Lipschitz selections of the diametric completion mapping in Minkowski spaces

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Abstract

We develop a constructive completion method in general Minkowski spaces, which successfully extends a completion procedure due to Bückner in two- and three-dimensional Euclidean spaces. We prove that this *generalized Bückner completion* is locally Lipschitz continuous, thus solving the problem of finding a continuous selection of the diametric completion mapping in finite dimensional normed spaces. The paper also addresses the study of an elegant completion procedure due to Maehara in Euclidean spaces, the natural setting of which are the spaces with a generating unit ball. We prove that, in these spaces, the *Maehara completion* is also locally Lipschitz continuous, besides establishing other geometric properties of this completion. The paper contains also new estimates of the (local) Lipschitz constants for the wide spherical hull.

1 Introduction

A bounded set M in a metric space X is called *diametrically complete* (or, briefly, *complete*) if it cannot be enlarged without increasing its diameter, thus, if $x \in X \setminus M$ implies

$$\operatorname{diam}(M \cup \{x\}) > \operatorname{diam} M.$$

This paper is concerned with diametrically complete sets in Minkowski spaces. Here, a Minkowski space is a finite-dimensional real normed vector space and can thus be represented as $(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. Occasionally, some results can be extended to real Banach spaces (of any dimension). Though not in a systematic way and only when the proof does not require much additional effort, we do so.

In a Euclidean space, the complete sets are precisely the well-known and widely studied convex bodies of constant width (surveys are given by Chakerian and Groemer [8] and by Heil and Martini [17]; a nice educational article on special aspects was written by Kawohl and Weber [21]). In a Banach space, every convex body (nonempty closed bounded convex set) of constant width is complete, and every complete set is a convex body, but generally not of constant width (see [30], for example, for more information; for earlier results, see the survey by Martini and Swanepoel [27]). Every bounded set of diameter d is contained in a complete set of diameter d, called a *completion* of the set. In general, a set has many different completions. For example, for a line segment of length d in Euclidean space, every suitably translated body of constant width d is a completion. As another example, Figure 4 in [31] refers to a certain three-dimensional Minkowski space with polyhedral norm and shows several completions of one and the same facet of the unit ball.

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The nonuniqueness of completions raises the question whether one can associate with every convex body one of its completions in a continuous way. In other words, does the set-valued mapping that associates with a convex body K the set $\gamma(K)$ of its completions, admit a continuous selection (with respect to the Hausdorff metric)? Denoting the set of convex bodies in $\mathbb{R}^n$ by $\mathcal{K}^n$, and the set of nonempty compact subsets of $\mathcal{K}^n$ by $\mathcal{C}(\mathcal{K}^n)$, the completion mapping γ maps $\mathcal{K}^n$ into $\mathcal{C}(\mathcal{K}^n)$. The Hausdorff metric δ on $\mathcal{K}^n$ (induced by the given norm) induces a Hausdorff metric Δ on $\mathcal{C}(\mathcal{K}^n)$, and it was shown in [29] that $\gamma : \mathcal{K}^n \to \mathcal{C}(\mathcal{K}^n)$ is locally Lipschitz continuous with respect to these metrics. From this it was deduced, for strictly convex norms, that the completion mapping admits a continuous selection. (Note that the set $\gamma(K)$ is generally not convex in $\mathcal{K}^n$, which prevents the application of standard results on the existence of continuous selections.) A more direct approach, valid for arbitrary norms, is desirable. The proofs for the existence of completions in Euclidean or Minkowski spaces, which are found in the literature (see, for example, Pál [32], Lebesgue [25], Bonnesen–Fenchel [5, p. 130], Eggleston [10, p. 126], Scott [42], Vrećica [43], Groemer [15], Sallee [40]), make use of infinite iteration procedures, with many free choices, or even of Zorn's lemma. Therefore, they are not suitable for providing continuous selections. What is rather needed, is a construction that is free of arbitrary choices, and therefore called canonical, which associates with every convex body a definite completion. It is the purpose of this paper to describe such canonical completions and to prove their continuity.

In Euclidean space, there is an elegant completion procedure, due to Maehara [26], and rediscovered by Polovinkin [34]. According to Sallee [39] and Polovinkin, it works in spaces whose unit ball is a generating set. In the third section, we prove the local Lipschitz continuity of the Maehara completion in Minkowski spaces with this property. The Lipschitz constant depends on the self-Jung constant and can be estimated by the dimension of the space. We also prove that the Maehara completion $\mu(K)$ plays a special role as a metric center within the family of all completions of a given convex body K. Finally, we discuss the smoothness properties of the Maehara completion in Euclidean spaces, showing that $\mu(K)$ is at least as smooth as K itself.

This completion method, however, does not extend to general Minkowski spaces. We shall show that another classical Euclidean construction of completions in finitely many steps does. This procedure is due to Bückner [7], for two- and three-dimensional Euclidean spaces, and in a special planar case earlier to Reinhardt [37]. (The essential step of this approach is also used by Lachand–Robert and Oudet [23], who apparently were not aware of Bückner's work.) We shall show in Section 4 that, with a suitable replacement of some of the arguments, this procedure can be extended to general Minkowski spaces, and that it yields a locally Lipschitz continuous selection of the completion mapping. As in the case of the Maehara completion, the Lipschitz constant depends only on the dimension of the space.

The paper contains also new estimates of the (local) Lipschitz constants for the wide spherical hull, following two different approaches involving the Jung and self-Jung constants of the space, respectively.

2 The wide spherical hull

This section begins with some preliminaries and then provides continuity estimates for the so-called wide spherical hull. These estimates will be used in later sections, but are also of independent interest and are, therefore, treated in slightly greater generality. We fix some notation and collect a few preparations. Then we have a closer look at spherical hulls, which are basic in the study of completions.

In the following, unless stated otherwise, $X = (\mathbb{R}^n, \|\cdot\|)$ is a Minkowski space of dimension $n \geq 2$. Its unit ball, $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, is a convex body with the origin as center of symmetry and interior point. We write $B(x,r) = rB + x$ for $x \in \mathbb{R}^n$ and $r > 0$ and call any such set a *ball* of radius r. We occasionally will consider also real Banach spaces (complete normed spaces whose dimension is not necessarily finite); to these, the introduced notions extend in a natural way.

The set $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ is equipped with the Hausdorff metric δ induced by the norm, that is,

$$\begin{aligned} \delta(K,L) &= \max\left\{\max_{x\in K}\min_{y\in L}\|x-y\|, \max_{x\in L}\min_{y\in K}\|x-y\|\right\} \\ &= \min\left\{r \geq 0 : K \subset L + rB,\ L \subset K + rB\right\} \end{aligned}$$

for $K, L \in \mathcal{K}^n$. In the case of general Banach spaces, the previous max and min must be replaced by sup and inf (except for the first max).

For a bounded subset M of X, the number $\operatorname{diam} M := \sup\{\|x-y\| : x, y \in M\}$ is the *diameter* of M. In the following, we often write $\operatorname{diam} M =: d_M$, in brief. The *Jung constant* $J(X)$ of the Minkowski space X is the smallest number c such that each convex body K of diameter d is contained in a ball of diameter cd. Every n-dimensional Minkowski space X satisfies $J(X) \leq 2n/(n+1)$, as was first proved by Bohnenblust [4]. The Jung constant of n-dimensional Euclidean space is $\sqrt{2n/(n+1)}$, which is due to Jung [19]; a proof is also found in Webster [44], Theorem 7.1.6. From the Jung constant, we have to distinguish the *self-Jung constant* $J_s(X)$, which is the smallest number c such that each convex body K of diameter d is contained in a ball of diameter cd which has its center in K. For Euclidean spaces and for two-dimensional Minkowski spaces, both constants coincide (cf. Klee [22]). For an n-dimensional Minkowski space X, also $J_s(X) \leq 2n/(n+1)$; see Amir [1], Proposition 2.12. Both notions, Jung and self-Jung constants, can be defined in a Banach space, replacing "the smallest number c" by "the infimum of all c" [1]. In the following, we denote the self-Jung constant of the space X under consideration by $2C_X^s$. It will be important below that $C_X^s < 1$ for the spaces X under consideration.

For sets $A, C \subset \mathbb{R}^n$, their Minkowski difference is the set

$$A \sim C := \bigcap_{c\in C}(A - c) = \{x \in \mathbb{R}^n : C + x \subset A\},$$

see, e.g., [41, p. 133]. For a set $M \subset \mathbb{R}^n$ of diameter d, let

$$\eta(M) := \bigcap_{x\in M} B(x,d) = dB \sim (-M).$$

The convex body $\eta(M)$ has been called the *wide spherical hull* of M, and

$$\theta(M) := \bigcap_{x\in\eta(M)} B(x,d) = dB \sim (-\eta(M))$$

is the *tight spherical hull* of M. The relevance of these operators for diametric completions may be seen from the fact that $\eta(M)$ is the union and $\theta(M)$ is the intersection of all completions of M (see [28, Prop. 2]).

We shall first obtain an estimate which implies, in particular, the local Lipschitz continuity of the wide spherical hull operator. We prove this estimate in a more general form than needed later, namely for the operator η_s defined by

$$\eta_s(K) := \bigcap_{x\in K} B(x,s)$$

for $K \in \mathcal{K}^n$ and fixed $s > 0$, since this operator has been considered before. Bavaud [3] called the set $\eta_s(K)$ the *s-adjoint transform* of K and studied η_s systematically for the Euclidean plane. The paper [3] contains further references.

Theorem 1 *Let $K, L \in \mathcal{K}^n$, and let s, t be positive numbers with*

$$\epsilon_0 := \min\{s,t\} - |s-t| - C_X^s \max\{d_K, d_L\} > 0. \tag{1}$$

If $\delta(K,L) \le \epsilon$ for some ϵ with $0 \le \epsilon \le \epsilon_0$, then

$$\delta(\eta_s(K), \eta_t(L)) \le \frac{\max\{s,t\}}{\epsilon_0} (\epsilon + |s-t|).$$

Proof. Since C_X^s is half the self-Jung constant of X, there exists a point $z \in K$ with

$$K \subset B(z, C_X^s d_K).$$

Let

$$\lambda = \frac{\epsilon + |s-t|}{s - C_X^s d_K}.$$

By assumption, $0 < \epsilon_0 \le s - C_X^s d_K$, hence $\lambda > 0$. Also by assumption, $\epsilon \le \epsilon_0 \le s - |s-t| - C_X^s d_K$, hence $\lambda \le 1$. Let $M := (1-\lambda)\eta_s(K) + \lambda z$. Let $x \in M$, $y \in K$. Then $x = (1-\lambda)a + \lambda z$ with $a \in \eta_s(K)$. This gives

$$\begin{aligned} \|x-y\| = \|(1-\lambda)a + \lambda z - y\| &\le (1-\lambda)\|a-y\| + \lambda\|z-y\| \\ &\le (1-\lambda)s + \lambda C_X^s d_K = s - \epsilon - |s-t| \\ &\le s - \epsilon - (s-t) = t - \epsilon. \end{aligned}$$

The inequality $\|x-y\| \le t-\epsilon$ holds for all $x \in M$ and all $y \in K$. Let $x \in M$ and $w \in K + \epsilon B$, thus $w = y + \epsilon b$ with $y \in K$ and $b \in B$. Therefore, $\|x-w\| \le \|x-y\| + \epsilon \le t$. Since $L \subset K + \epsilon B$, it follows that $\|x-w\| \le t$ for all $w \in L$. Thus, we have $M \subset \eta_t(L)$. Now let $u \in \eta_s(K)$. The point $v := (1-\lambda)u + \lambda z$ belongs to M, hence to $\eta_t(L)$, Thus, to each $u \in \eta_s(K)$ we have found a point $v \in \eta_t(L)$ with

$$\begin{aligned} \|u-v\| &= \|u - (1-\lambda)u - \lambda z\| = \lambda\|u-z\| \le \lambda s \\ &= \frac{s(\epsilon + |s-t|)}{s - C_X^s d_K} \le \frac{\max\{s,t\}}{\epsilon_0} (\epsilon + |s-t|). \end{aligned}$$

Since the assumptions of Theorem 1 and the last estimate are symmetric in the pairs (K, s) and (L, t), we conclude that

$$\delta(\eta_s(K), \eta_t(L)) \le \frac{\max\{s,t\}}{\epsilon_0} (\epsilon + |s-t|)$$

as stated. □

The estimate in the previous theorem becomes clearer in the special case where $s = d_K$, $t = d_L$, since then the number ϵ_0 given by (1) is equal to

$$E(K,L) := 2\min\{d_K, d_L\} - (1 + C_X^s)\max\{d_K, d_L\}.$$

Since $\delta(K,L) \le \epsilon$ implies $|d_K - d_L| \le 2\epsilon$, we obtain the following corollary.

Corollary 1 *Let $K, L \in \mathcal{K}^n$ be convex bodies satisfying $E(K,L) > 0$. If $\delta(K,L) \le \epsilon$ for some ϵ with $0 \le \epsilon \le E(K,L)$, then*

$$\delta(\eta(K), \eta(L)) \le \frac{3\max\{d_K, d_L\}}{E(K,L)}\,\epsilon.$$

Remark 1 If K and L have the same diameter d, then $E(K,L) = (1 - C_X^s)d > 0$, since $2C_X^s = J_s(X) < 2$. As a consequence, if

$$\delta(K,L) \le \epsilon \le (1 - C_X^s)d,$$

then Theorem 1 gives

$$\delta(\eta(K), \eta(L)) \le \frac{1}{1 - C_X^s}\,\epsilon.$$

We note that the condition $E(K,L) > 0$ of Corollary 1 is satisfied if the diameters of K and L are sufficiently close together. We can use this to eliminate $E(K,L)$ and thus give Corollary 1 a more convenient form.

Corollary 2 *Let $K, L \in \mathcal{K}^n$, let $0 < \alpha \le \beta \le 1 - 4\alpha$. If*

$$0 \le \epsilon \le \alpha(1 - C_X^s)\max\{d_K, d_L\}$$

and $\delta(K,L) \le \epsilon$, then

$$\delta(\eta(K), \eta(L)) \le \frac{3}{\beta(1 - C_X^s)}\,\epsilon.$$

For the proof, we may assume, without loss of generality, that $d_L \le d_K$. From $\delta(K,L) \le \epsilon$ we get $d_L \ge d_K - 2\epsilon \ge [1 - 2\alpha(1 - C_X^s)]d_K$. This gives

$$\begin{aligned}
E(K,L) &= 2d_L - (1 + C_X^s)d_K \\
&\ge 2[1 - 2\alpha(1 - C_X^s)]d_K - (1 + C_X^s)d_K \\
&= (1 - 4\alpha)(1 - C_X^s)d_K \ge \beta(1 - C_X^s)d_K \ge \alpha(1 - C_X^s)d_K \ge \epsilon
\end{aligned}$$

(and $E(K,L) > 0$ if $\epsilon = 0$). Hence, the assumptions of Corollary 1 are satisfied, and we obtain

$$\delta(\eta(K), \eta(L)) \le \frac{3d_K}{E(K,L)}\,\epsilon \le \frac{3}{\beta(1 - C_X^s)}\,\epsilon,$$

as asserted. □

By a different approach, we can prove an alternative version of Corollary 2, which is sometimes stronger. For this, we need some information on inradius and diameter of the wide spherical hull. The inradius r_K of a convex body $K \subset \mathbb{R}^n$ is the largest radius of a ball contained in K. In spaces of infinite dimension, r_K is defined as the supremum of the radii of all balls contained in K. We state the results for general Banach spaces with Jung constant less than 2, for later use in Hilbert spaces. In the following, we write $C_X = J(X)/2$.

Lemma 1 *Let X be a Banach space with $C_X < 1$. If $K \subset X$ is a convex body and $s > C_X d_K$, then $r_{\eta_s(K)} \ge s - C_X d_K$. In particular,*

$$r_{\eta(K)} \ge (1 - C_X)d_K. \tag{2}$$

Proof. If we fix $0 < \xi < s - C_X d_K$, we can assume that $K \subset (C_X + \xi)d_K B$. Any point $x \in K$ satisfies $\|x\| \le (C_X + \xi)d_K$, hence any point $y \in (s - (C_X + \xi)d_K)B$ satisfies $\|x - y\| \le s$.

Therefore, $(s-(C_X+\xi)d_K)B \subset B(x,s)$. Since $x \in K$ was arbitrary, this shows that $(s-(C_X+\xi)d_K)B \subset \eta_s(K)$. Since ξ was arbitrary, this yields the estimate for the inradius of $\eta_s(K)$. In the case of the wide spherical hull we obtain (2) by choosing $s = d_K$ in the previous estimate. □

Lemma 2 *If M is a convex body in a Banach space, $r_M > 0$ and $0 \le \epsilon < r_M$, then*

$$\delta(M, M \sim \epsilon B) \le \frac{d_M}{r_M}\,\epsilon. \tag{3}$$

If $M \in \mathcal{K}^n$, then the previous estimate remains true also for $\epsilon = r_M$.

Proof. Let $0 \le \epsilon < r_M$ and fix $0 < \xi < r_M - \epsilon$. By the definition of r_M there is a point $z \in M$ with $B(z, r_M - \xi) \subset M$. Let $x \in M$. Then, for $0 \le \lambda \le 1$,

$$B((1-\lambda)x + \lambda z, \lambda(r_M - \xi)) = (1-\lambda)x + \lambda B(z, r_M - \xi) \subset M.$$

With $\lambda = \epsilon/(r_M - \xi)$ this gives $B((1-\lambda)x + \lambda z, \epsilon) \subset M$, hence $(1-\lambda)x + \lambda z \in M \sim \epsilon B$. Since

$$\|x - (1-\lambda)x - \lambda z\| = \lambda\|x - z\| \le \frac{d_M}{r_M - \xi}\,\epsilon,$$

it follows that $x \in (M \sim \epsilon B) + (d_M/(r_M - \xi))\epsilon B$. Since $x \in M$ was arbitrary, this means that $\delta(M, M \sim \epsilon B) \le (d_M/(r_M - \xi))\,\epsilon$. Similarly, since $0 < \xi < r_M - \epsilon$ was also arbitrary, the assertion (3) follows. In the case of finite dimensional spaces, we can take $\xi = 0$, which also allows us to take $\epsilon = r_M$. □

Concerning the diameter of the wide spherical hull, we observe that

$$d_{\eta(K)} \le \tau_X d_K \tag{4}$$

with

$$\tau_X := \sup\{\operatorname{diam}(B \cap (B + x)) : \|x\| = 1\}.$$

In fact, since both sides of (4) are homogeneous of degree one, we may assume that $d_K = 1$. Then, if we fix $0 < \epsilon < 1$, K contains two points x, y with $\|x - y\| = 1 - \epsilon$. For simplicity, we may assume $x = 0$, therefore

$$\eta(K) \subset B \cap B(y,1) \subset \left(B \cap B\left(\frac{y}{1-\epsilon}, 1\right)\right) + \epsilon B$$

(the last inclusion, while not evident, is not difficult to prove) and so

$$\operatorname{diam} \eta(K) \le \operatorname{diam}\left(B \cap B\left(\frac{y}{1-\epsilon}, 1\right)\right) + 2\epsilon,$$

which gives (4). In finite dimensional spaces, equality in (4) holds for suitable segments. Clearly, $\tau_X \le 2$, and here equality holds for $X = \ell_1^2$.

Theorem 2 *Let X be a Banach space with $C_X < 1$, let $K, L \subset X$ be two convex bodies, and let A be a number for which $d_{\eta(K)} \le A d_K$, $d_{\eta(L)} \le A d_L$ (for example, $A = \tau_X$). Let*

$$0 \le \epsilon < \frac{1}{3}(1 - C_X)\min\{d_K, d_L\}. \tag{5}$$

If $\delta(K, L) < \epsilon$, then

$$\delta(\eta(K), \eta(L)) \le \frac{3A}{1 - C_X}\,\epsilon. \tag{6}$$

Proof. From (5) and (2) we have $3\epsilon < r_{\eta(K)}$. From (3), the assumption $d_{\eta(K)} \leq A d_K$, and (2) we get

$$\delta(\eta(K), \eta(K) \sim 3\epsilon B) \leq \frac{d_{\eta(K)}}{r_{\eta(K)}} 3\epsilon \leq \frac{3A}{1 - C_X} \epsilon.$$

Let $x \in \eta(K) \sim 3\epsilon B$ and $y \in K$, $y \neq x$. Then the point $z := x + 3\epsilon(x-y)/\|x-y\|$ satisfies $z \in \eta(K)$. Therefore, $\|z - y\| \leq d_K$, which gives $\|x - y\| \leq d_K - 3\epsilon$. If $y' \in L$, then by $\delta(K, L) < \epsilon$ there is some $y \in K$ with $\|y - y'\| < \epsilon$, and it follows that

$$\|x - y'\| \leq \|x - y\| + \|y - y'\| < d_K - 3\epsilon + \epsilon \leq d_L,$$

thus $x \in \eta(L)$. We have proved that $\eta(K) \sim 3\epsilon B \subset \eta(L)$. Therefore,

$$\eta(K) \subset (\eta(K) \sim 3\epsilon B) + \frac{3A}{1 - C_X} \epsilon B \subset \eta(L) + \frac{3A}{1 - C_X} \epsilon B.$$

An analogous inclusion holds with K and L interchanged. Both inclusions together yield the inequality of (6). $\square$

In the case of Minkowski spaces, we can replace $<$ by $\leq$ in (5) and, moreover, (6) is valid also for $\delta(K, L) = \epsilon$. Also, using that $A \leq 2$ and $C_X \leq n/(n+1)$, (6) can be replaced by the simpler and more elegant estimate

$$\delta(\eta(K), \eta(L)) \leq 6(n+1)\epsilon. \tag{7}$$

If X is a Hilbert space, we can use $J(X) = \sqrt{2}$ (Routledge [38]) and $A = \sqrt{3}$ to write (6) as

$$\delta(\eta(K), \eta(L)) \leq \frac{3\sqrt{6}}{\sqrt{2} - 1} \epsilon.$$

3 The Maehara completion

Let K be a convex body of diameter d. A convex body G is a *tight cover* of K if $K \subset G$ and $\operatorname{diam} G = d$. This notion was introduced by Groemer in [16] and studied in relation to completions of maximal volume in Minkowski spaces. The advantage of this notion is that the set of tight covers, in contrast to the set of completions, is convex. Like the notion of complete set, it has perfect sense in general Banach spaces. Our starting point will be to show that the spherical hulls can be used to obtain tight covers with some interesting properties, and then to discuss when these covers are actually completions.

The set

$$\mu(K) := \frac{1}{2}[\eta(K) + \theta(K)] \tag{8}$$

is called the *Maehara set* of K. The idea that motivates this definition comes from the expression $\theta(K) = dB \sim (-\eta(K))$ and the possibility of writing, in some cases, $\eta(K) - \theta(K) = dB$ which implies, as will be explained below, that $\mu(K)$ has constant width.

First notice that $K \subset \theta(K) \subset \eta(K)$, hence $K \subset \mu(K)$. Since $\theta(K) = \bigcap_{x \in \eta(K)} (x + dB)$, for $x \in \eta(K)$ and $y \in \theta(K)$ we have $\|x - y\| \leq d$. Therefore, writing $M - L := M + (-L)$,

$$\begin{aligned} \mu(K) - \mu(K) &= \frac{1}{2}[\eta(K) + \theta(K)] - \frac{1}{2}[\eta(K) + \theta(K)] \\ &= \frac{1}{2}[\eta(K) - \theta(K)] + \frac{1}{2}[\theta(K) - \eta(K)] \subset dB, \end{aligned} \tag{9}$$

showing that $\operatorname{diam} \mu(K) \leq d$ and hence $\operatorname{diam} \mu(K) = d$. Thus, $\mu(K)$ is a tight cover of K. It is a natural question to ask when $\mu(K)$ is diametrically complete, and thus a completion of K. This will certainly be the case if $\mu(K)$ has constant width, which is somewhat easier to check.

Recall that the convex body K is a *summand* of the convex body B if there is a convex body M such that $K + M = B$. A convex body B is called a *generating set* if any nonempty intersection of translates of B is a summand of B. The notion of 'generating set' was introduced by Balashov and Polovinkin [2], except that they required only that to each nonempty intersection of translates of B, say K, there exists a convex body M with $\overline{K + M} = B$. For our purposes, we can work with the formally more restrictive definition, since this does not restrict the known examples of Banach spaces with a generating unit ball. Infinite dimensional examples are Hilbert spaces, the spaces $\ell_\infty(I)$ (for both, see [2]), and $c_0(I)$, as follows from Proposition 3.1 in Granero, Moreno and Phelps [14]. (We remark that in a Hilbert space the vector sum of two weakly compact sets is weakly compact and hence norm closed, and in $c_0(I)$ an $\ell_\infty(I)$, the sum of two intersections of balls is an intersection of balls and hence closed.)

For the reader's convenience, we recall from [30] the following fact (a short version of arguments due to Maehara [26] and Sallee [40]). Suppose that the unit ball B of X is a generating set. Since $\eta(K)$ is an intersection of translates of dB, Lemma 3.1.8 of [41] (though stated for Minkowski spaces, it is valid for Banach spaces) says that $(dB \sim \eta(K)) + \eta(K) = dB$, hence

$$\eta(K) - \theta(K) = \eta(K) - (dB \sim (-\eta(K)) = \eta(K) + (dB \sim \eta(K)) = dB. \qquad (10)$$

Thus, in the terminology of Maehara and Sallee, $(\eta(K), \theta(K))$ is a pair of constant width. In particular, (9) holds with equality, showing that $\mu(K)$ is a body of constant width d. In this case, we call $\mu(K)$ the *Maehara completion* of K.

We add some historical remarks. In Euclidean spaces, the completion described above is due to Maehara [26]. It was extended by Sallee [39] to Minkowski spaces with the property that $\eta(A)$, for any set A of diameter one, is a summand of the unit ball. Some time before the introduction of generating sets by Balashov and Polovinkin [2], Geivaerts [13] (see also [41, Theorem 3.2.3]) had proven that two-dimensional convex bodies are generating sets (with different terminology, of course), and Maehara [26] had shown that Euclidean balls are generating sets. This was rediscovered by Polovinkin [33], and generalized to unit balls of Hilbert spaces by Balashov and Polovinkin [2]. It was repeatedly observed, probably first by Sallee [39], that generating sets are stable under nondegenerate linear maps and under direct sums. In finite dimensions, this exhausts the known examples of norm unit balls which are generating sets (there are some non-symmetric examples of generating sets in dimensions greater than two, but these are irrelevant in the context of Minkowski spaces).

The completion procedure of Maehara and Sallee was rediscovered by Polovinkin [34], more generally in reflexive Banach spaces. He called it *regular completion.* It was further investigated by Polovinkin and Sidenko [36], Polovinkin [35].

It is clear from the proof above that the Maehara completion does not require the full strength of the assumption that the unit ball B is generating, but only that $\bigcap_{x \in A}(B + x)$ is a summand of B whenever $\operatorname{diam} A \leq 1$. It was proved by Karasëv [20] (Theorem 1 and Lemma 7) that for this it is sufficient that $B \cap (B + x)$ is a summand of B whenever $\|x\| \leq 1$. Therefore, we define:

Definition A normed space, its norm $\|\cdot\|$ and its unit ball B have the *Karasëv property* if $B \cap (B + x)$ is a summand of B for every x with $\|x\| \leq 1$.

It seems to be unknown whether the Karasëv property of B implies that B is a generating set. In any case, we can formulate the following.

Proposition 1 *For a Minkowski space X, the following assertions are equivalent.*

(a) *X has the Karasëv property.*

(b) *$\mu(K)$ has constant width for every convex body $K \subset X$.*

(c) *$\mu(K)$ has constant width for every segment $K \subset X$.*

Proof. Suppose that (a) holds. If $K \subset X$ is a convex body of diameter 1 (without loss of generality), then $\eta(K) = \bigcap_{x\in K}(B + x)$ is a summand of B, by the result of Karasëv mentioned above. Hence, (9) holds with equality, showing that $\mu(K)$ is a body of constant width. Thus (b) holds.

To prove $(c) \Rightarrow (a)$, consider a segment $K \subset X$, without loss of generality $K = [0, x]$ with $\|x\| = 1$. By (c) we have $\mu(K) - \mu(K) = B$, so

$$\frac{1}{2}(\eta(K) - \theta(K)) + \frac{1}{2}(\theta(K) - \eta(K)) = B,$$

therefore $\eta(K) - \theta(K)$ has constant width 1. On the other hand, since $\|y - z\| \leq \operatorname{diam} K = 1$ for $y \in \eta(K)$ and $z \in \theta(K)$, we have also $\eta(K) - \theta(K) \subset B$ and hence

$$\eta(K) - \theta(K) = B, \tag{11}$$

which proves that $\eta(K)$ is a summand of B. What is left in order to complete the proof is to show that $\eta(K) = B \cap (B + x)$. For the nontrivial inclusion, it is easy to check that $B \cap (B + x) \subset \lambda x + B$ for every $\lambda \in [0, 1]$. $\square$

In the Minkowski space ℓ_1^3, an edge of its unit ball is an example of a convex body K for which the Maehara set $\mu(K)$ is not complete, since it has diameter 1 and is strictly contained in the unit ball (see Figure 1). This example shows that in arbitrary Minkowski spaces, Maehara sets cannot be used as canonical completions. We shall come back to this problem in Section 4.

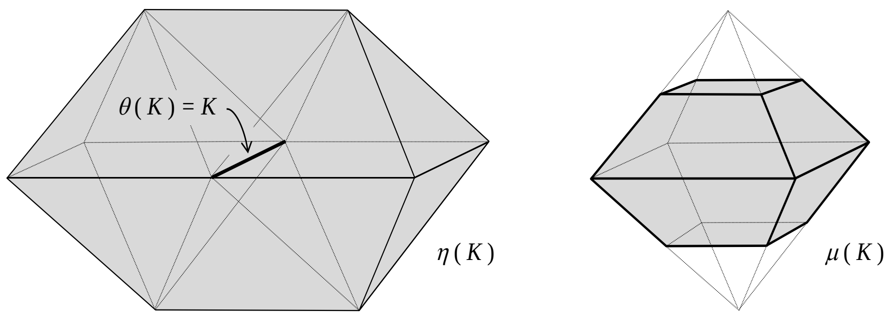

Fig. 1 The Maehara set $\mu(K)$ is not complete.

Our main interest in this section are the continuity properties of the Maehara completion. In this regard, we now show that it is locally Lipschitz continuous, with a constant depending only on the Jung constant of the space, hence, in finite dimensions, with a Lipschitz constant depending only on the dimension. Notice that, incidentally, we prove also the local Lipschitz continuity of the tight spherical hull $\theta(\cdot)$ in spaces with the Karasëv property.

Theorem 3 *Let X be a Banach space with $C_X < 1$ which has the Karasëv property. Let $K, L \subset X$ be convex bodies, let A be a number for which $d_{\eta(K)} \leq A d_K$, $d_{\eta(L)} \leq A d_L$ (for example, $A = 2$) and let*

$$0 \leq \epsilon < \frac{1}{3}(1 - C_X) \min\{d_K, d_L\}.$$

If $\delta(K, L) < \epsilon$, then

$$\delta(\mu(K), \mu(L)) \leq \left(\frac{3A}{1 - C_X} + 1 \right) \epsilon. \tag{12}$$

Proof. The Hausdorff distance between two convex bodies K, L can be expressed as

$$\delta(K, L) = \sup_{f \in S^*} |h(K, f) - h(L, f)|, \tag{13}$$

where S^* is the dual unit sphere and $h(\cdot, f)$ is the usual support function, defined by $h(K, f) = \sup\{f(x) : x \in K\}$ for every $f \in S^*$. In view of (8) and the local Lipschitz continuity of η proved in Theorem 2, we concentrate our attention on proving that θ is locally Lipschitz in spaces with the Karasëv property.

Let $f \in S^*$. Since the unit ball B has the Karasëv property, (10) holds, hence

$$h(\eta(K), -f) + h(\theta(K), f) = d_K h(B, -f) = d_K, \tag{14}$$

and therefore we can estimate

$$\begin{aligned} |h(\theta(K), f) - h(\theta(L), f)| &= |d_K - h(\eta(K), -f) - (d_L - h(\eta(L), -f))| \\ &\leq |d_K - d_L| + |h(\eta(K), -f) - h(\eta(L), -f)|. \end{aligned}$$

By assumption, we have $\epsilon < (1/3)(1 - C_X) \min\{d_K, d_L\}$, and Theorem 2 tells us that

$$\delta(\eta(K), \eta(L)) \leq \frac{3A}{1 - C_X} \epsilon$$

and hence

$$|h(\eta(K), -f) - h(\eta(L), -f)| \leq \frac{3A}{1 - C_X} \epsilon.$$

Since $|d_K - d_L| \leq 2\epsilon$, we conclude that

$$|h(\theta(K), u) - h(\theta(L), u)| \leq 2\,\epsilon + \frac{3A}{1 - C_X} \epsilon.$$

This yields

$$\begin{aligned} |h(\mu(K), f) - h(\mu(L), f)| &\leq \frac{1}{2}|h(\eta(K), f) - h(\eta(L), f)| + \frac{1}{2}|h(\theta(K), f) - h(\theta(L), f)| \\ &\leq \left(\frac{3A}{1 - C_X} + 1 \right) \epsilon, \end{aligned}$$

and this implies (12), completing the proof. □

In most cases where Theorem 3 is known to apply, we can estimate the Jung constant and thus simplify the result. For instance, if X is the n-dimensional Euclidean space, then $C_X = \sqrt{n/2(n+1)}$ and hence $1/\sqrt{3} \leq C_X < 1/\sqrt{2}$ for $n \geq 2$. Moreover,

$$d_{\eta(K)} \leq \sqrt{3}\, d_K.$$

In fact, let $x, y \in K$ be points with $\|x-y\| = d_K$. Then $\eta(K) \subset B(x, d_K) \cap B(y, d_K)$, and the latter set has diameter $\sqrt{3}\, d_K$. Hence, we can take $A = \sqrt{3}$.

Corollary 3 *Let X be a Euclidean space or a Hilbert space. Let $K, L \subset X$ be convex bodies and let $0 \leq \epsilon \leq 0.097 \min\{d_K, d_L\}$. If $\delta(K, L) \leq \epsilon$, then*

$$\delta(\mu(K), \mu(L)) \leq 20\, \epsilon.$$

Proof. If X is the n-dimensional Euclidean space, then $C_X = \sqrt{n/2(n+1)}$ and hence $C_X < 1/\sqrt{2}$ for $n \geq 2$. If X is a Hilbert space, $C_X = 1/\sqrt{2}$ (see [1] for several results on Jung and self-Jung contants). Moreover, in both cases

$$d_{\eta(K)} \leq \sqrt{3}\, d_K,$$

so we can take $A = \sqrt{3}$ in the last estimate obtained in Theorem 3. □

If X is a two-dimensional Minkowski space, then $1/2 \leq C_X \leq 2/3$. The best possible estimate for A is $A \leq 2$. Therefore, an immediate consequence of Theorem 3 in this context is the following.

Corollary 4 *Let X be a two-dimensional Minkowski space and let $K, L \in \mathcal{K}^2$. Then, if $0 \leq \epsilon \leq \frac{1}{9} \min\{d_K, d_L\}$ and $\delta(K, L) \leq \epsilon$, we have*

$$\delta(\mu(K), \mu(L)) \leq \frac{15}{2}\, \epsilon.$$

Next we show that, in spaces with generating unit ball, the Maehara completion plays a special role within the set of all completions of a given convex body. The set of completions of a convex body K is denoted by $\gamma(K)$. This is a compact subset of $\mathcal{K}^n$ in n-dimensional Minkowski spaces, and in spaces with the Karasëv property, it is convex (see Section 4 in [30]). Define the *radius of $\gamma(K)$ with respect to $C \in \gamma(K)$* by

$$r(C) := \sup\{\delta(C, M) : M \in \gamma(K)\}. \tag{15}$$

Theorem 4 *Let X be a Banach space with generating unit ball, and let $K \subset X$ be a convex body. Then $\mu(K)$ is a metric center of $\gamma(K)$, that is, $r(\mu(K)) \leq r(C)$ for all $C \in \gamma(K)$.*

The following lemma will be used in the proof of the theorem and may be of independent interest. As in Theorem 3, we denote by h the support function on convex bodies. Recall that a functional $f \in X^*$ attains its norm if there is $x \in B$ such that $h(B, f) = f(x)$. In the case of finite dimensional spaces, every functional attains its norm. In the general case, the Bishop–Phelps theorem states that the norm attaining functionals are always dense in the dual space.

Lemma 3 *Let X be a Banach space with generating unit ball, and let $K \subset X$ be a convex body. Every semispace defined by a norm attaining functional containing $\theta(K)$ contains also a completion of K.*

Proof. Let $f \in X^*$ be a norm attaining functional. Consider a semispace $f^{-1}(-\infty, \alpha]$ satisfying $\theta(K) \subset f^{-1}(-\infty, \alpha]$. We may assume that $\alpha = h(\theta(K), f)$. Let $z \in d_K B$ such that $f(z) = h(d_K B, f)$. Since $\theta(K)$ is a summand of $d_K B$, there is a convex body L satisfying

$\theta(K) + L = d_K B$ and, consequently, there are $x \in \theta(K)$ and $y \in L$ such that $x + y = z$. Obviously, $f(x) = h(\theta(K), f)$ and $f(y) = h(L, f)$. We can write

$$\big((x - z) + L\big) + \theta(K) = (x - z) + d_K B$$

and, since $(x - z) + y = 0$, we have $\theta(K) \subset (x - z) + d_K B$. It is easy to check that $(x - z) + d_K B \subset f^{-1}(-\infty, \alpha]$. Now consider the set $K' = \operatorname{conv}(K \cup \{x - z\})$ satisfying $K \subset K'$ and $\operatorname{diam} K' = d_K$, hence $\gamma(K') \subset \gamma(K)$. Since $x - z \in K'$, it is clear that $\eta(K') \subset (x - z) + d_K B$ and therefore any completion of K' is included in $(x - z) + d_K B$. We conclude that there is a completion of K included in the ball $(x - z) + d_K B$, hence also in the semispace $f^{-1}(-\infty, \alpha]$, which finishes the proof. $\square$

Proof of Theorem 4. Given a convex body $K \subset X$ and an arbitrary completion $C \in \gamma(K)$, we must show that $r(\mu(K)) \leq r(C)$. To this end, we need a suitable expression for the exact value of $r(C)$. First notice that, given $M \in \gamma(K)$, we have

$$\delta(C, M) = \sup_{f \in S^*} |h(C, f) - h(M, f)|, \tag{16}$$

where S^* is the dual unit sphere. Then, if we fix $f \in S^*$, using that $\theta(K) \subset M \subset \eta(K)$, we obtain

$$\begin{aligned} h(M, f) - h(C, f) &\leq h(\eta(K), f) - h(C, f), \\ h(C, f) - h(M, f) &\leq h(C, f) - h(\theta(K), f), \end{aligned}$$

and these estimates, together with (16), give

$$r(C) \leq \max \left\{ \sup_{f \in S^*} \{h(\eta(K), f) - h(C, f)\}, \sup_{f \in S^*} \{h(C, f) - h(\theta(K), f)\} \right\}. \tag{17}$$

Next we will see that the above inequality is, actually, an equality. According to the Bishop–Phelps theorem, if we fix $\epsilon > 0$, there is a norm attaining functional $f_0 \in S^*$ such that

$$h(\eta(K), f_0) - h(C, f_0) + \epsilon \geq \sup_{f \in S^*} \{h(\eta(K), f) - h(C, f)\}.$$

Since $\eta(K)$ is a summand of $d_K B$, every functional attaining its maximum on B attains its maximum also on $\eta(K)$, and so there exists $x \in \eta(K)$ such that $f_0(x) = h(\eta(K), f_0)$. Any completion of $K \cup \{x\}$ is a completion of K, since $\operatorname{diam}(K \cup \{x\}) = d_K$. Therefore, there exists a completion $M_0 \in \gamma(K)$ such that $x \in M_0$, and then necessarily $h(M_0, f_0) \geq f_0(x) = h(\eta(K), f_0)$. This gives

$$\begin{aligned} \delta(C, M_0) &\geq h(M_0, f_0) - h(C, f_0) = h(\eta(K), f_0) - h(C, f_0) \\ &\geq \sup_{f \in S^*} \{h(\eta(K), f_0) - h(C, f_0)\} - \epsilon. \end{aligned} \tag{18}$$

Again, according to the Bishop–Phelps theorem, there exists a norm attaining functional $f_1 \in S^*$ such that

$$h(C, f_1) - h(\theta(K), f_1) + \epsilon \geq \sup_{f \in S^*} \{h(C, f) - h(\theta(K), f)\}.$$

From Lemma 3 we know that there is $M_1 \in \gamma(K)$ satisfying $h(M_1, f_1) \leq h(\theta(K), f_1)$, and then necessarily $h(M_1, f_1) = h(\theta(K), f_1)$. This gives

$$\begin{aligned} \delta(C, M_1) &\geq h(C, f_1) - h(M_1, f_1) = h(C, f_1) - h(\theta(K), f_1) \\ &\geq \sup_{f \in S^*} \{h(C, f) - h(\theta(K), f)\} - \epsilon. \end{aligned} \tag{19}$$

From (17)–(19), we conclude that

$$r(C) = \max\left\{\sup_{f\in S^*}\{h(\eta(K),f) - h(C,f)\}, \sup_{f\in S^*}\{h(C,f) - h(\theta(K),f)\}\right\}. \tag{20}$$

Replacing in (20) the arbitrary completion C by the Maehara completion $\mu(K) = \frac{1}{2}\left(\eta(K) + \theta(K)\right)$, it follows that

$$r(\mu(K)) = \frac{1}{2}\sup_{f\in S^*}\{h(\eta(K),f) - h(\theta(K),f)\}.$$

Now let $f \in S^*$ be fixed. Using (20), and since $h(\theta(K),f) \leq h(C,f) \leq h(\eta(K),f)$, we have

$$\begin{aligned} r(C) &\geq \max\{h(\eta(K),f) - h(C,f), h(C,f) - h(\theta(K),f)\} \\ &\geq \frac{1}{2}\big(h(\eta(K),f) - h(\theta(K),f)\big). \end{aligned}$$

Taking the supremum in the right-hand side of the last inequality, we finally obtain $r(C) \geq r(\mu(K))$, which completes the proof. □

The question of covering a convex body K with a constant width set having no additional singularities to those of K has received some attention in the past. Let us say for the moment that a point x of a convex body $K \in \mathcal{K}^n$ of diameter $d > 0$ is a *d-singularity* of K if x is the endpoint of two different diameter segments of K. If K is of constant width d, then every singularity x of K (i.e., a point $x \in K$ lying in two different supporting hyperplanes of K) is a d-singularity of K. Conversely, answering (in a stronger form) a question of Danzer and Grünbaum (Problem 7 in [24]), Falconer [11] constructed, to any given $K \in \mathcal{K}^n$ of diameter $d > 0$, a completion K' with no 'additional singularities', in the sense that every singularaity of K' is already a d-singularity of K. The Maehara completion does not have this property, as shown, for example in the plane, by an isosceles triangle of diameter d without a d-singularity. On the other hand, Polovinkin and Sidenko [36] have shown that the Maehara completion of a smooth convex body is smooth. We strengthen this result in Euclidean spaces, showing that the Maehara completion of a convex body is at least as smooth as the body itself, in the sense that every normal cone of the Maehara completion $\mu(K)$ is contained in some normal cone of K. In the following, $N(K,x)$ denotes the normal cone of the convex body K at its boundary point x.

Theorem 5 *Let $K \in \mathcal{K}^n$ be a convex body in Euclidean space and $\mu(K)$ its Maehara completion. To each boundary point a of $\mu(K)$ there exists a boundary point b of K such that $N(\mu(K),a) \subset N(K,b)$.*

Proof. We shall make crucial use of the fact that K and any completion K' of K satisfy $K \subset \theta(K) \subset K' \subset \eta(K)$.

Let a be a boundary point of $\mu(K)$, without loss of generality a singular one. Since $\mu(K) = \frac{1}{2}[\theta(K) + \eta(K)]$, there are points $b \in \operatorname{bd}\theta(K)$ and $c \in \operatorname{bd}\eta(K)$ with $a = \frac{1}{2}(b + c)$. By [41, Theorem 2.2.1(a)],

$$N(\mu(K),a) = N(\theta(K),b) \cap N(\eta(K),c). \tag{21}$$

If we can show that $b \in K$, then $b \in \operatorname{bd} K$ and $N(\mu(K),a) \subset N(\theta(K),b) \subset N(K,b)$, which would finish the proof.

Now suppose, to the contrary, that $b \notin K$. Since $\dim N(\theta(K), b) \geq 2$ by (21), b is a singular point of $\theta(K)$. Since $\theta(K) = \bigcap_{x \in \eta(K)} B(x, d)$ and $x \in \eta(K) \Leftrightarrow K \subset B(x, d)$, we have

$$\theta(K) = \bigcap_{K \subset B(x,d)} B(x, d).$$

Since b is a singular boundary point of $\theta(K)$, it must be contained in the boundaries of two different balls $B(x, d), B(y, d)$ with $K \subset B(x, d)$, $K \subset B(y, d)$ and $\theta(K) \subset B(x, d) \cap B(y, d)$. Since $B(x, d), B(y, d)$ are different Euclidean balls of radius d with $b \in \operatorname{bd}[B(x, d) \cap B(y, d)]$, there is a ball $B(z, d)$ with $b \in \operatorname{bd} B(z, d)$ and $[B(x, d) \cap B(y, d)] \setminus \{b\} \subset \operatorname{int} B(z, d)$; in particular, $K \subset \operatorname{int} B(z, d)$. A suitable translate $B(z', d)$ of $B(z, d)$ still satisfies $K \subset B(z', d)$, but $b \notin B(z', d)$. Then $K \cup \{z'\}$ has diameter d. Any completion K' of this set is also a completion of K, it is contained in $B(z', d)$ and hence does not contain $\theta(K)$, a contradiction. This completes the proof. $\square$

Remark 2 In the proofs of Lemma 3 and Theorem 5, we have implicitly made use of a 'support principle' for intersections of balls of equal radius in spaces with a generating unit ball. For its general formulation, see Balashov and Polovinkin [2, Corollary 1.1]. For the case of Euclidean spaces, this principle appeared earlier in Frankowska and Olech [12].

4 The generalized Bückner completion

As we have seen, the Maehara completion exists only in very special Minkowski spaces. In contrast, Bückner's [7] completion procedure, suitably modified, can be extended to n-dimensional Minkowski spaces $X = (\mathbb{R}^n, \|\cdot\|)$, as we show in the following.

We begin with a preparatory remark. If $K \in \mathcal{K}^n$ is a convex body of diameter d, then any segment $[x, y]$ in K with $\|x - y\| = d$ is called a *diameter segment* of K. If $[x, y]$ is a diameter segment of K, then x and y are in the boundary of the ball $B' := (d/2)B + (x + y)/2$, hence there exists a pair H_x, H_y of parallel supporting hyperplanes of B' which are at distance d and pass through x and y, respectively. They are also supporting hyperplanes of K, since otherwise $\operatorname{diam} K > d$. Since the intersection of K with a supporting hyperplane of K contains an extreme point of K, it follows that K has always a diameter segment whose endpoints are extreme points of K.

First we observe that we may focus our attention on convex bodies K for which r_K/d_K is larger than some fixed positive constant. For given $K \in \mathcal{K}^n$, the set

$$\beta_0(K) := \frac{1}{2}K + \frac{1}{2}\eta(K)$$

has diameter d_K, since $K \subset \beta_0(K) \subset \mu(K)$ and $\operatorname{diam} \mu(K) = d_K$, as we saw in Section 3. Moreover,

$$r_{\beta_0(K)} \geq \frac{1}{2} r_{\eta(K)} \geq \frac{1}{2}(1 - C_X) d_K \geq \frac{d_K}{2(n+1)}$$

according to (2) and $C_X \leq n/(n+1)$. It is for this reason that in the following we restrict our considerations to the set

$$\mathcal{K}^{(n)} := \left\{ K \in \mathcal{K}^n : r_K \geq \frac{d_K}{2(n+1)} \right\}.$$

Before going ahead, we will give an intuitive explanation of the idea behind the Bückner construction. In Minkowski spaces, a convex body K is complete if and only if every boundary point is the endpoint of a diameter segment of K, as is easy to see. Therefore, completing a set is nothing else but finding a tight cover whose boundary points are endpoints of diameter segments. Now recall that a convex body K is complete if and only if $K = \bigcap_{x \in K} B(x, d_K)$, but if K is not complete, then $\eta(K) = \bigcap_{x \in K} B(x, d_K)$ has larger diameter. Bückner's main idea was to consider 'half of $\eta(K)$', which is a tight cover of K and is 'partially complete', in the sense that a large part of its boundary consists of diameter endpoints. The procedure is then iterated, until all boundary points are diameter endpoints.

Let $K \subset M$ be convex bodies such that $K \in \mathcal{K}^{(n)}$. Let $u \in \mathbb{R}^n \setminus \{o\}$ be a given vector. A point $x \in \operatorname{bd} M$ is called (K, u)-*directed* if $x + \lambda u \notin M$ for all $\lambda > 0$ and $x - \lambda u \in \operatorname{int} K$ for some $\lambda > 0$. The convex body M is called (K, u)-*complete* if each of its (K, u)-directed boundary points is the endpoint of a diameter segment of M.

For K and u as above, we define the set

$$Z^+(K, u) := \{x + \lambda u : x \in K,\ \lambda \geq 0\},$$

and we define an operator β_u on $\mathcal{K}^{(n)}$ by

$$\beta_u(K) := \eta(K) \cap Z^+(K, u).$$

Lemma 4 *Let $K \in \mathcal{K}^{(n)}$ have diameter $d > 0$, let $u \in \mathbb{R}^n \setminus \{o\}$. Then $\beta_u(K)$ is a tight cover of K, and it is (K, u)-complete. If also $v \in \mathbb{R}^n \setminus \{o\}$, then $\beta_v(\beta_u(K))$ is (K, u)-complete and (K, v)-complete.*

Proof. The inclusion $K \subset \beta_u(K)$ is clear. For the proof that $\operatorname{diam} \beta_u(K) = d$, let $x, y \in \beta_u(K)$. If $x, y \in K$, then $\|x - y\| \leq d$. If, say, $x \in K$ and $y \notin K$, then $y \in B(x, d)$ and hence $\|x - y\| \leq d$. Hence, we can assume that $x, y \in \beta_u(K) \setminus K$. Let E be a two-dimensional plane through x and y that is parallel to u. There are points $p, q \in K \cap E$ lying in different boundary lines g_1, g_2 of the strip $\{x + \mathbb{R}u : x \in K \cap E\}$. We have $\|p - q\| \leq d$, $\|p - x\| \leq d$, $\|p - y\| \leq d$, $\|q - x\| \leq d\|$, $\|q - y\| \leq d$. Assume that $\|x - y\| = D > d$. Let $P := \operatorname{conv}\{x, y, p, q\}$. The diameter of a polygon is the largest distance between vertices of the polygon, hence $\|x - y\| = \operatorname{diam} P$. Therefore, there are parallel support lines h_1, h_2 (in E) to P which have distance D and are such that, say, $x \in h_1$ and $y \in h_2$. Suppose that h_1, h_2 are not parallel to g_1, g_2. Since x, y lie between g_1 and g_2 and in the same open halfplane in E bounded by the line through p and q, and the points p, q lie between h_1 and h_2, this is impossible. Therefore, the pair (h_1, h_2) coincides with (g_1, g_2) or (g_2, g_1). Then g_1, g_2 have distance D, hence $\|p - q\| \geq D > d$, a contradiction. It follows that $\|x - y\| \leq d$. This proves that $\beta_u(K)$ has diameter d.

For the proof that $\beta_u(K)$ is (K, u)-complete, let x be a (K, u)-directed boundary point of $\beta_u(K)$. If x is not the endpoint of a diameter segment of $\beta_u(K)$, then, in particular, the distances of x from the points of K have an upper bound less than d, and a neighbourhood of x in $\operatorname{int} Z^+(K, u)$ is contained in $\eta(K)$ and hence in $\beta_u(K)$, a contradiction.

Let $v \in \mathbb{R}^n \setminus \{o\}$, and let $L := \beta_v(\beta_u(K))$. Then $\beta_u(K) \subset L$, and L has diameter d. As proved, L is $(\beta_u(K), v)$-complete. Any (K, v)-directed boundary point of L is also a $(\beta_u(K), v)$-directed boundary point of L, hence L is (K, v)-complete. To show that L is (K, u)-complete, let x be a (K, u)-directed boundary point of L. Then x lies on a ray $\{x' + \lambda u : \lambda \geq 0\}$ with some $x' \in \operatorname{int} K$. This ray intersects $\operatorname{bd} \beta_u(K)$ in a point y, which is a (K, u)-directed boundary point of $\beta_u(K)$ and hence is the endpoint of a diameter segment of $\beta_u(K)$. It cannot be an interior point of L, since otherwise $\operatorname{diam} L > d$. Thus, $x = y$, and x is the endpoint of a diameter segment of L. Therefore, L is (K, u)-complete. This finishes the proof of Lemma 4. $\square$

To construct now a completion of K, we extend and modify Bückner's iteration argument. Since later we want to prove continuity, we need a global construction, without any choices depending on the body under consideration. For this, we fix finitely many vectors $u_1, \dots, u_m \neq o$ with the property that

$$B \subset \bigcup_{i=1}^{m} \operatorname{int} Z^+ \left(\frac{1}{2(n+1)} B, u_i \right). \tag{22}$$

The number m can be chosen to depend only on the dimension n. For obtaining explicit bounds (if wanted), one can first use results on coverings of a Euclidean sphere by congruent balls of smaller radius, see [6] and [9], for example. The general case then follows from the fact that, in a sense, a Minkowski space is never too far from a Euclidean space. Indeed, by John's theorem ([18]), there is an ellipsoid E with $E \subset B \subset \sqrt{n}\, E$. Therefore,

$$\sqrt{n}\, E \subset \bigcup_{i=1}^{m} \operatorname{int} Z^+ \left(\frac{1}{2(n+1)} E, u_i \right)$$

for suitable $u_1, \dots, u_m \neq o$ (and m depending on n) is enough to ensure (22).

Consider now an arbitrary $K \in \mathcal{K}^n$ and the tight cover $\beta_0(K)$. Since $r_{\beta_0(K)}/d_K \geq 1/2(n+1)$, we can assume that $(d_K/2(n+1))B \subset \beta_0(K)$, then

$$d_K B \subset \bigcup_{i=1}^{m} \operatorname{int} Z^+(\beta_0(K), u_i). \tag{23}$$

Define

$$K_0 := \beta_0(K), \quad K_i := \beta_{u_i}(K_{i-1}) \text{ for } i = 1, \dots, m.$$

Let $x \in \operatorname{bd} K_m$. Since $\operatorname{bd} K_m \subset d_K B$, there is, by (23), a number $i \in \{1, \dots, m\}$ such that $x \in \operatorname{int} Z^+(\beta_0(K), u_i)$. The point x is a $(\beta_0(K), u_i)$-directed boundary point of K_m. Since K_m is $(\beta_0(K), u_i)$-complete by Lemma 4, there is a point $y \in K_m$ with $\|x - y\| = d_K$. Since x was an arbitrary boundary point of K_m, this shows that K_m is complete and is, therefore, a completion of K. Denoting K_m by $\beta_g(K)$, we have thus defined an operator β_g on $\mathcal{K}^n$, with the property that $\beta_g(K)$ is a completion of K. We call β_g the *generalized Bückner completion*.

Our final goal is to show that the generalized Bückner completion β_g is locally Lipschitz continuous. We shall deduce this from a corresponding result for β_u.

Lemma 5 *For $d > 0$, let $\mathcal{K}_d := \{K \in \mathcal{K}^{(n)} : d/2 \leq d_K \leq d\}$. Consider $K, L \in \mathcal{K}_d$ and $u \in \mathbb{R}^n \setminus \{o\}$. If*

$$\delta(K, L) \leq \epsilon \leq \frac{d}{8(n+1)} =: r, \tag{24}$$

then

$$\delta(\beta_u(K), \beta_u(L)) \leq 48(n+1)^2\, \epsilon. \tag{25}$$

Proof. Let $K, L \in \mathcal{K}_d$ and $\delta(K, L) \leq \epsilon$. By (24) (and using $C_X \leq n/(n+1)$ and $d_K, d_L \geq d/2$) we have $\epsilon \leq (1/3)(1 - C_X) \min\{d_K, d_L\}$, hence we can apply Theorem 2 in the version (7), which gives

$$\delta(\eta(K), \eta(L)) \leq 6(n+1)\, \epsilon =: \alpha. \tag{26}$$

Since $K \in \mathcal{K}_d \subset \mathcal{K}^{(n)}$, we have $r_K \geq d_K/2(n+1) \geq 2r$, hence we can assume, without loss of generality, that $2rB \subset K$. Since $\delta(K, L) \leq \epsilon \leq r$, we then have

$$rB \subset K \cap L. \tag{27}$$

Let $u \in \mathbb{R}^n \setminus \{o\}$ be given. To estimate $\delta(\beta_u(K), \beta_u(L))$, let $x \in \beta_u(K) = \eta(K) \cap Z^+(K, u)$. By (26) there is a point $y \in \eta(L)$ with $\|x - y\| \leq \alpha$. Since $x \in Z^+(K, u)$, there is a point $x' \in \operatorname{int} K$ with $x = x' + \lambda u$ for some $\lambda \geq 0$. Again by (26), there is a point $w' \in L$ with $\|x' - w'\| \leq \alpha$. Then $w := w' + \lambda u \in Z^+(L, u)$ and $\|x - w\| \leq \alpha$. Thus, we have $y, w \in \alpha B + x$. The convex hull of the ball rB and any point in the ball $\alpha B + x$ contains the point $z := r/(r + \alpha)x$. Indeed, if $a + x \in \alpha B + x$, then $\|a\| \leq \alpha$ and

$$z = \frac{r}{r+\alpha}(x + a) + \frac{\alpha}{r+\alpha}\left(-\frac{r}{\alpha}a\right), \quad \left\|-\frac{r}{\alpha}a\right\| \leq r.$$

Since $rB \subset L \subset \eta(L)$, $y \in \eta(L)$ and $w \in Z^+(L, u)$, we conclude that $z \in \eta(L) \cap Z^+(L, u) = \beta_u(L)$. We have

$$\|x - z\| = \frac{\alpha}{r+\alpha}\|x\| \leq \frac{\alpha}{r} d = 6(n+1)\frac{d}{r}\epsilon = 48(n+1)^2\,\epsilon.$$

Similarly, to given $z \in \beta_u(L)$ we can find a point $x \in \beta_u(K)$ satisfying this inequality. It follows that

$$\delta(\beta_u(K), \beta_u(L)) \leq 48(n+1)^2\,\epsilon,$$

completing the proof of Lemma 5. □

Theorem 6 *The generalized Bückner completion β_g is locally Lipschitz continuous with a constant which depends only on the dimension of the space.*

Proof. Writing $\beta_i := \beta_{u_i}$, we first prove that $\beta_m \circ \cdots \circ \beta_1$ is locally Lipschitz on $\mathcal{K}_d$ (defined in Lemma 5). To this end, we will show that there exists a number ϵ_0, depending only on d and the dimension n, such that the following holds: if $K, L \in \mathcal{K}_d$ and $\delta(K, L) \leq \epsilon \leq \epsilon_0$, then

$$\delta((\beta_m \circ \cdots \circ \beta_1)(K), (\beta_m \circ \cdots \circ \beta_1)(L)) \leq [48(n+1)^2]^m\,\epsilon. \tag{28}$$

We assume, first, that

$$\epsilon \leq \frac{d}{8(n+1)} =: r.$$

Let $K, L \in \mathcal{K}_d$ and $\delta(K, L) \leq \epsilon$. As in the proof of Lemma 5, we may assume that $rB \subset K \cap L$, then $K, L \subset dB$. Due to Lemma 5, we can apply (25) repeatedly and conclude that

$$\delta((\beta_m \circ \cdots \circ \beta_1)(K), (\beta_m \circ \cdots \circ \beta_1)(L)) \leq [48(n+1)^2]^m\epsilon.$$

For this to hold, at each step the upper bound for ϵ has to be decreased appropriately. Since we need m steps, we end up with a positive upper bound ϵ_0, which depends only on d and (since m depends only on n) on n.

Now, for $d > 0$, consider the open set $\beta_0^{-1}(\operatorname{int} \mathcal{K}_d)$. It is clear that $X = \bigcup_{d>0} \beta_0^{-1}(\operatorname{int} \mathcal{K}_d)$. Let $K, L \in \beta_0^{-1}(\operatorname{int} \mathcal{K}_d)$ and let

$$0 \leq \epsilon \leq \min\left\{\frac{1}{3(n+1)}\min\{d_K, d_L\}, \frac{\epsilon_0}{6(n+1)}\right\}.$$

Then, if $\delta(K, L) \leq \epsilon$, we can apply Theorem 2, version (6), to obtain that $\delta(\eta(K), \eta(L)) \leq 6(n+1)\epsilon \leq \epsilon_0$, hence $\delta(\beta_0(K), \beta_0(L)) \leq 6(n+1)\epsilon \leq \epsilon_0$. which together with (28) gives

$$\delta(\beta_g(K), \beta_g(L)) \leq 6 \cdot 48^m (n+1)^{2m+1}\epsilon. \tag{29}$$

This completes the proof. □

The estimate of the Lipschitz constant in (29) should be compared with the one obtained in [29] for the Lipschitz continuity of the diametric completion mapping γ (from which β_g is a selection) since, quite surprisingly, the latter is less than 1. It remains a challenge to decide whether, for all Minkowski spaces, there exists a completion procedure which is locally Lipschitz continuous, where the quantification of 'local' and the Lipschitz constant are independent of the dimension.

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