Extremal properties of random mosaics

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László Fejes Tóth's fascinating book [2] demonstrates in many ways the phenomenon that figures of discrete or convex geometry that are very economical, namely solving an extremal problem of isoperimetric type, often show a high degree of symmetry. Among the examples are also planar mosaics where, for instance, an extremal property leads to the hexagonal pattern. Mosaics, or tessellations, have become increasingly important for applications. Random tessellations in two or three dimensions have been suggested as models for various real structures. We refer, e.g., to chapter 10 of the book by Stoyan, Kendall, Mecke [31] and to the book by Okabe, Boots, Sugihara, and Chiu [25] on Voronoi tessellations, which also contains a chapter on random mosaics. Apart from possible applications, random mosaics are also an interesting object of study from a purely geometric point of view. Among the results of geometric appeal that have been obtained, some concern extremal problems for (roughly speaking) expected sizes of average cells under some side condition, leading to random mosaics with high symmetry or of a very simple type, namely made up of parallelepipeds only. High symmetry here means that the distribution of the random mosaic, which is usually assumed to be translation invariant, is also invariant under rotations. Extremal problems for the sizes of average cells (not taking expectations) seem senseless at first, since extrema cannot be attained. Nevertheless, in many problems, average cells of large size approximate certain definite shapes, for example balls, segments, regular simplices, with high probability.

The purpose of the following is a survey over results, older and more recent, that have been obtained on extremal properties of random mosaics. These random tessellations are mostly of special types, namely hyperplane, Voronoi or Delaunay mosaics generated by Poisson processes, and satisfying an assumption of homogeneity, also called stationarity.

1. Explanations

We work in Euclidean space $\mathbb{R}^d$, with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Its unit ball and unit sphere are denoted, respectively, by B^d and S^{d-1}. The set $\mathcal{K}$ of convex bodies (nonempty, compact, convex subsets) in $\mathbb{R}^d$ and its subset $\mathcal{P}$ of polytopes are equipped with the topology induced by the Hausdorff metric. Lebesgue measure on $\mathbb{R}^d$ is denoted by λ, and m-dimensional Hausdorff measure by $\mathcal{H}^m$. We write $\lambda(B^d) =: \kappa_d = \pi^{d/2}/\Gamma(1+d/2)$. For a topological space T, the σ-algebra of its Borel sets is denoted by $\mathcal{B}(T)$. A 'measure' on a topological space in the following is always a measure on its Borel σ-algebra.

The mosaics to be considered will be 'face-to-face'. Therefore, by a **mosaic** in $\mathbb{R}^d$, or a **tessellation** of $\mathbb{R}^d$, we understand here a locally finite set m of d-dimensional polytopes in

$\mathbb{R}^d$ with the following properties: the polytopes of m cover $\mathbb{R}^d$, and the intersection of any two different polytopes of m is either empty or a face of both polytopes. The polytopes of m are called its **cells**, and every k-dimensional face of some cell is, by definition, a **k-face** of m ($k = 0, \dots, d$). We denote by $\mathcal{F}_k(\mathsf{m})$ the set of all k-faces of m and by $\mathrm{skel}_k\mathsf{m}$ the union of these k-faces.

Since random mosaics will be modeled as special particle processes, we must explain these first. For a detailed introduction we refer to chapters 3 and 4 of the book [30]. Here we restrict ourselves to convex particles and simple processes. Thus, by a **particle process** X in $\mathbb{R}^d$ we understand a simple point process in $\mathcal{K}$, that is, a measurable mapping from some probability space $(\Omega, \mathsf{A}, \mathbb{P})$ into the space $\mathsf{N}_s(\mathcal{K})$ of simple, locally finite counting measures on $\mathcal{K}$, equipped with the usual σ-algebra. We identify a simple counting measure η with its support and often write $x \in \eta$ for $\eta(\{x\}) = 1$. Then we can view a realization of the particle process X as a locally finite system of (generally overlapping) convex bodies in $\mathbb{R}^d$. The particle process X is **stationary** (or homogeneous) if for any $t \in \mathbb{R}^d$ the process X and its translate $X + t$ (defined in the obvious way) have the same distribution. The **intensity measure** Θ of X is defined by $\Theta(A) := \mathbb{E}X(A)$ for $A \in \mathcal{B}(\mathcal{K})$, where $\mathbb{E}$ denotes mathematical expectation. We assume (this is part of the definition of a particle process) that Θ is locally finite with respect to the hit-or-miss topology, which means that $\Theta(\{K \in \mathcal{K} : K \cap C \neq \emptyset\}) < \infty$ for each $C \in \mathcal{K}$. From now on we assume that X is stationary and that Θ is not the zero measure; then the intensity measure has a decomposition

$$\int_{\mathcal{K}} f \,\mathrm{d}\Theta = \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} f(K + x)\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K),$$

for any Θ-integrable function f on $\mathcal{K}$. Here, γ is a positive number, the **intensity** of X, and $\mathbb{Q}$, the **grain distribution** of X, is a probability measure on $\mathcal{K}_0$, the space of convex bodies K with Steiner point $s(K)$ at the origin. (Other than in [30], we use here the Steiner point as center function, which is possible and convenient in the case of convex bodies.) If $A \in \mathcal{B}(\mathcal{K})$ and $B \in \mathcal{B}(\mathbb{R}^d)$ is a set with $\lambda(B) = 1$, then (denoting by $\mathbf{1}_A$ the indicator function of A)

$$\gamma \mathbb{Q}(A) = \mathbb{E} \sum_{K \in X,\, s(K) \in B} \mathbf{1}_A(K - s(K)),$$

which reveals the intuitive meaning of the intensity γ and the grain distribution $\mathbb{Q}$ (and incidentally shows that they are uniquely determined by X).

A random convex body with distribution $\mathbb{Q}$ is called the **typical grain** of X.

With a stationary particle process X one can associate two body-valued parameters, which comprise much information about the process. Let $S_{d-1}(K, \cdot)$ denote the surface area measure of the convex body K. The **Blaschke body** $B(X)$ of X is the unique body in $\mathcal{K}_0$ with

$$S_{d-1}(B(X), \cdot) = \gamma \int_{\mathcal{K}_0} S_{d-1}(K, \cdot)\, \mathbb{Q}(\mathrm{d}K);$$

it exists by a theorem going back to Minkowski. The projection body, $\Pi_{B(X)}$, of $B(X)$ is called the **associated zonoid** of X and is denoted by Π_X. Thus, its support function, denoted by h, is given by

$$\begin{aligned} h(\Pi_X, u) &= \frac{1}{2} \int_{S^{d-1}} |\langle u, v\rangle|\, S_{d-1}(B(X), \mathrm{d}v) = \frac{\gamma}{2} \int_{\mathcal{K}_0} \int_{S^{d-1}} |\langle u, v\rangle|\, S_{d-1}(K, \mathrm{d}v)\, \mathbb{Q}(\mathrm{d}K) \\ &= \gamma \int_{\mathcal{K}_0} h(\Pi_K, u)\, \mathbb{Q}(\mathrm{d}K). \end{aligned}$$

The formula

$$h(\Pi_X, u) = \frac{1}{2}\,\mathbb{E} \sum_{K \in X} \operatorname{card}([0,u] \cap \operatorname{bd} K), \qquad u \in \mathbb{R}^d,$$

where $[0,u]$ denotes the line segment with endpoints 0 and u, reveals the intuitive meaning of the associated zonoid (see [30], in particular section 4.6).

A further body-valued parameter is the set-valued expectation of the typical grain. Let Z be the typical grain of X. Its **set-valued expectation** (or Aumann expectation, also called selection expectation) is defined by

$$\mathbb{E}\, Z := \{\mathbb{E}\, \xi : \xi : \Omega \to \mathbb{R}^d \text{ is measurable and } \xi \in Z \text{ a.s.}\}$$

(a.s. stands for 'almost surely'). It is a convex body, and $h(\mathbb{E}\, Z, \cdot) = \mathbb{E}\, h(Z, \cdot)$, hence

$$h(\mathbb{E}\, Z, \cdot) = \gamma \int_{\mathcal{K}_0} h(K, \cdot)\, \mathbb{Q}(\mathrm{d}K).$$

In the plane, one has $S_{d-1}(K+M, \cdot) = S_{d-1}(K, \cdot) + S_{d-1}(M, \cdot)$ for $K, M \in \mathcal{K}$, from which one can deduce that

$$B(X) = \gamma \mathbb{E}\, Z \qquad \text{for } d = 2. \tag{1}$$

A **random mosaic** is now defined as a particle process which is almost surely a mosaic. Let X be a stationary mosaic in $\mathbb{R}^d$. For $k = 0, \ldots, d$, the set $\mathcal{F}_k(X)$ defines a stationary particle process, the process of k-faces of X, denoted by $X^{(k)}$. Assuming that it has locally finite intensity measure (this will be satisfied in the examples considered in later sections), we denote its intensity by $\gamma^{(k)}$ and its grain distribution by $\mathbb{Q}^{(k)}$.

There are several natural ways of defining 'average faces' of a stationary random mosaic. The **typical k-face** of X is, by definition, the random polytope $Z^{(k)}$ (unique up to stochastic equivalence) with distribution $\mathbb{Q}^{(k)}$. The intuitive idea behind this is that in every realization of the random mosaic one picks out a k-face at random, with equal chances for all the k-faces (which is, of course, only possible in a bounded region), and translates it to bring its Steiner point to the origin, to obtain a realization of $Z^{(k)}$. A more precise manifestation of this idea is given by the formula

$$\mathbb{P}\{Z^{(k)} \in A\} = \lim_{r \to \infty} \frac{\mathbb{E} \sum_{F \in X^{(k)},\, F \subset rW} \mathbf{1}_A(F - s(F))}{\mathbb{E} \sum_{F \in X^{(k)},\, F \subset rW} 1},$$

which holds for $A \in \mathcal{B}(\mathcal{P})$ and any $W \in \mathcal{K}$ with $\lambda(W) > 0$ (this follows from [30, Th. 4.1.3]). We write $Z^{(d)} =: Z$ and call this the **typical cell** of X.

Another way of defining an average cell of X consists in choosing the (almost surely unique) cell that contains a given point. By stationarity, it is inessential which point we choose; we choose 0 and call the cell containing this point the **zero cell** of X and denote it by Z_0. Up to translations, the distribution of the zero cell is the volume-weighted distribution of the typical cell. In fact, for every translation invariant, nonnegative, measurable function f of $\mathcal{P}$ we have $\mathbb{E} f(Z_0) = \mathbb{E}[f(Z)\lambda(Z)]/\mathbb{E}\lambda(Z)$ (see [30, Th. 10.4.1]).

For a stationary random mosaic X, the Blaschke body $B(X)$ is always centrally symmetric with respect to 0, and the support function of the associated zonoid is given by

$$h(\Pi_X, u) = \mathbb{E} \operatorname{card}([0,u] \cap \operatorname{skel}_{d-1} X).$$

For $d = 2$, $B(X) = \gamma \mathbb{E}\, Z$ and

$$\Pi_X = 2\vartheta_{\pi/2} B(X), \tag{2}$$

where $\vartheta_{\pi/2}$ denotes a rotation by $\pi/2$ (observe that $B(X) = -B(X)$). Thus, for a planar random mosaic, the three introduced parameter bodies differ from each other only by elementary transformations.

2. Poisson Hyperplane Tessellations

A particularly accessible class of random mosaics are those generated by Poisson hyperplane processes. They were already an essential topic in the early work on stochastic geometry by Miles and Matheron. In this section, we give a brief survey of the older extremal properties that have been shown for these random mosaics, and in the next sections we describe more recent results.

We denote by $G(d,k)$ the Grassmannian of k-dimensional linear subspaces and by $A(d,k)$ the affine Grassmannian of k-flats in $\mathbb{R}^d$, both with their usual topologies ($k = 1, \dots, d-1$). In particular, $A(d,d-1)$ is the space of hyperplanes. A hyperplane is often written in the form

$$H(u,\tau) = \{x \in \mathbb{R}^d : \langle x, u\rangle = \tau\}$$

with $u \in S^{d-1}$ and $\tau \in \mathbb{R}$. Note that $H(u,\tau) = H(-u,-\tau)$ (which will not cause ambiguities).

Let $\widehat{X}$ be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity measure $\widehat{\Theta} \neq 0$. This means that $\widehat{\Theta}$ is a translation invariant, locally finite measure on $A(d,d-1)$ and $\widehat{X}$ is a measurable mapping from some probability space $(\Omega, \mathsf{A}, \mathbb{P})$ into the measurable space $\mathsf{N}_s(A(d,d-1))$ of simple, locally finite counting measures on $A(d,d-1)$ with its usual σ-algebra, such that

$$\mathbb{P}\{\widehat{X}(A) = k\} = \mathrm{e}^{-\widehat{\Theta}(A)}\, \frac{\widehat{\Theta}(A)^k}{k!}, \qquad k = 0, 1, 2, \dots,$$

for $A \in \mathcal{B}(A(d,d-1))$ with $\widehat{\Theta}(A) < \infty$. In particular, $\widehat{\Theta}(A) = \mathbb{E}\,\widehat{X}(A)$. The measure $\widehat{\Theta}$ has a unique decomposition of the form

$$\int_{A(d,d-1)} f \,\mathrm{d}\widehat{\Theta} = \widehat{\gamma} \int_{S^{d-1}} \int_{-\infty}^{\infty} f(H(u,\tau))\,\mathrm{d}\tau\, \widehat{\varphi}(\mathrm{d}u)$$

for every nonnegative, measurable function f on $A(d,d-1)$, with a number $\widehat{\gamma} > 0$ and an even probability measure $\widehat{\varphi}$ on S^{d-1}. The number $\widehat{\gamma}$ is the **intensity** and $\widehat{\varphi}$ is the **spherical directional distribution** of $\widehat{X}$. We assume that $\widehat{X}$ is **nondegenerate**, which means that the measure $\widehat{\varphi}$ is not concentrated on a great subsphere. Again, simple counting measures are identified with their supports. Then a.s. every realization $\widehat{X}(\omega)$, $\omega \in \Omega$, can be viewed as a locally finite system of hyperplanes (i.e., every compact set is met by only finitely many of the hyperplanes). It defines a tessellation $X(\omega)$ of $\mathbb{R}^d$, the cells of which are the closures of the connected components of $\mathbb{R}^d \setminus X(\omega)$. Since $\widehat{X}$ is nondegenerate, the cells are a.s. bounded, hence in this way we define a random mosaic X. It is called the **Poisson hyperplane mosaic** induced by $\widehat{X}$. The random mosaic X is stationary.

For $k = 0, \dots, d$, the process $X^{(k)}$ of k-faces of X has a locally finite intensity measure (see [30, sect. 10.3]) and hence a positive, finite intensity $\gamma^{(k)}$ and a grain distribution $\mathbb{Q}^{(k)}$.

Further natural geometric parameters for the description of $\widehat{X}$ and hence X are obtained as follows. For $k \in \{0, \ldots, d-1\}$, let $\widehat{X}_{d-k}$ denote the intersection process of order $d-k$ of $\widehat{X}$ (see [30, sect. 4.4]). It is a stationary process in $A(d,k)$, defined by the intersections of any $d-k$ hyperplanes of $\widehat{X}$. We denote the intensity of $\widehat{X}_{d-k}$ by $\widehat{\gamma}_{d-k}$ (hence $\widehat{\gamma}_1 = \widehat{\gamma}$) and its directional distribution, which is a measure on $G(d,k)$, by $\widehat{\mathbb{Q}}_{d-k}$. The intuitive meaning of these parameters can be read off from

$$\widehat{\gamma}_{d-k}\widehat{\mathbb{Q}}_{d-k}(A) = \frac{1}{\kappa_{d-k}} \mathbb{E} \operatorname{card}\{E \in \widehat{X}_{d-k} : E \cap B^d \neq \emptyset,\, E_0 \in A\}$$

for $A \in \mathcal{B}(G(d,k))$, where $E_0 \in G(d,k)$ denotes the linear subspace parallel to the flat E.

The typical cell Z and the zero cell Z_0 of the stationary random mosaic X are random polytopes, which can be considered as 'average' cells of X, in different ways. We shall be interested in estimating the 'size' of these average cells, metrically or combinatorially. A quite general class of functionals for measuring the size of a d-polytope P (already considered by Miles [23] for typical cells) is given, for $0 \le r \le s \le d$, by

$$Y_{r,s}(P) := \sum_{F \in \mathcal{F}_s(P)} V_r(F),$$

where $\mathcal{F}_s(P)$ is the set of s-faces of P and V_r denotes the rth intrinsic volume. This comprises the following special cases.

- $Y_{r,d}(P) = V_r(P)$ is the rth intrinsic volume of P; in particular, $V_d(P)$ is the volume, $2V_{d-1}(P)$ is the surface area, $(2\kappa_{d-1}/d\kappa_d)V_1(P)$ is the mean width of P, and $V_0(P) = 1$.
- $Y_{s,s}(P) =: L_s(P)$ is the total s-dimensional volume of the s-faces of P.
- $Y_{0,s}(P) =: f_s(P)$ is the number of s-faces of P.

For $0 \le r \le s \le k$, the functional $Y_{r,s}$ is defined for k-polytopes. We want to evaluate $\mathbb{E}\, Y_{r,s}(Z^{(k)})$, its expectation for the typical k-face of the random mosaic X. Define the densities

$$d_r^{(k,s)} := \gamma^{(k)} \mathbb{E}\, Y_{r,s}(Z^{(k)}), \qquad d_r^{(k)} := d_r^{(k,k)} = \gamma^{(k)} \mathbb{E}\, V_r(Z^{(k)}). \tag{3}$$

It follows from [30], Theorem 10.1.2 and (10.9), that

$$d_r^{(k,s)} = 2^{k-s} \binom{d-s}{d-k} d_r^{(s)}, \tag{4}$$

where we have used that a.s. every s-face of X with $s \le k$ lies in precisely $2^{k-s}\binom{d-s}{d-k}$ k-faces of X. By [30, Th. 10.3.1],

$$d_r^{(s)} = \binom{d-r}{d-s} d_r^{(r)}, \tag{5}$$

in particular (case $j = 0$),

$$\gamma^{(s)} = \binom{d}{s} \gamma^{(0)} = \binom{d}{s} \widehat{\gamma}_d. \tag{6}$$

From (3) – (6) we get

$$\mathbb{E}\, Y_{r,s}(Z^{(k)}) = 2^{k-s} \binom{k}{r} \binom{k-r}{k-s} \mathbb{E}\, V_r(Z^{(r)}). \tag{7}$$

The case $r = 0$ gives

$$\mathbb{E}\, f_s(Z^{(k)}) = 2^{k-s}\binom{k}{s}. \tag{8}$$

So far, Poisson assumptions were not required, only some finiteness assumptions, which are satisfied in the Poisson case.

For a stationary Poisson hyperplane process $\widehat{X}$, the remaining essential parameters can be expressed as intrinsic volumes of an associated zonoid. This surprising and very useful fact was first discovered by Matheron.

Since X is a stationary particle process, its Blaschke body $B(X)$ and associated zonoid Π_X are defined as in Section 1. They can now be expressed more directly in terms of the data $\widehat{\gamma}$ and $\widehat{\varphi}$ of the hyperplane process $\widehat{X}$. Since $\widehat{\varphi}$ is an even measure on S^{d-1} and not concentrated on a great subsphere, there exists a unique 0-symmetric convex body $B(\widehat{X})$ with $S_{d-1}(B(\widehat{X}),\cdot) = \widehat{\gamma}\widehat{\varphi}$, the **Blaschke body** of the hyperplane process $\widehat{X}$. Its projection body $\Pi_{B(\widehat{X})} =: \Pi_{\widehat{X}}$, and thus the body with support function

$$h(\Pi_{\widehat{X}},\cdot) = \frac{\widehat{\gamma}}{2}\int_{S^{d-1}} |\langle \cdot, v\rangle|\, \widehat{\varphi}(\mathrm{d}v),$$

is the **associated zonoid** of $\widehat{X}$. It turns out ([30, p. 489]) that

$$B(X) = 2^{\frac{1}{d-1}} B(\widehat{X}), \qquad \Pi_X = 2\Pi_{\widehat{X}}. \tag{9}$$

The following remarkable relations can be stated (see [30], (4.63), (10.43), (10.44) for proofs and references). For $0 \le r \le d$,

$$\widehat{\gamma}_{d-r} = d_r^{(r)} = V_{d-r}(\Pi_{\widehat{X}}), \tag{10}$$

in particular,

$$\widehat{\gamma} = \widehat{\gamma}_1 = V_1(\Pi_{\widehat{X}}). \tag{11}$$

From (3), (6), (10) we obtain

$$\mathbb{E}\, V_r(Z^{(r)}) = \frac{V_{d-r}(\Pi_{\widehat{X}})}{\binom{d}{r} V_d(\Pi_{\widehat{X}})}. \tag{12}$$

In the plane, we can consider the set-valued expectation $\mathbb{E}\, Z$ of the typical cell Z. From (1), (2), (9) we get the nice formula

$$V_2(\mathbb{E}\, Z) = \mathbb{E}\, V_2(Z).$$

Since also

$$V_1(\mathbb{E}\, Z) = \mathbb{E}\, V_1(Z)$$

by linearity, the isoperimetric inequality gives

$$[\mathbb{E}\, V_1(Z)]^2 \ge \pi \mathbb{E}\, V_2(Z),$$

with equality if and only if $\widehat{X}$ is isotropic.

In higher dimensions, the Aleksandrov–Fenchel inequalities for intrinsic volumes can be used to obtain inequalities of isoperimetric type for the considered mosaics. A first example is a sharp inequality for the intersection densities (the intensities of the intersection processes), given the intensity, namely

$$\widehat{\gamma}_k \le \frac{\binom{d}{k}\kappa_{d-1}^k}{d^k \kappa_{d-k}\kappa_d^{k-1}}\,\widehat{\gamma}^k \tag{13}$$

for $k \in \{2,\dots,d\}$. Equality holds if and only if $\widehat{X}$ is isotropic (see [30], Section 4.6, also for references). Thus, *for given intensity, precisely the most symmetric, namely isotropic, hyperplane processes yield the highest kth intersection density.* The case $k = d$ together with (6) gives

$$\gamma^{(k)} \le \frac{\binom{d}{k}\kappa_{d-1}^d}{d^d \kappa_d^{d-1}}\,\widehat{\gamma}^d, \tag{14}$$

where again equality holds precisely in the isotropic case. This shows that *for given intensity $\widehat{\gamma}$, the intensity of the k-faces of the mosaic X is maximal in the isotropic case.*

In contrast to these inequalities, the following one was not mentioned in [21] or [30]. Let $0 < k < d-1$. From the inequalities (6.4.6) in [26], where we replace i,j,k by $0,k,d-1$ and observe that

$$W_r = \frac{\kappa_r}{\binom{d}{r}} V_{d-r},$$

we obtain

$$\left(\frac{V_{d-k}(\Pi_{\widehat{X}})}{\binom{d}{k}V_d(\Pi_{\widehat{X}})}\right)^{d-1} \ge \frac{\kappa_{d-1}^k}{\kappa_k^{d-1}}\left(\frac{V_1(\Pi_{\widehat{X}})}{dV_d(\Pi_{\widehat{X}})}\right)^k \tag{15}$$

and thus, by (12),

$$\left[\mathbb{E}\,V_k(Z^{(k)})\right]^{d-1} \ge \frac{\kappa_{d-1}^k}{\kappa_k^{d-1}}\left[\mathbb{E}\,V_{d-1}(Z^{(d-1)})\right]^k. \tag{16}$$

Supppose that equality holds in (16). Then (see [26, p. 334]) equality must hold in the inequality $W_{d-2}(\Pi_{\widehat{X}})^2 \ge W_{d-3}(\Pi_{\widehat{X}})W_{d-1}(\Pi_{\widehat{X}})$. Since $\Pi_{\widehat{X}}$ is centrally symmetric and d-dimensional, this implies by [26, Th. 6.6.19] that $\Pi_{\widehat{X}}$ is a 1-tangential body, and hence a cap body, of a ball. This means that $\Pi_{\widehat{X}}$ is the convex hull of a ball B and a (possibly empty) set M of points outside B with the property that the segment joining any two points of M intersects B. If M is not empty, then $\Pi_{\widehat{X}}$ has uncountably many exposed faces which are segments of different directions. Since $\Pi_{\widehat{X}}$ is a zonoid, each of its faces is a summand of $\Pi_{\widehat{X}}$. This leads to a contradiction, hence $\Pi_{\widehat{X}}$ is a ball, and $\widehat{X}$ is isotropic.

Eliminating $V_d(\Pi_{\widehat{X}})$ from the right side of (15) by means of the inequality connecting V_1 and V_d (i.e., [30, (14.31)] for $j = 1$, $k = d$), we obtain

$$\mathbb{E}\,V_k(Z^{(k)}) \ge \frac{d^k\kappa_d^k}{\kappa_{d-1}^k\kappa_k}\,\frac{1}{\widehat{\gamma}^k}, \tag{17}$$

which together with (5) gives [21, Satz 3.12.3]. Equality holds precisely if $\widehat{X}$ is isotropic. Thus, *for given intensity $\widehat{\gamma}$, the isotropic mosaics yield the smallest expected volume of the typical k-face.*

We turn to the zero cell Z_0 of the mosaic X. In contrast to the case of the typical cell Z, useful explicit representations for the expectations of the functionals $Y_{r,s}(Z_0)$ are only known in the following few special cases (for proofs and references, see [30, sect. 10.4]). The first of these is

$$\mathbb{E}\,V_d(Z_0) = \frac{d!}{2^d}V_d(\Pi^o_{\widehat{X}}),$$

where $\Pi^o_{\widehat{X}} := (\Pi_{\widehat{X}})^o$ denotes the polar body of the associated zonoid $\Pi_{\widehat{X}}$. The expected total k-volume of the k-faces can be reduced to this, namely by

$$\mathbb{E}\, L_k(Z_0) = d_k^{(k)} \mathbb{E}\, V_d(Z_0) = \frac{d!}{2^d} V_{d-k}(\Pi_{\widehat{X}}) V_d(\Pi^o_{\widehat{X}}),$$

for $k = 0, \dots, d-1$. The particular case $k = 0$ gives the expected number of vertices,

$$\mathbb{E}\, f_0(Z_0) = \frac{d!}{2^d} V_d(\Pi_{\widehat{X}}) V_d(\Pi^o_{\widehat{X}}).$$

Since the zero polytope is a.s. simple (that is, each vertex lies in precisely d facets), we have

$$\mathbb{E}\, f_1(Z_0) = \frac{d}{2} \mathbb{E}\, f_0(Z_0),$$

and for $d = 3$ we can use Euler's relation to get

$$\mathbb{E}\, f_2(Z_0) = 2 + \frac{1}{2} \mathbb{E}\, f_0(Z_0).$$

Sharp inequalities are known for the volume and the vertex number, namely

$$\mathbb{E}\, V_d(Z_0) \geq d!\kappa_d \left(\frac{2\kappa_{d-1}}{d\kappa_d} \widehat{\gamma} \right)^{-d}, \tag{18}$$

with equality if and only if $\widehat{X}$ is isotropic, and

$$2^d \leq \mathbb{E}\, f_0(Z_0) \leq \frac{d!\kappa_d^2}{2^d}. \tag{19}$$

Equality on the left side of (19) holds if and only if X is a **parallel mosaic**, which means that the hyperplanes of $\widehat{X}$ belong to d fixed translation classes. Equality on the right holds if and only if $\widehat{X}$ is **affinely isotropic**. This means that there exists a nondegenerate affine transformation α of $\mathbb{R}^d$ such that the hyperplane process $\alpha\widehat{X}$ is isotropic. For the proofs, we refer to [30, Th. 10.4.9].

The extremal property of isotropic mosaics exhibited by (18) goes much farther. *For given intensity $\widehat{\gamma}$, every moment $\mathbb{E}\, V_d(Z_0)^k$, and also every moment $\mathbb{E}\, V_d(Z)^k$, for $k \in \mathbb{N}$, attains its minimum precisely if $\widehat{X}$ is isotropic.* The proof of this result, which is due to Mecke, is also reproduced in [30, Th. 10.4.9].

3. Nonstationary Hyperplane Tessellations

We mention briefly (following [27]) how a few of the preceding results can be extended to nonstationary Poisson hyperplane tessellations. We assume again that $\widehat{X}$ is a Poisson hyperplane process in $\mathbb{R}^d$ with a locally finite intensity measure $\widehat{\Theta}$. Since $\widehat{X}$ need not be stationary now, the measure $\widehat{\Theta}$ is not necessarily translation invariant. We assume however, that it is **translation regular**, which means that it is absolutely continuous with respect to some translation invariant, locally finite measure $\widetilde{\Theta}$ on $A(d, d-1)$. For simplicity, we restrict ourselves here to the case where $\widehat{\Theta}$ has a continuous density with respect to $\widetilde{\Theta}$. In that case, the constant intensities that exist in the stationary case are replaced by measurable intensity functions, which admit intuitive interpretations. The **intensity function** $\widehat{\gamma}$ of $\widehat{X}$ can be defined by

$$\widehat{\gamma}(z) = \lim_{r \to 0} \frac{1}{V_d(rK)} \mathbb{E} \sum_{H \in \widehat{X}} \mathcal{H}^{d-1}(H \cap (rK + z))$$

for $z \in \mathbb{R}^d$, where K is any convex body with $V_d(K) > 0$. It has also the representation

$$\widehat{\gamma}(z) = \lim_{r\to 0} \frac{1}{2r}\,\mathbb{E}\,\mathrm{card}\{H \in \widehat{X} : H \cap (rB^d + z) \neq \emptyset\}.$$

For $k \in \{0, \dots, d-1\}$, the intersection process $\widehat{X}_{d-k}$ of order $d-k$ is defined as in the stationary case; it is obtained by taking the intersections of any $d-k$ hyperplanes of $\widehat{X}$ and is a.s. a simple process of k-flats. It has an intensity function given by

$$\widehat{\gamma}_{d-k}(z) = \lim_{r\to 0} \frac{1}{\kappa_{d-k} r^{d-k}}\,\mathbb{E}\,\mathrm{card}\{E \in \widehat{X}_{d-k} : E \cap (rB^d + z) \neq \emptyset\}$$

for $z \in \mathbb{R}^d$. The inequality (13) extends to an inequality holding at every point, namely

$$\widehat{\gamma}_k(z) \le \frac{\binom{d}{k}\kappa_{d-1}^k}{d^k \kappa_{d-k}\kappa_d^{k-1}}\,\widehat{\gamma}(z)^k \qquad \text{for } z \in \mathbb{R}^d. \tag{20}$$

Equality for all z holds if and only if the hyperplane process $\widehat{X}$ is stationary and isotropic. Thus, here an extremal property implies the invariance of the distribution under the full group of rigid motions!

To study the tessellation induced by $\widehat{X}$, we need first a suitable notion of nondegeneracy. We say that $\widehat{X}$ is **nondegenerate** if the zero cell Z_0 is bounded with positive probability and if the following holds. Whenever $U \subset S^{d-1}$ is a measurable set and $\widehat{X}$ contains with positive probability a hyperplane with normal vector in U, then $\widehat{X}$ contains with positive probability infinitely many such hyperplanes. If $\widehat{X}$ is nondegenerate, then it can be shown that the cells induced by $\widehat{X}$ constitute a random mosaic X and that the process $X^{(k)}$ of its k-faces ($k \in \{0, \dots, d\}$) has a locally finite intensity measure, which is also translation regular, in the sense that it is absolutely continuous with respect to some translation invariant, locally finite measure on the space of polytopes.

In the stationary case, we have defined in (3) a density $d_r^{(k)} = \gamma^{(k)}\mathbb{E}\,V_r(Z^{(k)})$, the **specific** r**th intrinsic volume** of the stationary k-face process $X^{(k)}$. In the nonstationary case, where no typical k-face exists, this definition cannot be used, but it can be generalized by defining

$$d_r^{(k)}(z) := \lim_{r\to 0} \frac{1}{V_d(rB^d)}\,\mathbb{E} \sum_{K \in X^{(k)}} \Phi_r(K, rB^d + z)$$

for λ-almost all $z \in \mathbb{R}^d$. Here $\Phi_r(K, \cdot)$ is the rth curvature measure of the convex body K. It can then be shown that the relations

$$d_r^{(k)} = \binom{d-r}{d-k} d_r^{(r)}, \qquad d_r^{(r)} = \widehat{\gamma}_{d-r},$$

and hence the inequality

$$d_r^{(k)} \le \binom{d-r}{d-k}\binom{d}{r} \frac{\kappa_{d-1}^{d-r}}{d^{d-r}\kappa_r \kappa_d^{d-r-1}}\,\widehat{\gamma}^{d-r}, \tag{21}$$

hold almost everywhere. Equality in (21) holds if and only if $\widehat{X}$ is stationary and isotropic.

4. Weighted Faces

We return to the stationary Poisson hyperplane process $\widehat{X}$ and its induced tessellation X, as studied in Section 2. The results of that section reveal a clear distinction between the zero cell Z_0 and the typical cell Z, though either of them provides a natural notion of 'average cell'. Heuristically, the zero cell can also be obtained, up to translations, if in a large bounded region of space we choose a uniformly distributed random point (with respect to Lebesgue measure) and take the almost surely unique cell of X containing that point. A similar procedure makes also sense for k-faces: in a large bounded region of space we choose a random point, uniformly distributed on the k-skeleton $\mathrm{skel}_k X$ of X, with respect to the k-dimensional Hausdorff measure, and take the (almost surely unique) k-face of X containing that point. This leads (up to translations) to the notion of the weighted typical k-face. A precise formal definition using Palm theory may be sketched as follows. On $\mathsf{N}_s(A(d,d-1))$, we can define a probability measure $\mathbb{P}_k^o$ by

$$\widehat{\gamma}_{d-k}\mathbb{P}_k^o(\mathcal{A}) = \mathbb{E}\int_{\mathrm{skel}_k X} \mathbf{1}_B(x)\mathbf{1}_{\mathcal{A}}(\widehat{X}-x)\,\mathcal{H}^k(\mathrm{d}x),$$

where B is any Borel set in $\mathbb{R}^d$ with $\lambda(B)=1$. (Details, as well as the results below, are found in [28]). There exists a hyperplane process Y_k with distribution $\mathbb{P}_k^o$, and we denote by $Z_0^{(k)}$ the (always existing and almost surely unique) k-face of the tessellation induced by Y_k that contains the origin 0. In particular, $Z_0^{(d)}$ is stochastically equivalent to the zero cell Z_0. The random polytope $Z_0^{(k)}$ is uniquely determined up to stochastic equivalence and is called the **volume-weighted typical k-face**, or briefly the **weighted typical k-face**, of X. This terminology is justified, since the distribution of $Z_0^{(k)}$ is, if translations are disregarded, the volume-weighted distribution of the typical k-face $Z^{(k)}$. In fact, for every translation invariant, nonnegative, measurable function f on $\mathcal{P}$ one has

$$\mathbb{E}\,f(Z_0^{(k)}) = \frac{1}{\mathbb{E}\,V_k(Z^{(k)})}\,\mathbb{E}\left[f(Z^{(k)})V_k(Z^{(k)})\right].$$

(Recall that $V_k(K)=\mathcal{H}^k(K)$ for a k-dimensional convex body K.) Another justification of the terminology, and at the same time a precise version of the intuitive approach with which we started, is provided by the following formula. Here W can be any convex body with positive volume, and every polytope that appears is replaced by its translate with Steiner point at the origin. For $A\in\mathcal{B}(\mathcal{P})$, we have

$$\mathbb{P}\{Z_0^{(k)}-s(Z_0^{(k)})\in A\} = \lim_{r\to\infty}\frac{\mathbb{E}\sum_{F\in X^{(k)},\,F\subset rW}\mathbf{1}_A(F-s(F))V_k(F)}{\mathbb{E}\sum_{F\in X^{(k)},\,F\subset rW}V_k(F)}.$$

It is a useful consequence of the Poisson property of $\widehat{X}$ that the distribution of the weighted typical k-face can be determined from the distribution of the zero cell Z_0, in the following way. Recall that $\widehat{\mathbb{Q}}_{d-k}$ ist the directional distribution of the intersection process $\widehat{X}_{d-k}$ of order $d-k$ and is a probability measure on the Grassmannian $G(d,k)$. Now, for $k\in\{1,\dots,d-1\}$ and every $A\in\mathcal{B}(\mathcal{P})$ we have

$$\mathbb{P}\{Z_0^{(k)}\in A\} = \int_{G(d,k)}\mathbb{P}\{Z_0\cap L\in A\}\,\widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L). \tag{22}$$

This can also be interpreted as follows. If we choose a random k-dimensional linear subspace $\mathcal{L}$ of $\mathbb{R}^d$ with distribution $\widehat{\mathbb{Q}}_{d-k}$ such that $\widehat{X}$ and $\mathcal{L}$ are independent, then the intersection $Z_0\cap\mathcal{L}$ is stochastically equivalent to the weighted k-face $Z_0^{(k)}$.

The representation (22) allows us to extend the inequalities (19) to the weighted k-face. In fact, for the vertex number f_0 it yields

$$\mathbb{E} f_0(Z_0^{(k)}) = \int_{G(d,k)} \mathbb{E} f_0(Z_0 \cap L)\, \widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L).$$

For fixed $L \in G(d,k)$, the polytope $Z_0 \cap L$ appearing in the integrand can also be obtained as follows. If we intersect each hyperplane of $\widehat{X}$ with L, we obtain the section process $\widehat{X} \cap L$ (see [30, sect. 4.4]), which is a stationary Poisson hyperplane process with respect to the space L. The set $Z_0 \cap L$ is the zero cell of the tessellation of L induced by $\widehat{X} \cap L$. The associated zonoid (in L) of $\widehat{X} \cap L$ is given by $\Pi_{\widehat{X}}|L$, the orthogonal projection of the associated zonoid $\Pi_{\widehat{X}}$ to L; see [30, (4.61)]. Hence, it follows from [30, Th. 10.4.9] (applied in L) that

$$\mathbb{E}\, f_0(Z_0^{(k)}) = 2^{-k} k! \int_{G(d,k)} V_k(\Pi_{\widehat{X}}|L) V_k((\Pi_{\widehat{X}}|L)^o)\, \widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L). \tag{23}$$

Here, $(\Pi_{\widehat{X}}|L)^o$ denotes the polar body of $\Pi_{\widehat{X}}|L$ in the subspace L. Since $\Pi_{\widehat{X}}|L$ is a zonoid, the inequalities

$$\frac{4^k}{k!} \le V_k(\Pi_{\widehat{X}}|L) V_k((\Pi_{\widehat{X}}|L)^o) \le \kappa_k^2 \tag{24}$$

hold. Equality on the right side (the Blaschke–Santaló inequality) holds if and only if $\Pi_{\widehat{X}}|L$ is an ellipsoid; equality on the left side (Reisner's inequality) holds if and only if $\Pi_{\widehat{X}}|L$ is a parallelepiped. From (23) and (24) we obtain the inequalities

$$2^k \le \mathbb{E}\, f_0(Z_0^{(k)}) \le 2^{-k} k! \kappa_k^2 \tag{25}$$

for $k \in \{2, \dots, d-1\}$. Equality on the left side holds if and only if $\widehat{X}$ is a parallel mosaic. Equality on the right side holds if $\widehat{X}$ is affinely isotropic, but we don't know whether this is the only case. For the characterization of the equality case on the left side, the following geometric result is needed. Let $K \subset \mathbb{R}^d$ be a 0-symmetric zonoid with generating measure ρ, thus the support function of K is represented by

$$h(K,x) = \int_{S^{d-1}} |\langle x, v \rangle|\, \rho(\mathrm{d}v), \qquad x \in \mathbb{R}^d,$$

and ρ is a finite, even measure on S^{d-1}. Suppose that K has the property that for any $d-k$ linearly independent vectors $v_1, \dots, v_{d-k}$ in the support of the measure ρ, the orthogonal projection of K to $v_1^\perp \cap \dots \cap v_{d-k}^\perp$ is a parallelepiped. Then K is itself a parallelepiped. This can be proved. The analogous assertion, with 'parallelepiped' replaced by 'ellipsoid', would settle the equality case on the right side of (25), but this has not been proved so far.

The inequalities (25) have been extended in [29]. Instead of weighting the typical face by the volume, we can use for weighting the total j-dimensional volume of its j-faces. Thus, for a polytope P and for $0 \le j \le \dim P$ we put

$$L_j(P) := \sum_{F \in \mathcal{F}_j(P)} V_j(F),$$

and we define the **L_j-weighted typical k-face** $Z_{k,j}$ of the random mosaic X as the random polytope with distribution given by

$$\mathbb{P}\{Z_{k,j} \in A\} = \frac{1}{\mathbb{E}\, L_j(Z^{(k)})}\, \mathbb{E}\left[\mathbf{1}_A(Z^{(k)}) L_j(Z^{(k)})\right]$$

for $A \in \mathcal{B}(\mathcal{P})$. Then the inequalities

$$2^k \le \mathbb{E} f_0(Z_{k,j}) \le 2^{j-2k} \sum_{i=0}^{k-j} 2^{2i} \binom{k-j}{i} (k-i)! \kappa_{k-i}^2$$

are valid. The equality cases are the same as for (25).

5. Large Cells in Poisson Hyperplane Mosaics

As we have seen, isoperimetric problems for sizes of average cells or faces of random mosaics can be stated, and in some cases solved, if one asks for expected values of functionals measuring the size. To ask just for the shape of average cells with extremal sizes is not a meaningful question, since such extrema will not be attained. Surprisingly, however, such questions can make perfect sense if one asks for the asymptotic shape of average cells under the condition that their size (in some sense) is large. The origin of such questions is a conjecture ventured by D.G. Kendall (in the 1940s, and later popularized by him in the foreword to the first edition of [31], which appeared in 1987). Kendall considered the zero cell Z_0 of a stationary and isotropic Poisson line process in the plane and conjectured that the conditional law for its shape, under the condition of given area, converges weakly, as the area tends to infinity, to the degenerate law concentrated at the circular shape. This has been verified in various extended versions and in general dimensions, beginning with solutions and analogs in the planar case by Kovalenko [9, 10, 11] and Miles [24]. In this and the two subsequent sections we describe the main results that have been obtained on generalized versions of Kendall's problem.

The first higher-dimensional version of problems of this type was studied by Mecke and Osburg [20]. For the special case of a stationary Poisson hyperplane mosaic with spherical directional distribution concentrated (with equal masses) in $\{\pm u_1, \dots, \pm u_d\}$, where $(u_1, \dots, u_d)$ is an orthonormal basis of $\mathbb{R}^d$, they obtained that zero cells of large volume approximate cubical shape. This was made precise in several ways, involving monotonicity, stochastic order, and limit relations. The results were transferred to affine images of such hyperplane tessellations.

Hyperplane tessellations with more general directional distributions were investigated in [3] – [8], as we now explain. First, we consider again a stationary Poisson hyperplane process $\widehat{X}$ in $\mathbb{R}^d$, with intensity $\widehat{\gamma}$ and spherical directional distribution $\widehat{\varphi}$. We ask for the shape of its zero cell Z_0, under the condition that the zero cell is large. Here 'large' can be interpreted in terms of volume, or diameter, or many other reasonable functionals. We can put this axiomatically. By a **size functional** we understand any continuous real function $\Sigma \not\equiv 0$ on the space $\mathcal{K}_{(0)}$ of convex bodies in $\mathbb{R}^d$ containing 0 which is increasing under set inclusion and is homogeneous of some degree $k > 0$. For many such functionals, there are precise asymptotic results about the shape of Z_0 under the condition that $\Sigma(Z_0)$ is large. It turns out that such asymptotic shapes are determined by an isoperimetric inequality that connects the size functional with the **hitting functional** of the process $\widehat{X}$. This is the function Φ defined by

$$\Phi(K) := \frac{1}{2\widehat{\gamma}} \mathbb{E} \operatorname{card}\{H \in \widehat{X} : H \cap K \neq 0\}, \qquad K \in \mathcal{K}.$$

Explicitly, it is given by

$$\Phi(K) = \int_{S^{d-1}} h(K, u) \, \widehat{\varphi}(\mathrm{d}u).$$

Note that Φ is continuous and homogeneous of degree one. It follows from continuity and compactness that among all convex bodies $K \in \mathcal{K}_{(0)}$ with a given positive value of $\Phi(K)$, there exist convex bodies for which Σ becomes maximal. Hence, by homogeneity there is an isoperimetric-type inequality

$$\Phi(K) \geq \tau\Sigma(K)^{1/k} \tag{26}$$

holding for all $K \in \mathcal{K}_{(0)}$, and with equality holding for some convex bodies; these are called **extremal bodies**. It turns out (in many cases) that the shapes of Σ-large zero cells approximate the shapes of extremal bodies. Since, however, these shapes can in general not be attained, we need to measure how close a shape comes to that of an extremal body. Therefore, we define, for given Σ and Φ, a **deviation functional** as any function ϑ on the space $\{K \in \mathcal{K}_{(0)} : \Sigma(K) > 0\}$ which is continuous, nonnegative, homogeneous of degree zero, and satisfies $\vartheta(K) = 0$ if and only if K is an extremal body. The existence follows by continuity, and for the same reason there exist continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(0) = 0$ and $f(x) > 0$ for $x > 0$ such that the following stability version of the inequality (26) holds:

$$\Phi(K) \geq (1 + f(\epsilon))\tau\Sigma(K)^{1/k} \qquad \text{whenever } \vartheta(K) \geq \epsilon. \tag{27}$$

For the geometrically most interesting concrete size functionals Σ, simple explicit functions ϑ and f can be provided.

We can now estimate $\mathbb{P}\{\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a\}$, the conditional probability that the zero cell deviates in shape by at least ϵ from an extremal body, under the condition that its Σ-size it at least a. We assume that Σ, ϑ, f with the properties listed above are given.

Theorem. *For given $\epsilon > 0$ and $a > 0$, there exist positive constants c (depending on $\widehat{X}, \Sigma, f, \epsilon$) and c_0 such that*

$$\mathbb{P}\{\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a\} \leq c \exp\left(-c_0 f(\epsilon)\widehat{\gamma} a^{1/k}\right). \tag{28}$$

We can also condition by $\Sigma(Z_0) = a$, instead of $\Sigma(Z_0) \geq a$. Namely, the random polytope Z_0 takes its values in $\mathcal{K}$. Since this is a Polish space, the regular conditional probability distribution of Z_0 with respect to $\Sigma(Z_0)$ exists. Similarly to (28), we have

$$\mathbb{P}\{\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) = a\} \leq c \exp\left(-c_0 f(\epsilon)\widehat{\gamma} a^{1/k}\right). \tag{29}$$

The role of the isoperimetric inequality (26) and its strengthening (27) can be explained as follows. By definition,

$$\mathbb{P}\{\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a\} = \frac{\mathbb{P}\{\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \geq a\}}{\mathbb{P}\{\Sigma(Z_0) \geq a\}}, \tag{30}$$

and this has to be estimated from above. Let $\epsilon > 0$ and $a > 0$ be given. Let B be an extremal body of (26). Since dilates of B are also extremal bodies and Σ is homogeneous of degree $k > 0$, we can assume that $\Sigma(B) = a$ and hence that $\Phi(B) = \tau\Sigma(B)^{1/k} = \tau a^{1/k}$. If $H \cap B = \emptyset$ for all $H \in \widehat{X}$, then $B \subset Z_0$ (since $0 \in B$), hence $\Sigma(Z_0) \geq \Sigma(B) = a$. Therefore, the denominator of (30) can be estimated by

$$\begin{aligned} \mathbb{P}\{\Sigma(Z_0) \geq a\} &\geq \mathbb{P}\{\mathrm{card}\{H \in \widehat{X} : H \cap B \neq \emptyset\} = 0\} \\ &= \exp(-\Phi(B)2\widehat{\gamma}) = \exp\left(-2\tau\widehat{\gamma} a^{1/k}\right). \end{aligned}$$

The estimation of the numerator we explain only heuristically. Let K be a convex body satisfying

$$\vartheta(K) \geq \epsilon, \quad \Sigma(K) \geq a.$$

Then, using (27) instead of (26),

$$\mathbb{P}\{\mathrm{card}\{H \in \widehat{X} : H \cap K \neq \emptyset\} = 0\} = \exp(-\Phi(K)2\widehat{\gamma}) \leq \exp\left(-(1+f(\epsilon))2\tau\widehat{\gamma}a^{1/k}\right).$$

An only slightly weaker inequality can be proved if the deterministic convex body K is replaced by the random polytope Z_0, namely

$$\mathbb{P}\{\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \geq a\} \leq c\exp\left(-(1+c_1 f(\epsilon))2\tau\widehat{\gamma}a^{1/k}\right) \tag{31}$$

with positive constants c, c_1. Division now gives

$$\mathbb{P}\{\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a\} \leq c\exp\left(-c_1 f(\epsilon)2\tau\widehat{\gamma}a^{1/k}\right).$$

The bulk of the work, of course, consists in the proof of the estimate (31).

The first concrete example is the case where the size functional Σ is given by the volume V_d. Denoting by

$$B = B(\widehat{X})$$

the Blaschke body of the hyperplane process $\widehat{X}$, we can express the hitting functional as a mixed volume. Since $\widehat{\varphi}$ is now the surface area measure of B, we have $\Phi(K) = dV(K, B, \dots, B)$. Minkowski's inequality

$$V(K, B, \dots, B) \geq V_d(B)^{1-1/d} V_d(K)^{1/d} \tag{32}$$

is the inequality (26) in this case (with $\tau = dV_d(B)^{1-1/d}$), and it is well known that its extremal bodies are all homothetic to B. For a deviation functional we can choose

$$\vartheta(K) := \inf\{\beta/\alpha : \alpha, \beta > 0,\ \alpha B \subset K + t \subset \beta B \text{ for some } t \in \mathbb{R}^d\}.$$

A stability version (27) of inequality (32) is known with $f(x) = \mathrm{const} \cdot x^{d+1}$; hence we obtain

$$\mathbb{P}\{\vartheta(Z_0) \geq \epsilon \mid V_d(Z_0) \geq a\} \leq c\exp\left(-c_0\epsilon^{d+1}\widehat{\gamma}a^{1/d}\right). \tag{33}$$

Thus, *the Blaschke body provides the shape of zero cells of large volume.* We can also deduce the existence of a (degenerate) limit distribution for the shape. By $\mathcal{S}_\mathsf{H}$ we denote the quotient space of $\mathcal{K}$ with respect to the equivalence relation given by homothety. The equivalence class of a convex body K is denoted by $s_\mathsf{H}(K)$ and is called the homothetic shape of K. We define the conditional law of the homothetic shape of Z_0, given the lower bound $a > 0$ for the volume, as the probability measure μ_a on $\mathcal{S}_\mathsf{H}$ with

$$\mu_a(A) := \mathbb{P}\{s_\mathsf{H}(Z_0) \in A \mid V_d(Z_0) \geq a\}$$

for $A \in \mathcal{B}(\mathcal{S}_\mathsf{H})$. Then

$$\lim_{a\to\infty} \mu_a = \delta_{s_\mathsf{H}(B)} \qquad \text{weakly,} \tag{34}$$

with $\delta_{s_\mathsf{H}(B)}$ denoting the Dirac measure concentrated at $s_\mathsf{H}(B)$.

If $\widehat{X}$ is isotropic, then the Blaschke body $B(\widehat{X})$ is a ball, hence the asymptotic shape is given by the class of balls, as in Kendall's original problem.

The inequality (33) was proved in [3]. It was also remarked there that this inequality remains true if the zero cell is replaced by the typical cell. This is due to the fact that the distribution of the zero cell is, up to translations, the volume-weighted distribution of the typical cell.

The setting for extended Kendall problems was considerably broadened in [7]. Besides admitting general size functionals Σ, a wider class of Poisson hyperplane processes $\widehat{X}$ was considered, namely those with an intensity measure $\widehat{\Theta}$ of the form

$$\widehat{\Theta}(A) = 2\widehat{\gamma} \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(H(u,t))t^{r-1}\,\mathrm{d}t\,\widehat{\varphi}(\mathrm{d}u) \tag{35}$$

for $A \in \mathcal{B}(A(d, d-1))$. Here $r \geq 1$, $\widehat{\gamma} > 0$, and $\widehat{\varphi}$ is a finite (not necessarily even) measure on the unit sphere, not concentrated on a closed hemisphere. The case of a stationary hyperplane process $\widehat{X}$ is obtained if $r = 1$ and $\widehat{\varphi}$ is even. As explained in the next section, the case of $r = d$ and rotation invariant $\widehat{\varphi}$ allows one to treat the typical cell of a stationary Poisson–Voronoi tessellation in a similar way. In [7], also general versions of (29) and (34) were obtained. For stationary Poisson hyperplane tessellations, the following special size functionals were treated. The different cases may require different notions of shape, since the extremal bodies of the corresponding crucial inequality (26) may be equivalent to a fixed convex body with different meanings of 'equivalent', for example, homothetic, or equivalent by positive dilatation, or similar. If the size is measured by the diameter, then the limit shape of the zero cell is provided by the class of segments. If size is measured by thickness (minimal width), then the whole class of bodies of constant width can be considered as the asymptotic shape of the zero cell. Further size functionals studied in [7] are the inradius, the centered inradius, and the width in a given direction.

For the typical cell instead of the zero cell and for size functionals different from the volume, no such simple transfer argument as mentioned above is possible. For stationary, isotropic Poisson hyperplane tessellations, results on the asymptotic shapes of large typical cells were obtained in [8]. If the size is measured by the kth intrinsic volume, $k \in \{2, \ldots, d\}$, asymptotic shapes are balls. (For the zero cell, the same was proved in [4].) For the diameter as size functional, one obtains segments as asymptotic shapes. The proof makes use of a special representation of the distribution of the typical cell with respect to the highest vertex as center function (see [30, Theorem 10.4.7]).

6. Large Cells in Poisson–Voronoi Mosaics

A much studied class of tessellations are the Voronoi or Dirichlet mosaics. If $S \subset \mathbb{R}$ is a nonempty, locally finite set and $x \in S$, then the **Voronoi cell** $C(x, S)$ of x (with respect to S) is defined as the set of all points in $\mathbb{R}^d$ for which x is a nearest point in S, thus

$$C(x, S) = \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - s\| \text{ for all } s \in S\}.$$

The point x is called the **nucleus** of the Voronoi cell $C(x, S)$.

Now let $\widetilde{X}$ be a stationary Poisson point process in $\mathbb{R}^d$ with intensity $\widetilde{\gamma} > 0$ (and, as always assumed, locally finite intensity measure). Then the collection $\{C(x, \widetilde{X}) : x \in \widetilde{X}\}$ is a stationary random mosaic. It is called the **Poisson–Voronoi mosaic** induced by $\widetilde{X}$. We denote it by X. Since the intensity measure of $\widetilde{X}$ is locally finite and translation invariant, it is a constant multiple of Lebesgue measure and hence is invariant under rotations. Therefore, the random mosaic X is isotropic.

We are interested in the asymptotic shape of the typical cell Z of X, under the condition that it is large. It follows from Slivnyak's theorem on Poisson processes that the typical cell Z is stochastically equivalent to $C(0, \widetilde{X} \cup \{0\})$, the Voronoi cell of 0 for the point process $\widetilde{X} \cup \{0\}$, which is obtained from $\widetilde{X}$ by adding a point at 0. For $x \in \mathbb{R}^d$, let $H(x)$ be the mid-hyperplane of 0 and x, that is, the set of all points having equal distance from 0 and x. For $x \neq 0$, let $H^-(x)$ be the closed halfspace bounded by $H(x)$ that contains 0. By the definition of Voronoi cells, we have

$$C(0, \widetilde{X} \cup \{0\}) = \bigcap_{x \in \widetilde{X}} H^-(x);$$

hence $C(0, \widetilde{X} \cup \{0\})$ is the zero cell, Z_0, of the mosaic induced by the hyperplane process $\widehat{X} := \{H(x) : x \in \widetilde{X}\}$. This is a (non-stationary) Poisson process, and its intensity measure $\widehat{\Theta}$ can be represented by

$$\widehat{\Theta}(A) = 2^d \widetilde{\gamma} \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(H(u,t)) t^{d-1} \, \mathrm{d}t \, \sigma(\mathrm{d}u) \tag{36}$$

for $A \in \mathcal{B}(A(d, d-1))$, where σ denotes spherical Lebesgue measure. This is of type (35), with $\widehat{\gamma} = 2^d d\kappa_d \widetilde{\gamma}$. Therefore, the methods described in [7] can be used to obtain results on asymptotic shapes for Z_0 (which is stochastically equivalent to the typical cell Z of X), under the condition that it has large size. In [4], the following results were obtained.

For $K \in \mathcal{K}_{(0)}$ with $K \neq \{0\}$, we measure the deviation from a centered ball by the function

$$\vartheta(K) := \frac{R_0 - r_0}{R_0 + r_0},$$

where R_0 is the radius of the smallest ball with center 0 containing K and r_0 is the radius of the largest ball with center 0 contained in K. Let $k \in \{1, \dots, d\}$, let V_k denote the kth intrinsic volume. If $\epsilon \in (0,1)$ and if $a > 0$ is sufficiently large, then

$$\mathbb{P}\{\vartheta(Z_0) \geq \epsilon \mid V_k(Z_0) \geq a\} \leq c \exp\left(-c_0 \epsilon^{(d+3)/2} \widetilde{\gamma} a^{d/k}\right),$$

where the constant c depends on d and ϵ, while c_0 depends only on d. A similar result was obtained in [4], with the size of Z_0 measured by the centered inradius.

The general methods of [7] allow also the treatment of size functionals where the resulting asymptotic shapes are of lower dimension. For example, if the size of Z_0 is measured by the largest distance of a vertex from the nucleus, then the limit shape is given by the class of all segments with one endpoint at the origin.

7. Large Cells in Poisson–Delaunay Mosaics

As in the previous section, let $\widetilde{X}$ be a stationary Poisson point process in $\mathbb{R}^d$ with intensity $\widetilde{\gamma} > 0$. Together with the Voronoi mosaic induced by $\widetilde{X}$ comes a certain dual of it, the Delaunay mosaic. We recall here its definition without recourse to the Voronoi mosaic. With probability one, any $d+1$ points of $\widetilde{X}$ lie on a unique sphere. If the open ball bounded by this sphere does not contain a point of $\widetilde{X}$, then the convex hull of the $d+1$ points is called a **cell**. The collection of all cells obtained in this way is a tessellation of $\mathbb{R}^d$ into simplices. In this way, a stationary random mosaic Y is defined, which is called the **Poisson–Delaunay mosaic** induced by $\widetilde{X}$.

For a d-dimensional simplex S, there is a unique sphere through its vertices, and we denote by $z(S)$ the center of this sphere, also called the **circumcenter** of S, and by $R(S)$ the radius of the sphere. Let Z be the typical cell of Y with respect to the center function z. Then Z is a d-dimensional random simplex with circumcenter 0. For its distribution, there is an explicit integral representation due to Miles; see [30, Theorem 10.4.4]. This was used in [5], [6] to obtain results on asymptotic shapes of large typical cells. We describe briefly a general result obtained in [6]. Let Δ_0 be the subspace of $\mathcal{P}$ consisting of all d-dimensional simplices with circumcenter 0. By a **size functional** we understand now a positive, continuous function Σ on Δ_0 which is homogeneous of some degree $k > 0$ and which, if restricted to the simplices S with $R(S) = 1$, has the property that Σ attains a maximum (denoted by τ) and that $V_d/\Sigma^{1/k}$ is bounded. By homogeneity, we then have

$$\Sigma(S) \leq \tau R(S)^{1/k} \tag{37}$$

for all $S \in \Delta_0$. Every simplex S for which (37) holds with equality is called an **extremal simplex**. For given Σ, a **deviation functional** is defined as a nonnegative, continuous function ϑ on Δ_0 which is homogeneous of degree zero and satisfies $\vartheta(S) = 0$ if and only if S is an extremal simplex. For given Σ and ϑ, a **stability function** is a continuous function $f : [0, 1] \to [0, 1]$ with the properties that $f(0) = 0$, $f(x) > 0$ for $x > 0$ and

$$\Sigma(S) \leq (1 - f(\epsilon))\tau R(S)^{1/k} \qquad \text{whenever } \vartheta(S) \geq \epsilon. \tag{38}$$

Now suppose that Σ, ϑ, f with these properties are given. If $\epsilon \in (0, 1)$ and if $a > 0$ is sufficiently large, then

$$\mathbb{P}\{\vartheta(Z) \geq \epsilon \mid \Sigma(Z) \geq a\} \leq c \exp\left(-c_0 f(\epsilon)\widetilde{\gamma} a^{d/k}\right), \tag{39}$$

with constants c, c_0 independent of a.

For concrete size functionals Σ, this yields results on asymptotic shapes if the extremal simplices of the isoperimetric inequality (37) can be determined. This can be surprisingly difficult; for example, it is still not known whether the simplices of extremal mean width inscribed to the unit sphere are regular. Other cases are simpler. For $\Sigma = V_d$, the volume, it is easy to see that all maximal simplices are regular, and in [5] a stability result of type (38), with a stability function of optimal order, was obtained, for the following deviation functional. For a simplex $S \in \Delta_0$, let $\vartheta(S)$ be the smallest number η for which there exists a regular simplex $T \in \Delta_0$ with $R(S) = 1$ such that for each vertex p of $R(S)^{-1}S$ there is a vertex q of T with $\|p - q\| \leq \eta$, and conversely. The version of (39) proved in [5] reads

$$\mathbb{P}\{\vartheta(Z) \geq \epsilon \mid V_d(Z) \geq a\} \leq c \exp\left(-c_0 \epsilon^2 \widetilde{\gamma} a\right),$$

where c depends only on d and ϵ and c_0 depends on d.

Further, in [6] the following concrete cases of the general result (39) were treated. For each of the size functionals: surface area, inradius, minimal width, the asymptotic shape of the typical cell is that of the regular simplices. For the case of the surface area, it follows from a more general result of Tanner [32] that the extremal simplices of (37) are the regular ones. The case of the inradius is easier, and for the minimal width, a result of Alexander [1] was used. If the diameter is chosen as size functional, then the asymptotic shapes of large typical cells are provided by the diametral simplices. A simplex S is called diametral if a longest edge of S is a diameter of the circumsphere of S. In the plane, these are the right-angled triangles.

8. General Mosaics

For random mosaics that are not of the special types described in the previous sections, only very few extremal results have been obtained. In the plane, we can mention inequalities due to Mecke [13], which are parallel to results on deterministic mosaics by L. Fejes Tóth. Let X be a stationary random mosaic in $\mathbb{R}^2$. Let Λ denote the mean total length of the edges of X per unit area; with the notation of [30, sect. 10.1], this is $\Lambda = d_1^{(1)} = \overline{V}_1(X^{(1)})$. Further, let N ($= n_{20} = n_{21}$ in the notation of [30]) be the mean number of edges of the typical cell of X. Mecke has proved the following two theorems.

Suppose that all cells of X have the same area F. Then

$$\Lambda^2 \geq \frac{N}{F} \tan \frac{\pi}{N}.$$

Equality holds if and only if $N \in \{3, 4, 6\}$ and all cells of X are regular N-gons.

Suppose that all cells of X have the same perimeter U. Then

$$\Lambda \geq \frac{2N}{U} \tan \frac{\pi}{N}.$$

Equality holds if and only if $N \in \{3, 4, 6\}$ and all cells of X are regular N-gons.

For general mosaics in higher dimensions, we know only of one result exhibiting an extremal property. Let X be a stationary random mosaic in $\mathbb{R}^d$. An inequality of Wieacker [33] connects the volume of the Blaschke body $B(X)$ with the expected volume of the zero cell Z_0, namely

$$V_d(B(X))^{d-1} \mathbb{E} V_d(Z_0) \geq 1. \tag{40}$$

Since this result is not mentioned in [30], we present here the proof in the style of [30]. First we show that

$$\int_{S^{d-1}} f(u)\, S_{d-1}(B(X), \mathrm{d}u) = \mathbb{E}\left(V_d(Z_0)^{-1} \int_{S^{d-1}} f(u)\, S_{d-1}(Z_0, \mathrm{d}u) \right) \tag{41}$$

for any nonnegative measurable function f on S^{d-1}. In fact, writing

$$g(K) := \int_{S^{d-1}} f(u)\, S_{d-1}(K, \mathrm{d}u) \qquad \text{for } K \in \mathcal{K},$$

we get from [30, Th. 10.4.1] and Fubini's theorem for kernels (with $\mathbb{Q}^{(d)}, \gamma^{(d)}, Z$ as in Section 2)

$$\mathbb{E}\left(V_d(Z_0)^{-1} \int_{S^{d-1}} f(u)\, S_{d-1}(Z_0, \mathrm{d}u) \right) = \mathbb{E}(V_d(Z_0)^{-1} g(Z_0)) = \gamma^{(d)} \mathbb{E}\, g(Z)$$

$$= \gamma^{(d)} \int_{\mathcal{K}_0} g(K)\, \mathbb{Q}^{(d)}(\mathrm{d}K) = \gamma^{(d)} \int_{\mathcal{K}_0} \int_{S^{d-1}} f(u)\, S_{d-1}(K, \mathrm{d}u)\, \mathbb{Q}^{(d)}(\mathrm{d}K)$$

$$= \int_{S^{d-1}} f(u)\, S_{d-1}(B(X), \mathrm{d}u),$$

which proves (41). Now we use (23), together with Minkowski's inequality for mixed volumes and Jensen's inequality for concave functions, to obtain

$$\begin{aligned} V_d(B(X)) &= \frac{1}{d}\int_{S^{d-1}} h(B(X),u)\, S_{d-1}(B(X),\mathrm{d}u) \\ &= \frac{1}{d}\,\mathbb{E}\left(V_d(Z_0)^{-1}\int_{S^{d-1}} h(B(X),u)\, S_{d-1}(Z_0,\mathrm{d}u)\right) \\ &= \mathbb{E}\left(V_d(Z_0)^{-1} V(B(X),Z_0,\dots,Z_0)\right) \\ &\geq V_d(B(X))^{\frac{1}{d}}\,\mathbb{E}\left(V_d(Z_0)^{-\frac{1}{d}}\right) \\ &\geq V_d(B(X))^{\frac{1}{d}}\left(\mathbb{E}V_d(Z_0)\right)^{-\frac{1}{d}}. \end{aligned}$$

This gives (40). Equality holds if and only if there exists a convex body K such that a.s. every realization of X consists of translates of K (so the randomness affects only the translations).

References

[1] Alexander, R.: The width and diameter of a simplex. Geom. Dedicata, **6**, 87–94 (1977)

[2] Fejes Tóth, L.: Reguläre Figuren. Akadémiai Kiadó, Budapest, and Teubner, Leipzig (1965)

[3] Hug, D., Reitzner, M., Schneider, R.: The limit shape of the zero cell in a stationary Poisson hyperplane tessellation. Ann. Probab., **32**, 1140–1167 (2004)

[4] Hug, D., Reitzner, M., Schneider, R.: Large Poisson–Voronoi cells and Crofton cells. Adv. Appl. Prob. (SGSA), **36**, 667–690 (2004)

[5] Hug, D., Schneider, R.: Large cells in Poisson–Delaunay tessellations. Discrete Comput. Geom., **31**, 503–514 (2004)

[6] Hug, D., Schneider, R.: Large typical cells in Poisson–Delaunay mosaics. Rev. Roumaine Math. Pures Appl., **50**, 657–670 (2005)

[7] Hug, D., Schneider, R.: Asymptotic shapes of large cells in random tessellations. Geom. Funct. Anal., **17**, 156–191 (2007)

[8] Hug, D., Schneider, R.: Typical cells in Poisson hyperplane tessellations. Discrete Comput. Geom., **38**, 305–319 (2007)

[9] Kovalenko, I.N.: A proof of a conjecture of David Kendall on the shape of random polygons of large area. (Russian) Kibernet. Sistem. Anal. 1997, 3–10, 187; Engl. transl.: Cybernet. Systems Anal., **33**, 461–467 (1997)

[10] Kovalenko, I.N.: An extension of a conjecture of D.G. Kendall concerning shapes of random polygons to Poisson Voronoï cells. In: Engel, P. *et al.* (eds) Voronoï's impact on modern science, Book I. Transl. from the Ukrainian, Kyiv: Institute of Mathematics. Proc. Inst. Math. Natl. Acad. Sci. Ukr., Math. Appl., **212**, 266–274 (1998)

[11] Kovalenko, I.N.: A simplified proof of a conjecture of D.G. Kendall concerning shapes of random polygons. J. Appl. Math. Stochastic Anal., **12**, 301–310 (1999)

[12] Mecke, J.: Inequalities for intersection densities of superpositions of stationary Poisson hyperplane processes. In: Jensen, E.B., Gundersen, H.J.G. (eds.) Proc. Second Int. Workshop Stereology, Stochastic Geometry, pp. 115–124, Aarhus (1983)

[13] Mecke, J.: Isoperimetric properties of stationary random mosaics. Math. Nachr., **117**, 75–82 (1984)

[14] Mecke, J.: On some inequalities for Poisson networks. Math. Nachr., **128**, 81–86 (1986)

[15] Mecke, J.: An extremal property of random flats. J. Microsc., **151**, 205–209 (1988)

[16] Mecke, J.: On the intersection density of flat processes. Math. Nachr., **151**, 69–74 (1991)

[17] Mecke, J.: Inequalities for the anisotropic Poisson polytope. Adv. Appl. Prob. (SGSA), **27**, 56–62 (1995)

[18] Mecke, J.: Inequalities for mixed stationary Poisson hyperplane tessellations. Adv. Appl. Prob. (SGSA), **30**, 921–928 (1998)

[19] Mecke, J.: On the relationship between the 0-cell and the typical cell of a stationary random tessellation. Pattern Recognition, **32**, 1645–1648 (1999)

[20] Mecke, J., Osburg, I.: On the shape of large Crofton parallelotopes. Math. Notae, **41** (2001/02), 149–157 (2003)

[21] Mecke, J., Schneider, R., Stoyan, D., Weil, W., *Stochastische Geometrie.* DMV-Seminar **16**, Birkhäuser, Basel (1990)

[22] Mecke, J., Thomas, C.: On an extreme value problem for flat processes. Commun. Stat., Stochastic Models (2), **2**, 273–280 (1986)

[23] Miles, R.E.: Random polytopes: the generalisation to n dimensions of the intervals of a Poisson process. Ph.D. Thesis, Cambridge University (1961)

[24] Miles, R.E.: A heuristic proof of a long-standing conjecture of D.G. Kendall concerning the shapes of certain large random polygons. Adv. Appl. Prob. (SGSA), **27**, 397–417 (1995)

[25] Okabe, A., Boots, B., Sugihara, K., Chiu, S.N.: Spatial Tessellations; Concepts and Applications of Voronoi Diagrams. 2nd ed., Wiley, Chichester (2000)

[26] Schneider, R.: Convex Bodies: the Brunn–Minkowski Theory. Cambridge University Press, Cambridge (1993)

[27] Schneider, R., Nonstationary Poisson hyperplanes and their induced tessellations. Adv. Appl. Prob. (SGSA), **35**, 139–158 (2003)

[28] Schneider, R.: Weighted faces of Poisson hyperplane tessellations. Adv. Appl. Prob. (SGSA), to appear

[29] Schneider, R.: Vertex numbers of weighted faces in Poisson hyperplane mosaics. Submitted

[30] Schneider, R., Weil, W., Stochastic and Integral Geometry. Springer, Berlin, Heidelberg (2008)

[31] Stoyan, D., Kendall, W.S., Mecke, J.: Stochastic Geometry and Its Applications. 2nd ed., Wiley, Chichester (1995)

[32] Tanner, R.M.: Some content maximizing properties of the regular simplex. Pacific J. Math., **52**, 611–616 (1974)

[33] Wieacker, J.A.: Geometric inequalities for random surfaces. Math. Nachr., **142**, 73–106 (1989)