta L_2)}((C(\rho A)+\vartheta C(B))\cap S^{n-1})[F,\vartheta G]\,d\nu(\vartheta)\\
&=\int_{SO_n}\sigma^{(L_1+\rho^{-1}\vartheta L_2)}((C(A)+\rho^{-1}\vartheta C(B))\cap S^{n-1})[F,\rho^{-1}\vartheta G]\,d\nu(\vartheta)\\
&=\varphi(A).
\end{aligned}$$

Since spherical Lebesgue measure is uniquely determined, up to a factor, by its rotation invariance (and finiteness), the measure φ must be a constant multiple of $\sigma^{(L_1)}$. Analogously we deduce that for fixed $A \in \mathcal{B}(L_1 \cap S^{n-1})$ the measure $I(A,\cdot)$ must be a constant multiple of $\sigma^{(L_2)}$. Both results together yield that

$$I(A,B)=\alpha(L_1,L_2)\sigma^{(L_1)}(A)\sigma^{(L_2)}(B)$$

for all $A \in \mathcal{B}(L_1 \cap S^{n-1})$, $B \in \mathcal{B}(L_2 \cap S^{n-1})$; here $\alpha(L_1,L_2)$ is a constant depending only on L_1 and L_2. If we choose $A = L_1 \cap S^{n-1}$, $B = L_2 \cap S^{n-1}$ and observe the invariance properties of the functional I following from its definition, we see that $\alpha(L_1, L_2)$ depends only on the dimensions n, j, k. Therefore, there is a constant β_{njk} so that

$$I(A, B) = \beta_{njk}\sigma^{(L_1)}(A)\sigma^{(L_2)}(B).$$

In particular, this shows that

$$I(N(K, F) \cap S^{n-1}, N(M, G) \cap S^{n-1})) = \beta_{njk}\gamma(F, K)\gamma(G, M).$$

This is the assertion of Theorem 9.2, except that it remains to show that $\beta_{njk} = \alpha_{njk}$.

Collecting the results obtained so far, we have proved the following kinematic formula for polytopes $K, M \in \mathcal{K}^n$:

$$\begin{aligned}
&\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(g) \\
&\quad = \sum_{k=j}^{n} \beta_{njk} \sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{n+j-k}(M)} \gamma(F, K)\gamma(G, M)\lambda_F(A)\lambda_G(B) \\
&\quad = \sum_{k=j}^{n} \beta_{njk}\Phi_k(K, A)\Phi_{n+j-k}(M, B).
\end{aligned}$$

If we choose $A = B = \mathbb{R}^n$, the obtained formula must coincide with that of Theorem 7.2, for all polytopes K, M. This shows that $\beta_{njk} = \alpha_{njk}$ and thus completes the proof of Theorem 9.2. ■

For arbitrary convex bodies K, M, the general kinematic formula (48) is now obtained by approximation, using the weak continuity of the curvature measures. An extension to polyconvex sets K, M is easily achieved by additivity, as in the case of Theorem 7.2.

Also the Crofton formula of Theorem 7.2 has a local counterpart. We collect both results in the following theorem.

9.3 Theorem. *Let $K, M \subset U\mathcal{K}^n$ be polyconvex sets, let $j \in \{0, \dots, n\}$, and let $A, B \in \mathcal{B}(\mathbb{R}^n)$ be Borel sets. Then the principal kinematic formula*

$$\begin{aligned}
&\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(G) \qquad (50) \\
&\quad = \sum_{k=j}^{n} \alpha_{njk}\Phi_j(K, A)\Phi_{n+j-k}(M, B)
\end{aligned}$$

holds. For $k \in \{1,\dots,n-1\}$ and $j \le k$ the Crofton formula

$$\int_{\mathcal{E}_k^n} \Phi_j(K \cap E, A \cap E)\, d\mu_k(E) = \alpha_{njk}\Phi_{n+j-k}(K,A) \tag{51}$$

holds. In both cases, the coefficients α_{njk} are those given in Theorem 7.2.

Proof. It remains to prove formula (51). Here we can assume that K is a convex body, since the general case is then obtained by additivity. We deduce (51) from (50), by a similar but simpler argument as used in the proof of Theorem 7.1.

Let $L_k \in \mathcal{L}_k^n$ be a fixed subspace; then $\mu_k = \gamma_k(\lambda^{(n-k)} \otimes \nu)$, as in Section 3. Let W be a unit cube in L_k. Let $A \in \mathcal{B}(\mathbb{R}^n)$. By (50) we have

$$\begin{aligned} J &:= \int_{G_n} \Phi_j(L_k \cap gK, W \cap gA)\, d\mu(g) \\ &= \sum_{m=j}^{n} \alpha_{njm}\Phi_m(L_k, W)\Phi_{n+j-m}(K,A) \end{aligned}$$

with

$$\Phi_m(L_k, W) = \begin{cases} \lambda_{L_k}(W) = 1 & \text{for } m = k, \\ 0 & \text{for } m \neq k, \end{cases}$$

hence

$$J = \alpha_{njk}\Phi_{n+j-k}(K,A).$$

On the other hand,

$$\begin{aligned} J &= \int_{SO_n}\int_{\mathbb{R}^n} \Phi_j(L_k \cap (\vartheta K + x), W \cap (\vartheta A + x))\, d\lambda(x)\, d\nu(\vartheta) \\ &= \int_{SO_n}\int_{L_k^\perp}\int_{L_k} \Phi_j(L_k \cap (\vartheta K + x_1 + x_2), W \cap (\vartheta A + x_1 + x_2)) \\ &\quad d\lambda^{(k)}(x_2)\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta). \end{aligned}$$

For the computation of the inner integral, we put

$$\Phi_j(L_k \cap (\vartheta K + x_1), \cdot) =: \varphi, \qquad \vartheta A + x_1 =: A'.$$

Then

$$\int_{L_k} \Phi_j(L_k \cap (\vartheta K + x_1 + x_2), W \cap (\vartheta A + x_1 + x_2))\, d\lambda^{(k)}(x_2)$$

$$
\begin{aligned}
&= \int_{L_k} \varphi((W - x_2) \cap A')\, d\lambda^{(k)}(x_2) \\
&= \varphi(A')\lambda^{(k)}(W) \\
&= \Phi_j(L_k \cap (\vartheta K + x_1), L_k \cap (\vartheta A + x_1)),
\end{aligned}
$$

where Theorem 2.1 was used. This yields

$$
\begin{aligned}
J &= \int_{SO_n} \int_{L_k^\perp} \Phi_j(L_k \cap (\vartheta K + x_1), L_k \cap (\vartheta A + x_1))\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta) \\
&= \int_{SO_n} \int_{L_k^\perp} \Phi_j(K \cap \vartheta(L_k + x), A \cap \vartheta(L_k + x))\, d\lambda^{(n-k)}(x)\, d\nu(\vartheta) \\
&= \int_{\mathcal{E}_k^n} \Phi_j(K \cap E, A \cap E)\, d\mu_k(E),
\end{aligned}
$$

where we have used the rigid motion covariance of the curvature measures as well as the inversion invariance of the measures $\lambda^{(n-k)}$ and ν. The two representations obtained for J together prove the assertion. ■

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