

Diametrically complete sets in Minkowski spaces

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Abstract

We obtain a new characterization of the diametrically complete sets in Minkowski spaces, by modifying two well-known characteristic properties of bodies of constant width. We also get sharp inequalities for the circumradius and inradius of a diametrically complete set of given diameter. Strengthening former work of D. Yost, we show that in a generic Minkowski space of dimension at least three the set of diametrically complete sets is not closed under the operation of adding a ball. We conclude with new results about Eggleston's problem of characterizing the Minkowski spaces in which every diametrically complete set is of constant width.

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1 Introduction

A bounded set in a metric space is called diametrically complete if it cannot be enlarged without increasing its diameter. In a Euclidean space and in two-dimensional Minkowski spaces, the diametrically complete sets are precisely the convex bodies of constant width. In an arbitrary Minkowski space (a finite dimensional real normed space), it is known that every body of constant width is a complete set, but the converse does not hold generally, although this was believed for a long time, until Eggleston [5] gave counterexamples. Constant width sets have a long and rich history, which goes back to Euler, and they have been intensively studied since then. In contrast, diametrically complete sets, although they were introduced as early as 1911 (by Meissner [12]), have received less attention, and many of their basic properties are still far from being well understood. Also the question, raised by Eggleston [5] in 1965, of characterizing the Minkowski spaces in which every complete set is of constant width, remains essentially open.

In this paper, based on a new characterization of complete sets, we investigate structural properties of the set of all complete sets and obtain, for general norms, a new necessary condition for the spaces satisfying Eggleston's condition mentioned above.

A *Minkowski space* can be represented as \mathbb{R}^n with a norm $\|\cdot\|$, for some $n \in \mathbb{N}$ (we assume $n \geq 2$). In the following, all metric notions, like distance, diameter, width, etc., refer to this norm. The *unit ball* of the norm $\|\cdot\|$ is the set $B := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Any set $B(z, r) := rB + z$ with $r \geq 0$ and $z \in \mathbb{R}^n$ is called a *ball*. By \mathcal{K}^n we denote the space of convex bodies (nonempty, compact, convex subsets) of \mathbb{R}^n , equipped with the Hausdorff metric induced by some norm. \mathcal{K}_0^n is the subset of bodies with interior points.

A convex body $K \in \mathcal{K}^n$ is said to be of *constant width* if any two parallel supporting hyperplanes of K have the same distance. This distance is then equal to $\text{diam } K$, the diameter of K . The body K is of constant width if and only if $K - K$, the Minkowski sum of K and of $-K$, its reflection in the origin o , is a ball. A bounded set M is called *diametrically complete*, or briefly *complete* (also called *diametrically maximal*), if $\text{diam}(M \cup \{x\}) > \text{diam } M$ for each $x \in \mathbb{R}^n \setminus M$. Clearly, a complete set is compact and convex. For surveys about previous results on constant width and complete sets, we refer the reader to Chakerian and Groemer [4], Heil and Martini [6], Martini and Swanepoel [10].

Eggleston [5, p. 169] considered the following possible properties of a Minkowski space. He said that a normed space has

Property (A), if every complete set in the space is of constant width,

Property (B), if every complete set in the space is a ball,

Property (C), if every convex body of constant width in the space is a ball.

A space $(\mathbb{R}^n, \|\cdot\|)$ not having property (C) contains a non-symmetric (that is, not centrally symmetric) convex body K such that $K - K$ is a ball. Therefore, the unit ball B is *reducible*, that is, it has a non-symmetric summand. As Yost [25] noted, for $n \geq 3$ most (in the Baire category sense) o -symmetric convex bodies are irreducible (not reducible). He deduced (Corollary 22 and its proof) that most n -dimensional Minkowski spaces ($n \geq 3$) have a smooth and strictly convex unit ball, have property (C), and hence do not have property (A). On the other hand, for a space $(\mathbb{R}^n, \|\cdot\|)$ to have property (B), it is necessary and sufficient that its unit ball is a parallelepiped. The sufficiency was shown by Eggleston [5] and the necessity by Soltan [24].

Every bounded set M in $(\mathbb{R}^n, \|\cdot\|)$ is contained in a complete set of the same diameter. Any such complete set is called a *completion* of M ; it is in general not unique. The *diametric completion map* γ associates with every convex body $K \in \mathcal{K}^n$ the set $\gamma(K)$ of its completions.

The results quoted above show clearly that the class of bodies of constant width, which has such a rich theory in Euclidean spaces, is rather poor in typical normed spaces, and that the class of complete sets should be studied instead. The first aim of this note is to characterize complete sets by properties which are obtained by modifying characteristic properties of bodies of constant width.

The second part of the note is motivated by the study of some structural properties of the set of complete bodies. Depending on the nature of a given Minkowski space, the following properties may or may not be satisfied.

Property (D): The sum of any two complete sets is complete.

Property (E): The diametric completion map γ is convex-valued.

Property (F): The sum of any complete set and a ball is complete.

By (E) we mean, of course, that for each $C \in \mathcal{K}^n$ and any two completions K, K' of C , the Minkowski combination $\alpha K + (1 - \alpha)K'$ is a completion of C , for all $\alpha \in [0, 1]$. The first example of a Minkowski space not satisfying property (F) was given by Naszodi and Visy [17].

Proposition 1. *For a space $(\mathbb{R}^n, \|\cdot\|)$, properties (A), (D) and (E) are equivalent. Property (F) is strictly weaker.*

The implication (A) \Rightarrow (D) is trivial. Suppose that (D) holds. Let K be a complete convex body. Then also $-K$ is complete. By (D), $K - K$ is complete. Since it is centrally symmetric, one easily proves (or see [2, Proposition 32.4]) that it must be a ball. Hence, K is of constant width. Thus (A) holds. The equivalence of (A) and (E) was proved in [16, Prop. 6]. Trivially, (A) implies (F). That property (F) is strictly weaker than (A), is shown by the example of the space ℓ_1^3 . It is known that it does not satisfy (A) (see [5]). That this space satisfies (F), was deduced in [14] from some earlier results. We shall give a short direct proof at the beginning of Section 3.

The fact that most n -dimensional Minkowski spaces (for $n \geq 3$) do not have properties (A), (D), (E), shows that the concept of diametrically complete sets is not well compatible with the linear structure of a vector space. The second aim of this note is to demonstrate an even stronger incompatibility: we strengthen the result of Yost, replacing property (A) by the weaker property (F). Note that (F) no longer deals with bodies of constant width; it expresses a pure property of the system of complete sets.

The question, already raised by Eggleston [5], of characterizing the Minkowski spaces which have property (A) (and thus also (D) and (E)), remains open. The third aim of this note is to obtain some new restrictions for these spaces. The unit balls of such spaces have special intersection properties. In Section 4, the available information is supplemented by a new necessary condition for general norms, with an improvement for polyhedral norms.

2 Characterizations of complete sets

In this section, we work in a given Minkowski space $(\mathbb{R}^n, \|\cdot\|)$, with unit ball B .

Complete sets have the following characterization, also known as the *spherical intersection property* ([5]). A convex body K of diameter d is complete if and only if

$$K = \bigcap_{x \in K} B(x, d). \tag{1}$$

Another characterization ([5, p. 167]) says that K is complete if and only if each boundary point of K is at distance d from some other point of K .

Further characterizations require some notation. Let $K \in \mathcal{K}^n$. If H is a hyperplane, we denote by $w(K, H)$ the distance between the two supporting hyperplanes of K parallel to H and call this the *width* of K in direction H . It is well known (e.g., [2, Th. 32.2]) that the diameter of K is equal to its maximal width. The set Σ bounded by two parallel supporting hyperplanes H, H' of K is called a *supporting slab* of K ,

and $w(K, H) = w(K, H')$ is also the width of Σ . Every segment with one endpoint in $K \cap H$ and the other in $K \cap H'$ is called a *diametrical chord* of K , and it is said to be *generated by Σ* .

A convex body $K \in \mathcal{K}^n$ of diameter d is of constant width if and only if every supporting slab of K has width d . The following is proved, for example, in [4, p. 55, (IV')]. *The convex body K is of constant width d if and only if all diametrical chords of K have length d .* To obtain a counterpart of these two characterizations for complete sets, we say that a supporting slab Σ of K is *regular* if at least one of its bounding hyperplanes contains a smooth boundary point of K . (A smooth, or regular, point of K is a boundary point at which there is a unique supporting hyperplane of K .) A diametrical chord is called *regular* if it is generated by some regular supporting slab.

We note that every convex body $K \in \mathcal{K}_0^n$ is the intersection of its regular supporting slabs. This follows from the fact that the smooth points are dense in the boundary of K (a consequence of Reidemeister's theorem, see, for example, [23], Note 2 on page 79), hence to every point $x \in \mathbb{R}^n \setminus K$ there exists a supporting hyperplane H of K that contains a smooth boundary point of K and is such that x lies in the open halfspace bounded by H that is disjoint from K . From this it also follows that through every boundary point of K there passes a supporting hyperplane which is a limit of supporting hyperplanes through smooth boundary points of K .

A supporting slab of a convex body K is called *B-regular* if it is parallel to a regular supporting slab of B . The following remark shows the relevance of the *B-regular* supporting slabs.

Remark. The diameter of a convex body K is the supremum of the widths of all *B-regular* supporting slabs of K . For the proof, let $\text{diam } K = d$. Then d is the maximal width of K and hence is equal to half the diameter of $K - K$. Since $K - K$ is α -symmetric, we have $K - K \subset dB$, and $K - K$ and dB have a common boundary point x . As noted above, through x there exists a supporting hyperplane H of dB (and hence of $K - K$) which is the limit of a sequence $(H_j)_{j \in \mathbb{N}}$ of supporting hyperplanes of dB through smooth boundary points of dB . It follows that

$$2d = w(dB, H) = w(K - K, H) = \lim_{j \rightarrow \infty} w(K - K, H_j) = 2 \lim_{j \rightarrow \infty} w(K, H_j),$$

which gives the assertion.

The fact observed in the remark is specially useful when the unit ball B is a polytope, since then it has only a finite number of regular supporting slabs and hence the supremum becomes a maximum.

Now we turn to the announced characterizations of complete sets.

Theorem 1. *For a convex body $K \in \mathcal{K}_0^n$ of diameter d the following assertions are equivalent.*

- (a) *K is complete.*
- (b) *Every regular supporting slab of K has width d .*
- (c) *Every regular diametrical chord of K has length d .*

Proof. Suppose that (b) holds. Let $y \in \mathbb{R}^n \setminus K$. As noted above, there exists a regular supporting slab Σ of K with $y \notin \Sigma$. By assumption, the slab Σ has width d . Therefore, there exists a point $x \in K$ with $\|y - x\| > d$. Since $y \notin K$ was arbitrary, this shows that K is complete, thus (a) holds.

Conversely, suppose that (a) holds. Assume that K has a regular supporting slab Σ of width $< d$. Let H_1, H_2 be the bounding hyperplanes of Σ and let $x \in H_1$, say, be a smooth boundary point of K . By continuity, there are a number $\epsilon > 0$ and a neighbourhood \mathcal{N} of H_1 (with respect to the usual topology on the space of hyperplanes) such that each hyperplane $H \in \mathcal{N}$ satisfies $w(K, H) \leq d - \epsilon$. We assert that there exists a positive number $\epsilon' \leq \epsilon$ such that the distance of x from any supporting hyperplane of K not in \mathcal{N} is greater than ϵ' . Suppose this is false. Then there exists a sequence $(H^j)_{j \in \mathbb{N}}$ of supporting hyperplanes of K , not contained in \mathcal{N} , such that the distance of x from H^j converges to 0 for $j \rightarrow \infty$. A subsequence of $(H^j)_{j \in \mathbb{N}}$ converges to a hyperplane H , necessarily a supporting hyperplane of K and passing through x . Since $H^j \notin \mathcal{N}$ for all j , we have $H \neq H_1$. This contradicts the fact that x is a smooth boundary point of K . This contradiction proves the existence of ϵ' .

Let $K' := \text{conv}[K \cup ((\epsilon'/2)B + x)]$. Let H, H' be two parallel supporting hyperplanes of K' . If one of the hyperplanes, say H , belongs to \mathcal{N} , then $w(K, H) \leq d - \epsilon$ and hence $w(K', H) \leq d - \epsilon + \epsilon' \leq d$. Otherwise, the distance of x from H, H' is greater than ϵ' , hence the ball $\epsilon'B + x$ lies in the interior of the slab bounded by H, H' . Therefore, $w(K', H) = w(K, H) \leq d$. We conclude that $\text{diam } K' \leq d$. Hence, K is not complete, a contradiction. Thus, (b) holds.

Suppose that (b) holds. Let Σ be a regular supporting slab of K , with boundary hyperplanes H_1, H_2 . If $p \in K \cap H_1$ and $q \in K \cap H_2$, then $\|p - q\| \leq d$, since $\text{diam } K = d$, but also $\|p - q\| \geq w(K, H_1) = d$, since (b) holds. Thus, each regular diametrical chord of K has length d . This is property (c).

Conversely, suppose that (c) holds. Let Σ be a regular supporting slab of K , with boundary hyperplanes H_1, H_2 . Let $q \in H_2$, say, be a smooth boundary point of K , and let $p \in K \cap H_1$. By the assumption, $\|p - q\| = d$. We have $K \subset B(p, d)$, and q is a boundary point of $B(p, d)$. Hence, there exists a supporting hyperplane H of $B(p, d)$ through q . It is also a supporting hyperplane of K . Since q is a smooth boundary point of K , we conclude that $H = H_2$. It follows that the slab Σ has width d . Hence, (b) holds. \square

A corollary of Theorem 1 is the result, first proved by Naszódi and Visy [17], that every smooth complete set is of constant width.

We mention two inequalities. Recall that the circumradius R of a convex body K is defined by $R = \min\{s \geq 0 : K \subset sB + z \text{ for some } z \in \mathbb{R}^n\}$, and the inradius r of K by $r = \max\{s \geq 0 : K \supset sB + z \text{ for some } z \in \mathbb{R}^n\}$.

Theorem 2. *The circumradius R and the inradius r of a complete convex body K of diameter d satisfy*

$$R \leq \frac{n}{n+1} d, \quad (2)$$

$$r \geq \frac{1}{n+1} d. \quad (3)$$

Both inequalities are sharp. If equality holds in one of the inequalities, then K is a simplex.

Proof. The proof is just a combination of known results. Inequality (2) holds for arbitrary convex bodies in any Minkowski space. A proof with complete discussion of the equality case is found in Leichtweiss [8]. Equality holds if and only if K is a simplex. The possible unit balls in the equality case are not uniquely determined (see [8]), but the difference body of an extremal simplex K is among them. In that case, K is a body of constant width and thus complete; hence, even for complete bodies, (2) cannot be improved.

By a result of Sallee [21] and in view of the characterization (1), the circumradius R and the inradius r of any complete convex body of diameter d satisfy $R + r = d$. This yields the remaining assertions. \square

3 Complete sets in typical Minkowski spaces

As announced at the end of Section 1, we first show that the space ℓ_1^3 , that is, \mathbb{R}^3 with unit ball B given by an octahedron with centre at the origin, satisfies property (F). Let K be a complete convex body of diameter d in this space. By (1), K is a polytope whose facets are parallel to facets of B . Therefore, K has at most four regular supporting slabs. It cannot have fewer regular supporting slabs, since any intersection of halfspaces which are translates of the supporting halfspaces of B with more than one symmetric pair missing, is unbounded, and with one symmetric pair missing, is a parallelepiped, which has diameter larger than d if it satisfies condition (b) of Theorem 1. Let $r > 0$. The sum $K + rB$ has four regular supporting slabs parallel to those of K , each of width $d + 2r$. Suppose that $K + rB$ has an additional regular supporting slab. Then $K + rB$ has a facet F which is not parallel to a facet of B and this is not possible since $K + rB$ is an intersection of closed balls (a more general result is Theorem 3.3 in [15]). Hence, all regular supporting slabs of $K + rB$ have the same width, and by Theorem 1, $K + rB$ is complete. Thus, the space satisfies property (F).

For convenience, we introduce on \mathbb{R}^n an auxiliary scalar product $\langle \cdot, \cdot \rangle$. We use it, for example, in the following notation for hyperplanes and closed halfspaces. For vectors $u \neq o$ and for $\tau \in \mathbb{R}$ we write

$$H(u, \tau) := \{x \in \mathbb{R}^n : \langle x, u \rangle = \tau\}, \quad H^-(u, \tau) := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq \tau\}.$$

The supporting function of a convex body K is defined by

$$h(K, u) := \max\{\langle x, u \rangle : x \in K\},$$

and $H(K, u) = H(u, h(K, u))$ is the supporting hyperplane of K with outer normal vector u . The set \mathcal{K}^n of convex bodies is equipped with the Hausdorff metric ρ that is induced by the Euclidean metric corresponding to the scalar product.

We turn to properties of typical Minkowski spaces. For this, we denote by \mathcal{S}_0^n the set of all o -symmetric convex bodies in \mathcal{K}_0^n . Equipped with the Hausdorff metric, this is

a complete metric space. Every $B \in \mathcal{S}_0^n$ is the unit ball of a norm on \mathbb{R}^n , and conversely. If (P) is a property of n -dimensional Minkowski spaces, we say that B has property (P) if \mathbb{R}^n with the norm defined by the unit ball B has this property. For convenience, we use \mathcal{S}_0^n with the Hausdorff metric to formulate results on the space of n -dimensional Minkowski spaces. It would not be difficult to reformulate our results in terms of the space of isometry classes of n -dimensional Minkowski spaces with the Banach–Mazur metric.

Theorem 3. *Let $n \geq 3$. The set of all unit balls in \mathcal{S}_0^n not having property (F) is open and dense in \mathcal{S}_0^n .*

Corollary 1. *Let $n \geq 3$. The set of all unit balls in \mathcal{S}_0^n which are smooth and strictly convex, have property (C) and do not have property (F), is a dense G_δ set.*

Corollary 1 follows from Theorem 3 by using known results and Baire’s Category Theorem, as in Yost [25]. He proved a similar assertion, with property (A) instead of property (F).

Unit balls $B, B_i \in \mathcal{S}_0^n$ determine norms $\|\cdot\|$ and $\|\cdot\|_i$ on \mathbb{R}^n , and the corresponding diameters are denoted by $\text{diam}_{\|\cdot\|} = D$ and $\text{diam}_{\|\cdot\|_i} = D_i$, respectively. By $\mathcal{D}(B)$ we denote the set of diametrically complete bodies in $(\mathbb{R}^n, \|\cdot\|)$. The bodies in $\mathcal{D}(B)$ are called B -complete. The following result includes the fact that $\mathcal{D}(B)$ is closed, but is more general.

Lemma 1. *Let $B_i, B \in \mathcal{S}_0^n$ be unit balls with $B_i \rightarrow B$, let $C_i, C \in \mathcal{K}$ be convex bodies with $C_i \rightarrow C$. If C_i is B_i -complete for each i , then C is B -complete.*

Proof. Since $B_i \rightarrow B$ in the Hausdorff metric and o is in the interior of B and B_i , there exist numbers $0 < \lambda_i < 1$ with $\lambda_i \rightarrow 1$ and

$$\lambda_i B \subset B_i \subset \lambda_i^{-1} B, \quad \text{hence} \quad \lambda_i \|\cdot\| \leq \|\cdot\|_i \leq \lambda_i^{-1} \|\cdot\|.$$

Let $x \in \text{bd } C$. Since $C_i \rightarrow C$, there exist points $x_i \in \text{bd } C_i$ with $x_i \rightarrow x$. Since C_i is B_i -complete, there exists $y_i \in C_i$ with $\|x_i - y_i\|_i = D_i(C_i)$. After going over to a subsequence and changing the notation, we can assume that the sequence $(y_i)_{i \in \mathbb{N}}$ converges to some point y . Then $y \in C$. From $x_i - y_i \rightarrow x - y$, $\|\cdot\|_i \rightarrow \|\cdot\|$ and $C_i \rightarrow C$ we conclude that $\|x - y\| = D(C)$. Since x was an arbitrary boundary point of C , the body C is B -complete. \square

Proof of Theorem 3.

First we show that the set of convex bodies in \mathcal{S}_0^n having property (F) is closed in \mathcal{S}_0^n . Suppose that $B_i, B \in \mathcal{S}_0^n$, that B_i has property (F), and $B_i \rightarrow B$. Let $C \in \mathcal{D}(B)$ and $r > 0$. We have to show that $C + rB \in \mathcal{D}(B)$. It would be possible to derive this from continuity properties of the diametric completion map with respect to Hausdorff metrics that were proved in [16] (Theorem 1 and Proposition 4), but since their proofs are more complicated, we give here a short direct proof.

In \mathbb{R}^n with a given norm, complete sets have interior points, and any set of diameter d is contained in a ball of diameter $(2n/(n+1))d$, as follows from (2). Therefore, we may assume that o is an interior point of C and that $C \subset \tau D(C)B$ with a number $\tau < 1$.

Let $0 < \lambda < 1$ and $\mu > 1$ be given, and choose $\alpha > 1$ with

$$[\mu - (\mu - \lambda)\tau] > \alpha^2\lambda.$$

Since $B_i \rightarrow B$ in the Hausdorff metric and o is in the interior of B , there exists $i_0 \in \mathbb{N}$ with

$$\alpha^{-1}B \subset B_i \subset \alpha B, \quad \text{hence} \quad \alpha^{-1}\|\cdot\| \leq \|\cdot\|_i \leq \alpha\|\cdot\|, \quad \text{for } i \geq i_0. \quad (4)$$

Now let $i \geq i_0$ be given. Let C_i be a B_i -completion of λC . Then $\lambda C \subset C_i$ and $D_i(C_i) = D_i(\lambda C)$.

Let $y \in \text{bd } \mu C$, then $\mu^{-1}y \in \text{bd } C$. Since C is B -complete, there exists a point $z \in C$ with $\|\mu^{-1}y - z\| = D(C)$, hence $\|y - \mu z\| = \mu D(C)$. This gives

$$\begin{aligned} \mu D(C) &= \|y - \mu z\| \leq \|y - \lambda z\| + \|\lambda z - z\| + \|z - \mu z\| \\ &= \|y - \lambda z\| + (1 - \lambda)\|z\| + (\mu - 1)\|z\| \leq \|y - \lambda z\| + (\mu - \lambda)\tau D(C) \end{aligned}$$

and hence

$$\|y - \lambda z\| \geq [\mu - (\mu - \lambda)\tau]D(C) > \alpha^2\lambda D(C) \geq D(C_i),$$

the latter because of

$$D(C_i) \leq \alpha D_i(C_i) = \alpha D_i(\lambda C) \leq \alpha^2 D(\lambda C) = \alpha^2 \lambda D(C),$$

where (4) was used. Since $\lambda z \in C_i$, it follows that $y \notin C_i$. Since y was an arbitrary boundary point of μC , this shows that $C_i \subset \mu C$.

We have proved that $\lambda C \subset C_i \subset \mu C$ for all $i \geq i_0$. Since $\lambda < 1$ and $\mu > 1$ were arbitrary, this shows that $C_i \rightarrow C$. Since also $B_i \rightarrow B$, we have $C_i + rB_i \rightarrow C + rB$. Since C_i is B_i -complete and B_i has property (F), the body $C_i + rB_i$ is B_i -complete. From Lemma 1 it follows that $C + rB$ is B -complete. Hence, B has property (F). This completes the proof of the fact that the set of bodies in \mathcal{S}_0^n with property (F) is closed.

From now on we assume that $n \geq 3$. We have to show that the set of unit balls in \mathcal{S}_0^n not having property (F) is dense.

Let a unit ball $K \in \mathcal{S}_0^n$ and a number $\epsilon > 0$ be given. Recall that the set of o -symmetric polytopes is dense in \mathcal{S}_0^n . Therefore, we can choose an n -dimensional polytope $P \in \mathcal{S}_0^n$ with $\rho(P, K) < \epsilon$. By $N(P)$ we denote the system of normal vectors (that is, of outer unit normal vectors of the facets) of P .

We choose a unit vector e and a number τ such that

$$P \subset H^-(e, \tau) \quad \text{and} \quad H(e, \tau) \cap P = \{p\}$$

for some vertex p of P . Then we choose a number $\alpha > 0$ so small that the hyperplane $H(e, \tau - \alpha)$ meets the relative interior of each facet of P that contains p , and does not

meet any other facet of P . We put $P_\alpha := P \cap H^-(e, \tau - \alpha)$. By choosing α sufficiently small, we achieve that $\rho(P_\alpha \cap -P_\alpha, P) < \epsilon$. We choose a point $z \in \text{int}(P \setminus P_\alpha)$.

Let e_1, e_2 be unit vectors orthogonal to each other and to e . With a number $\lambda > 0$ we define

$$v_1 := e + \lambda e_1, \quad v_2 := e - \lambda e_1, \quad v_3 := e + \lambda e_2, \quad v_4 := e - \lambda e_2. \quad (5)$$

Since $H(e, \langle z, e \rangle) \cap P_\alpha = \emptyset$, we can choose the number $\lambda > 0$ so small that

$$H(v_i, \langle z, v_i \rangle) \cap P_\alpha = \emptyset \quad \text{for } i = 1, \dots, 4.$$

Then we can choose a number $\eta > 0$ such that for $|\tau| \leq \eta$ we still have

$$H(v_i, \langle z, v_i \rangle + \tau) \cap P_\alpha = \emptyset \quad \text{for } i = 1, \dots, 4 \quad (6)$$

and

$$H(v_3, \langle z, v_3 \rangle + \tau) \cap H(v_4, \langle z, v_4 \rangle + \tau) \cap \text{int } P \neq \emptyset. \quad (7)$$

With numbers $0 < \tau_1 < \tau_2 < \eta$ we define the polytopes

$$B := P \cap \bigcap_{i=1}^2 \pm H^-(v_i, \langle z, v_i \rangle - \tau_1) \cap \bigcap_{i=3}^4 \pm H^-(v_i, \langle z, v_i \rangle) \quad (8)$$

and

$$C := P \cap \bigcap_{i=1}^2 \pm H^-(v_i, \langle z, v_i \rangle - \tau_1) \cap \bigcap_{i=3}^4 H^-(v_i, \langle z, v_i \rangle - \tau_2) \cap \bigcap_{i=3}^4 -H^-(v_i, \langle z, v_i \rangle + \tau_2). \quad (9)$$

We can choose η so small that each of the polytopes B and C has the same system of normal vectors as the polytope

$$P \cap \bigcap_{i=1}^4 \pm H^-(v_i, \langle z, v_i \rangle)$$

and hence

$$N(B) = N(C) = N(P) \cup \{v_i\}_{i=1}^4.$$

Further, we note that $\rho(P_\alpha \cap -P_\alpha, P) < \epsilon$ implies $\rho(B, P) < \epsilon$ and hence $\rho(B, K) < 2\epsilon$. The polytope B is o -symmetric and thus is the unit ball of a norm $\|\cdot\|$ on \mathbb{R}^n . In the following, the Minkowski space notions diameter, width, $\mathcal{D}(B)$ refer to this norm.

Intuitively speaking, the polytope C is obtained from the polytope B by pushing the supporting hyperplanes with outer normal vectors v_3, v_4 inwards and pulling the supporting hyperplanes with outer normal vectors $-v_3, -v_4$ outwards. The aim of this construction is to achieve that, with respect to the norm with unit ball B , the body C is complete, whereas the sum $C + B$ is not complete. To show this, we shall make use of Theorem 1.

First we show that $C \in \mathcal{D}(B)$. Notice that every regular supporting slab of P is also a regular supporting slab of B and a regular supporting slab of C . The supporting slabs

of C with normal vectors v_1, v_2 are also supporting slabs of B . The supporting slab of C with normal vector v_3 arises from the parallel supporting slab of B by a translation (namely, by the vector $-(\tau_2/\langle v_3, v_3 \rangle)v_3$), and similarly for the supporting slab of C with normal vector v_4 . Since C has no other regular supporting slabs, it follows that each regular supporting slab of C has width 2. In particular, $\text{diam } C \geq 2$. Since the norm is polyhedral, the diameter of C is the maximum of the widths of its supporting slabs which are parallel to the facets of the unit ball, B . Since $N(B) = N(C)$, we conclude that $\text{diam } C = d$ and then, from Theorem 1, that C is B -complete.

Now we consider the sum $C + B$. Each regular supporting slab of $C + B$ that is parallel to a supporting slab of B has width 4. But $C + B$ has an additional regular supporting slab, namely the one with normal vector e . In fact, let

$$H(v_1, \langle z, v_1 \rangle - \tau_1) \cap H(v_2, \langle z, v_2 \rangle - \tau_1) =: E.$$

This is an $(n - 2)$ -dimensional affine subspace parallel to $\text{lin}\{e, e_1\}^\perp$, which satisfies $E \subset H(B, e)$ and $\dim(E \cap B) = n - 2$. Further, let

$$H(v_3, \langle z, v_3 \rangle - \tau_2) \cap H(v_4, \langle z, v_4 \rangle - \tau_2) =: F.$$

Then F is an $(n - 2)$ -dimensional affine subspace parallel to $\text{lin}\{e, e_2\}^\perp$, satisfying $F \subset H(C, e)$ and $\dim(F \cap C) = n - 2$ (here (7) is used). It follows that

$$H(C + B, e) \cap (C + B) = (F \cap C) + (E \cap B),$$

and this set has dimension $n - 1$ and thus is a facet of $C + B$. Therefore, the supporting slab of $C + B$ with normal vector e is regular. However, the assumption $\tau_1 < \tau_2$ and the way the vectors v_i were defined imply that

$$h(C, e) < h(B, e)$$

whereas

$$h(C, -e) = h(B, -e).$$

Therefore, the supporting slab of $C + B$ with normal vector e is properly contained in the parallel supporting slab of $2B$ and hence has width less than 4. By Theorem 1, the body $C + B$ is not B -complete.

To the given unit ball $K \in \mathcal{S}_0^n$ and the given number $\epsilon > 0$, we have constructed a unit ball $B \in \mathcal{S}_0^n$ with $\rho(B, K) < 2\epsilon$ and such that there exists a convex body $C \in \mathcal{D}(B)$ for which $C + B \notin \mathcal{D}(B)$. This completes the proof of Theorem 3. \square

4 Perfect norms and generating sets

A Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ has Eggleston's property (A) if in this space every complete set is necessarily of constant width, or equivalently, if every set of diameter d is contained in a set of constant width d . Eggleston's statement ([5], p. 169), "It is not easy to see what simple property distinguishes the spaces that have property (A) from those which do not have property (A)", is still true 45 years later. Chakerian and Groemer [4],

p. 62, explicitly mentioned the problem of characterizing those Minkowski spaces for which the concepts of completeness and constant width coincide. Adopting terminology introduced by Karasëv [7], we say that a Minkowski space with this property and its norm are *perfect*. After recalling some known facts on perfect norms, we conclude this note with necessary conditions.

A convex body $K \subset \mathbb{R}^n$ is called a *generating set* (a terminology introduced by Polovinkin) if any nonempty intersection of translates of K is a summand of K .

For a nonempty bounded set $A \subset \mathbb{R}^n$, let

$$\eta(A) := \bigcap_{x \in A} B(x, \text{diam}A), \quad \theta(A) := \bigcap_{x \in \eta(A)} B(x, \text{diam}A).$$

In [16], the convex body $\eta(A)$ was called the *wide spherical hull* and $\theta(A)$ the *tight spherical hull* of A . It was shown in [13], Proposition 2, that $\eta(A)$ is the union and $\theta(A)$ is the intersection of all complete sets of diameter $d = \text{diam}A$ containing A . With ‘complete sets’ replaced by ‘bodies of constant width’ and under the assumption that the unit ball is a generating set, this was proved in [20], Theorem 3.

The following proposition and its consequences show how these spherical hulls are related to perfect norms. Although the proposition is essentially known, we present the short proof, referring to known properties of the Minkowski difference, and then explain its history. Recall that the *Minkowski difference* of the sets $X, T \subset \mathbb{R}^n$ is defined by

$$X \sim T := \bigcap_{t \in T} (X - t) = \{y \in \mathbb{R}^n : T + y \subset X\}.$$

In particular, for a set A of diameter d we have

$$\eta(A) = dB \sim (-A), \quad \theta(A) = dB \sim (-\eta(A)) = dB \sim (dB \sim A),$$

where the definition of the Minkowski difference together with $B = -B$ was used.

Proposition 2. *Let A be a set of diameter 1 in the space $(\mathbb{R}^n, \|\cdot\|)$ with unit ball B . If one of the sets $\eta(A)$, $\theta(A)$ is a summand of B , then the set*

$$\frac{1}{2}[\eta(A) + \theta(A)]$$

is a body of constant width 1 that contains A .

Proof. We have $\eta(A) - \theta(A) = \eta(A) - (B \sim (-\eta(A))) = \eta(A) + (B \sim \eta(A))$. If now $\eta(A)$ is a summand of B , then it follows from Lemma 3.1.8 in [23] that $\eta(A) + (B \sim \eta(A)) = B$, hence $\eta(A) - \theta(A) = B$. This yields

$$\frac{1}{2}[\eta(A) + \theta(A)] - \frac{1}{2}[\eta(A) + \theta(A)] = \frac{1}{2}[\eta(A) - \theta(A)] + \frac{1}{2}[\theta(A) - \eta(A)] = B,$$

thus $\frac{1}{2}[\eta(A) + \theta(A)]$ is a body of constant width 1.

Second, suppose that $\theta(A)$ is a summand of B . Then there is a convex body K with $\theta(A) + K = B$, which gives $K = B \sim \theta(A) = B \sim (B \sim (B \sim A))$. Since $B \sim A$

is an intersection of translates of B , Lemma 3.2.4 of [23] gives $B \sim (B \sim (B \sim A)) = B \sim A = -\eta(A)$. Thus, $\theta(A) - \eta(A) = B$, and we can conclude as before. \square

The assumption of Proposition 2 is *a fortiori* satisfied if the unit ball B is a generating set. If a Minkowski space has the property that every set of diameter 1 is contained in a set of constant width 1, then the norm is perfect.

In Euclidean spaces, the fact that $\frac{1}{2}[\eta(A) + \theta(A)]$ is a set of constant width equal to $\text{diam } A$ and containing A , was first proved (in different terminology) by Maehara [9]. Sallee [22] extended the result to the Minkowski spaces satisfying an assumption equivalent to the fact that $\eta(A)$, for any set A of diameter 1, is a summand of the unit ball. This result was later rediscovered by Polovinkin [19] (more generally, in reflexive Banach spaces); he assumed that the unit ball is a generating set but needed in his proof only the mentioned condition.

The preceding results have raised interest in the class of generating sets, since they yield examples of perfect norms. It has been known for a long time that two-dimensional convex bodies are generating sets (see Theorem 3.2.3 in [23] and the references given there in Note 3 for Section 3.2). Maehara [9] was the first to prove that Euclidean balls are generating sets. This was rediscovered by Polovinkin [18] (a different proof appears in [7], and an extension to Hilbert spaces in [1]). It is easy to see that the system of generating sets is stable under linear transformations and direct sums. From [11] (see assertions (36), (37), (43)), the n -dimensional convex polytopes ($n \geq 3$) which are generating sets are explicitly known in the following cases: in dimension three (see also [3]), in dimension n if they have at most $n + 3$ facets, in dimension n if they are centrally symmetric. A centrally symmetric polytope is a generating set if and only if it is a direct sum of polygons and, in odd dimension, a segment. New examples of generating sets (under the name of ‘Sallee sets’) in \mathbb{R}^3 were constructed by Borowska and Grzybowski [3]; none of their examples is centrally symmetric.

Proposition 2 has two corollaries (due to the definition of $\eta(A)$ and the fact that $\text{diam } \theta(A) = \text{diam } A$). The first was proved in a different way by Karasëv [7]; the second seems to be new.

Corollary 2. *Suppose that the unit ball B of the norm has the property that every intersection*

$$\bigcap_{y \in T} (B + y) \quad \text{with} \quad \text{diam } T \leq 1$$

is a summand of B . Then the norm is perfect.

Corollary 3. *Suppose that the unit ball B of the norm has the property that every intersection*

$$\bigcap_{y \in T} (B + y) \quad \text{of diameter } 1$$

is a summand of B . Then the norm is perfect.

Karasëv [7] has strengthened Corollary 2 by part (a) of the following proposition, showing that one need only consider intersections of two translates at a time. Part (b) is a partial converse, also due to Karasëv [7].

Proposition 3. (a) *If $B \cap (B + u)$ is a summand of B for all $u \in \mathbb{R}^n$ with $\|u\| \leq 1$, then the norm is perfect.*

(b) *If a norm is strictly convex and perfect, then $B \cap (B + u)$ is a summand of B whenever $\|u\| \leq 1$.*

The sufficient condition for a perfect norm given by part (a) is not necessary, as shown by an example due to Karasëv [7]. We modify and explain this example below, after the proof of Theorem 4. The unit ball B of the perfect norm in Karasëv's example, however, still has the property that $B \cap (B + u)$, for $\|u\| \leq 1$, is positively homothetic to a summand of B . This observation leads us to a necessary condition for general perfect norms. For its proof, we need the following strengthening of the sufficiency part of a known criterion for summands (Theorem 3.2.2 in [23]).

Lemma 2. *Let $K, L \in \mathcal{K}^n$. Suppose that for each supporting hyperplane H of K there exist a point $x \in H \cap K$ and a vector $t \in \mathbb{R}^n$ such that $x \in L + t \subset K$. Then L is a summand of K .*

Proof. From the assumption it follows that $K \sim L \neq \emptyset$, and from (3.1.12) in [23] it then follows that $(K \sim L) + L \subset K$. Suppose this inclusion is proper. Then K has a supporting hyperplane H such that $H \cap ((K \sim L) + L) = \emptyset$. By assumption, there exist $x \in H \cap K$ and $t \in \mathbb{R}^n$ with $x \in L + t \subset K$. This implies that $t \in K \sim L$ and hence $x \in L + (K \sim L)$, a contradiction. Thus $(K \sim L) + L = K$, which shows that L is a summand of K . \square

Theorem 4. *If the norm on \mathbb{R}^n with unit ball B is perfect, then $\frac{1}{2}(B \cap (B + u))$ is a summand of B whenever $\|u\| \leq 1$.*

Proof. We modify the proof of Theorem 3 in [7]. Let $u \in \mathbb{R}^n$ be a vector with $\|u\| \leq 1$. Let H be an arbitrary supporting hyperplane of $B \cap (B + u)$ and choose a point $y \in H \cap B \cap (B + u)$. Put

$$S := \frac{1}{2}((B \cap (B + u)) - y) + y,$$

thus S arises from $B \cap (B + u)$ by a dilatation with centre y and homothety factor $1/2$. We have $S \subset B \cap (B + u)$ and $\text{diam } S \leq 1$, hence the set $S \cup \{o, u\}$ has diameter 1 (note that y has distance 1 from o and from u). Let C be a completion of this set. Then $S \subset C$ and $C \subset B \cap (B + u)$, since $o, u \in C$ and $\text{diam } C = 1$. Hence, H is a supporting hyperplane to C that contains y . Let H' be the other supporting hyperplane of C parallel to H . Since the norm is perfect, the set C is of constant width, hence H' is at distance 1 from H . There is a point $y' \in C \cap H'$, and we have $C \subset B + y'$. The hyperplane H supports $B + y'$, since it has distance 1 from y' . Since the direction of the hyperplane H was arbitrary, we have shown that the bodies $K = B$ and $L = \frac{1}{2}(B \cap (B + u))$ satisfy the assumption of Lemma 2. The assertion follows. \square

By a modification of Karasëv's [7] example we show that the homothety factor $1/2$ in Theorem 4 is best possible. In \mathbb{R}^3 , let the unit ball B be obtained from the cube with vertices $(\pm 1, \pm 1, \pm 1)$ by cutting off the vertex $(1, 1, 1)$ by the plane through the points $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, and cutting off the opposite vertex symmetrically

(note that this yields Dürer's octahedron from his MELENCOLIA I). The norm with unit ball B is perfect. We prove this as follows. Let K be a complete convex body with respect to B , say of diameter 1. By (1), K is an intersection of translates of B , hence $K = \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \cap \Sigma_4$, where Σ_i , $i = 1, \dots, 4$, are the supporting slabs of K parallel to the facets of B , with Σ_4 corresponding to the triangular facets. Suppose, first, that $\Sigma_1, \Sigma_2, \Sigma_3$ are regular slabs of K . Then, by Theorem 1, they have width 1. Their intersection is a cube C . Clearly, Σ_4 has width ≤ 1 , since otherwise $\text{diam } K > 1$. If Σ_4 has width < 1 , then K is not complete, since suitable points from C outside Σ_4 can be added without increasing the diameter. On the other hand, suppose that one of the supporting slabs $\Sigma_1, \Sigma_2, \Sigma_3$, say Σ_1 , is not regular. Then $K = \Sigma_2 \cap \Sigma_3 \cap \Sigma_4$ and, since K has nonempty interior, Σ_2, Σ_3 and Σ_4 are regular, hence they have width 1. This implies that K has diameter > 1 , a contradiction. We have proved that every B -regular supporting slab of K is regular. Further, K has the following property. Whenever H, H' are two parallel supporting planes of K such that $H \cap K$ is an edge of K , then $H' \cap K$ is a parallel edge of K or is one-pointed. Therefore, $K - K$ and B have the same regular supporting slabs, and since $K - K \subset B$, they are identical, thus K has constant width. This shows that the norm with unit ball B is perfect. For the vector $u = (1, -1, 0)$ of length 1, the intersection $B \cap (B + u)$ has an edge of length 2 for which the corresponding edge of B has length 1. Hence, no homothet $\alpha(B \cap (B + u))$ with $\alpha > 1/2$ can be a summand of B .

For suitable polyhedral norms, however, an improvement is possible.

Theorem 5. *Suppose that the norm $\|\cdot\|$ on \mathbb{R}^n is perfect and its unit ball B is a polytope. Let D denote the maximal diameter of the proper faces of B and put*

$$\alpha := \min\{1/D, 1\} \quad (\text{so that } \alpha \geq 1/2).$$

Then $\alpha(B \cap (B + u))$ is a summand of B whenever $\|u\| \leq 1$.

Proof. Again we modify the proof of Theorem 3 in [7], but in a different way. Let $u \in \mathbb{R}^n$ be a vector with $\|u\| \leq 1$ and put $Y := B \cap (B + u)$. Let E be an edge of the polytope Y and H a supporting hyperplane of Y such that $H \cap Y = E$. Let $S \subset E$ be a subsegment which is a translate of αE . Since the edge E is the intersection of a k -face of B and a $(n + 1 - k)$ -face of $B + u$, for some $k \in \{1, \dots, n\}$, it has diameter at most D . Hence, S has diameter at most 1. The set $S \cup \{o, u\}$ has diameter 1; let C be a completion of this set. Then $S \subset C$ and $C \subset Y$. Hence, H is a supporting hyperplane to C that contains S . Let H' be the other supporting hyperplane of C parallel to H . As in the previous proof, H' is at distance 1 from H , and for a point $y' \in C \cap H'$ we have $C \subset B + y'$. The hyperplane H supports $B + y'$. Thus, the support set $(B + y') \cap H$ contains S . It follows that the support set $B \cap (H - y')$ of B contains a translate of αE . Since E was an arbitrary edge of Y and H was an arbitrary supporting hyperplane of Y with $H \cap Y = E$, it follows from Theorem 3.2.8 in [23] that the polytope αY is a summand of B . \square

A complete classification of the convex bodies with the property of Theorem 4 seems to be difficult. Even the corresponding question for polytopes is open. Some light is shed on the difficulty of this question if one observes that Karasëv's example can be modified with considerable freedom. From the given cube, one can cut off parts by

several planes (up to eight) in a centrally symmetric way, each plane cutting off one or two vertices. If the cut-offs are sufficiently close to the edges of the cube, and different parts cut off have distance larger than one from each other, then one can show that the norm with the obtained unit ball is perfect.

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