Some geometry of convex bodies in $C(K)$ spaces

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Dedicated to the memory of Robert R. Phelps

Abstract

We deal with some problems related to vector addition and diametric completion procedures of convex bodies in $C(K)$ spaces. We prove that each of the following properties of convex bodies in $C(K)$ characterizes the underlying compact Hausdorff space K as a Stonean space: (i) $C(K)$ has a generating unit ball; (ii) all Maehara sets in $C(K)$ are complete; (iii) the set D_d of all complete sets of diameter d in $C(K)$ is convex. In contrast to these results, we further show the following for all $C(K)$ spaces: (a) a weaker version of the generating unit ball property is satisfied; (b) there is a Maehara-like completion procedure; (c) D_d is starshaped with respect to any ball of radius $d/2$. The proofs are based on a systematic investigation of generalized order intervals and intersections of balls, which is carried out in the first part of the paper, and of diametrically complete sets. A new characterization of spaces ℓ_∞^n among all n-dimensional real normed spaces, in terms of summands and intersections of translates of the unit ball, is finally provided.

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1 Introduction

The geometry of convex bodies in Banach spaces poses some interesting questions about interrelations between different ways of generating convex bodies, in particular intersections of balls and vector sums, and special metric notions such as constant width and diametric completeness. One instance are generating sets and their application to metric completions. The unit ball of a Banach space X is a *generating set* (introduced by Balashov and Polovinkin [2] for reflexive Banach spaces, but the notion extends) if any nonempty intersection of translates of the unit ball is a summand of it. Using suitable intersections of balls and vector addition, one can associate (as explained in Section 2) with every subset $A \subset X$ of diameter $d > 0$ a certain set $\mu(A)$ of diameter d containing A, called the *Maehara set* of A. If X is a reflexive Banach space with generating unit

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ball, then it was shown by Polovinkin [26] (and for Euclidean spaces earlier by Maehara [14]) that $\mu(A)$ is of constant width and hence is a diametric completion of A.

The study of diametrically complete sets has its origin in the investigation of sets of constant width in Euclidean spaces. This is a classical topic of convex geometry, with a voluminous literature (see, e.g., the survey [4]), and still actively studied. It is known since Meissner [15] that in a Euclidean space the sets of constant width are identical with the diametrically complete sets, and since Eggleston [6] that in a Minkowski space (a finite-dimensional normed space) every set of constant width is diametrically complete, but not conversely. In fact, in a normed space, the family of sets of constant width can be rather poor. For example, Yost [29] has shown that for $n \geq 3$ in most (in the sense of Baire category) n-dimensional Minkowski spaces the only sets of positive constant width are the balls. This is one of the reasons why the study of constant width sets in Euclidean spaces should in normed spaces be replaced by the study of diametrically complete sets. Such sets are characterized by the 'spherical intersection property': a set of diameter $d > 0$ in a normed space is diametrically complete if and only if it is the intersection of all closed balls with centre in the set and with radius d.

We mention that intersections of balls in relation to vector sums are studied in [9] and (for finite-dimensional polyhedral spaces) in [20], and that the behaviour of diametrically complete sets under vector addition is investigated in [18]. Diametrically complete sets in finite-dimensional normed spaces, in relation to vector addition, are the subject of [22] and [23].

The present paper is devoted to similar investigations in $C(K)$ spaces, with special attention to the interplay between the topology of K and the convex geometry of $C(K)$. Given a compact Hausdorff space K, the space $C(K)$ is the Banach space of continuous real functions on K with the supremum norm. These spaces, which are reflexive only when K is finite, show some interesting features with respect to the notions mentioned above, and surprisingly many specific properties lead to a characterization of $C(K)$ over a Stonean (that is, extremally disconnected) space K. Here we may mention the result, due to Davis [5] (see also Theorem II.4.8 in [13]), that a Banach space has Jung constant 1 if and only if it is $C(K)$ for some Stonean K. Especially in $C(K)$ spaces, it was shown in [16], among other results, that the set of intersections of balls is stable under vector addition if and only if K is Stonean. The complete hull mapping in $C(K)$ spaces was investigated in [17], where it was shown, for example, that the completion map in $C(K)$ is convex-valued if and only if K is Stonean.

The preliminary section 2 of the present paper provides more detailed explanations. Section 3 collects what we need about semicontinuous functions, which are an essential tool in what follows. Section 4 treats a special class of convex sets in $C(K)$, the (generalized) intervals. Section 5 is devoted to intersections of balls. After these preparatory sections, which contain also new results, the main results are proved in Sections 6 to 8. Theorem 6.1 says that $C(K)$ has a generating unit ball if and only if K is a Stonean space. Theorem 6.2 shows that, in contrast to this, all $C(K)$ spaces have the property that a nonempty intersection of two translates of the unit ball is a summand of the unit ball (a property which a reflexive Banach space has if and only if its unit ball is a generating set, as proved by Karasëv [11]). More precisely, this theorem states that in $C(K)$ a convex body is a summand of the unit ball if and only if it is an intersection of two translates of the unit ball. In Section 8 it is proved that this property characterizes the space ℓ_∞^n among all n-dimensional normed spaces. The main results of Section 7 are the characterization of Stonean K as precisely the spaces for which all Maehara sets in $C(K)$ are complete, and the observation that, nevertheless, in all $C(K)$ spaces a slight modification of the Maehara sets yields complete sets. This seems to be the first example of an explicit completion procedure that yields complete sets without necessarily yielding bodies of constant width. As a matter of fact, it was proved in [18] that in $C(K)$ the only closed sets of positive constant width are balls. Also proved in Section 7 is the fact that the set of all diametrically complete sets of given positive diameter in $C(K)$ is always starshaped, but is convex if and only if K is Stonean.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. The set $B(c,r) = \{x \in X : \|x-c\| \le r\}$, where $c \in X$ and $r > 0$, is the ball with centre c and radius r (all balls occurring in the following are closed). In particular, $B = B(0,1)$ is the unit ball of X. By a *convex body* in X we understand a nonempty bounded closed convex subset of X. For convex bodies $C, D \subset X$, their *vector sum* is the set

$$C + D = \{x + y : x \in C,\ y \in D\}.$$

This set is again bounded and convex, but not necessarily closed (except in reflexive Banach spaces, see [2], Lemma 1.1). We call the convex body $\overline{C+D}$ (where $\overline{M}$ denotes the closure of the set M) the *Minkowski sum* of C and D. The convex body C is called a *summand* of the convex body A if there is another convex body D such that A is the Minkowski sum of C and D.

The *diameter* of a bounded set $A \subset X$ is the number $\operatorname{diam} A = \sup_{x,y\in A} \|x-y\|$. The set A is *diametrically complete* (or briefly 'complete'), also called *diametrically maximal*, if

$$\operatorname{diam}(A \cup \{x\}) > \operatorname{diam} A \quad \text{for } x \in X \setminus A.$$

If $A \subset X$ is a bounded set, then any complete set $A' \subset X$ containing A and with the same diameter as A is called a *completion* of A. Every bounded set has at least one completion, but in general many. The *completion map* γ associates with A the set of all its completions. Lipschitz continuous selections of the completion map (with respect to suitable Hausdorff metrics) were studied in [21] and [24].

For $\lambda > 0$, a closed set $A \subset X$ is *of constant width* λ if

$$\sup_{x\in A}\langle x^*, x\rangle - \inf_{x\in A}\langle x^*, x\rangle = \lambda$$

for all $x^* \in X^*$ with $\|x^*\| = 1$. If this holds, then A is complete and $\operatorname{diam} A = \lambda$.

With a nonempty bounded set $A \subset X$ one can associate several intersections of balls. The *ball hull* $\beta(A)$ is the intersection of all balls containing A. Continuity properties of the ball hull mapping (with respect to the natural Hausdorff metric) were investigated in [19], where it was proved that β need not be continuous, even in three-dimensional normed spaces. For a set $A \subset X$ of diameter $d > 0$, its *wide spherical hull* is the set

$$\eta(A) = \bigcap_{x \in A} B(x, d),$$

and its *tight spherical hull* is defined by

$$\theta(A) = \bigcap_{x \in \eta(A)} B(x, d).$$

In contrast to the lack of continuity of β, the map η is locally Lipschitz continuous in Minkowski spaces ([21], [24]); the problem of the continuity of θ is still open. Under different names, the mappings η and θ have been studied in various contexts; see the references in the introduction to [21]. Our present interest in these maps stems from the fact that they lead to the Maehara set. The set

$$\mu(A) = \frac{1}{2}(\eta(A) + \theta(A))$$

contains A and has diameter d. We remark that the proof given for this fact in [24], Section 3, works in general normed spaces. The set $\mu(A)$ has been called the *Maehara set* of A, since Maehara [14] has shown for Euclidean spaces that $\mu(A)$ is of constant width and hence a completion of A. The same result for reflexive Banach spaces with a generating unit ball was proved by Polovinkin [26].

In the following, K is a compact Hausdorff space and $C(K)$ is the Banach space of continuous real functions on K with the supremum norm $\|\cdot\|_\infty$. The space K is called *Stonean* if it is extremally disconnected, which means that the closure of every open set in K is open.

The following will be used repeatedly. Suppose that K is not Stonean. Then there is an open set $G \subset K$ such that $\overline{G}$ is not open (in particular, $G \neq \emptyset$ and $\overline{G} \neq K$). Since $\overline{G}$ is not open, there exists a point $x_0 \in \overline{G}$ that is not an interior point of $\overline{G}$. Hence, if K is not Stonean, then there are

$$\text{an open set } G \subset K \text{ and a point } x_0 \in (\overline{G} \setminus G) \cap (\overline{K \setminus \overline{G}}). \tag{2.1}$$

3 Semicontinuous functions

For geometric investigations in $C(K)$ spaces, semicontinuous functions are an indispensable tool. In this section, we collect what we need about semicontinuous functions. We consider only bounded functions.

Let $f : K \to \mathbb{R}$ be bounded. For $x \in K'$, the set of accumulation points of K, we use the definitions

$$\underline{f}(x) = \liminf_{y \to x} f(y) = \sup_{u \in \mathcal{U}(x)} \inf_{y \in U \setminus \{x\}} f(y),$$

$$\overline{f}(x) = \limsup_{y \to x} f(y) = \inf_{u \in \mathcal{U}(x)} \sup_{y \in U \setminus \{x\}} f(y),$$

where $\mathcal{U}(x)$ denotes the system of neighbourhoods of x. (Note that some authors, for instance Bourbaki [3], use a non-equivalent definition.) Since real functions on K are continuous at each isolated point of K, it will be consistent in the following if for each isolated point x of K we define

$$\underline{f}(x) = \liminf_{y \to x} f(y) = f(x) = \limsup_{y \to x} f(y) = \overline{f}(x) \tag{3.1}$$

and then in the statements do not distinguish between accumulation points and isolated points.

We recall that a bounded function $f : K \to \mathbb{R}$ is *lower semicontinuous* (in the following abbreviated by l.s.c.) if

$$\underline{f} \geq f$$

(here and in the following, such relations between functions are meant pointwise). Equivalent to this is each of the following conditions (a) and (b). (a) For each $x \in K$ and each real $r < f(x)$ there is a neighbourhood U of x with $r < f(y)$ for all $y \in U$. (b) For each $r \in \mathbb{R}$, the set $\{x \in K : f(x) \leq r\}$ is closed. The function f is called *upper semicontinuous* (abbreviated by u.s.c.) if the function $-f$ is lower semicontinuous. The counterparts to conditions (a) and (b) are obvious, and f is u.s.c. if and only if

$$\overline{f} \leq f.$$

If $f : K \to \mathbb{R}$ is an arbitrary bounded function, then the function

$$f^\wedge = \sup\{h \in C(K) : h \leq f\}$$

is l.s.c. and is called the *lower semicontinuous envelope* of f. Similarly, the function

$$f^\vee = \inf\{h \in C(K) : h \geq f\}$$

is u.s.c. and is called the *upper semicontinuous envelope* of f. Since K is a normal space, we have

$$f^\wedge = \min\{f, \underline{f}\}, \qquad f^\vee = \max\{f, \overline{f}\}. \tag{3.2}$$

In fact, let $x \in K$ and let $s = \min\{f(x), \liminf_{y \to x} f(y)\}$. For given $\varepsilon > 0$, there is an open neighbourhood U of x with $f(y) > s - \varepsilon$ for $y \in U$. There is a constant $c < s - \varepsilon$ with $f \geq c$. By Urysohn's lemma, there is a continuous function $h : K \to [c, s - \varepsilon]$ with $h(x) = s - \varepsilon$ and $h(y) = c$ for $y \in K \setminus U$. It follows that $f^\wedge(x) \geq h(x) = s - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $f^\wedge(x) \geq s$. Suppose that $f^\wedge(x) > s$. Then there is a function $h \in C(K)$ with $f \geq h$ and $h(x) > s$. There are a number $\varepsilon > 0$ and a neighbourhood U of x with $h(y) \geq s + \varepsilon$ for $y \in U$. Then $f(y) \geq s + \varepsilon$ for $y \in U$ and hence $\min\{f(x), \liminf_{y \to x} f(y)\} \geq s + \varepsilon$, a contradiction. This proves that $f^\wedge(x) = s$ and thus the first equality of (3.2), and the second is obtained similarly.

If f is l.s.c., it follows from (3.2) that $f^\wedge = f$. Similarly, if f is u.s.c., then $f^\vee = f$.

For a function $f : K \to \mathbb{R}$, we denote by $\mathcal{D}_f$ the set of continuity points of f. If f is (lower or upper) semicontinuous, then $\mathcal{D}_f$ is a residual set (Fort [8]), hence it is dense in K, since K is a Baire space.

For a bounded function $f : K \to \mathbb{R}$, it follows again from the fact that K is a normal space (or from (3.2)) that

$$f^\wedge(x) = f(x) = f^\vee(x) \qquad \text{for all } x \in \mathcal{D}_f. \tag{3.3}$$

Theorem 3.1. *The space K is Stonean if and only if for any l.s.c. function $f : K \to \mathbb{R}$, the u.s.c. envelope $f^\vee$ is continuous. (Here l.s.c. and u.s.c. may be interchanged.)*

Proof. Suppose first that K is Stonean. Let $f : K \to \mathbb{R}$ be l.s.c. and suppose that $f^\vee$ is not continuous at some point $x_0 \in K$. Then we can choose a number r with

$$\liminf_{x \to x_0} f^\vee(x) < r < \limsup_{x \to x_0} f^\vee(x). \tag{3.4}$$

Since f is l.s.c., the set $M := \{x \in K : f(x) > r\}$ is open, and since $f^\vee$ is u.s.c., the set $N := \{x \in K : f^\vee(x) < r\}$ is open. If $x \in M$, then $f^\vee(x) \geq f(x) > r$, hence $x \notin N$. Thus $M \cap N = \emptyset$. Since K is extremally disconnected, also $\overline{M} \cap \overline{N} = \emptyset$ (see [7], 6.2.26). Because of (3.4), every neighbourhood of x_0 contains points x with $f(x) > r$ and also points y with $f^\vee(y) < r$, thus $x_0 \in \overline{M} \cap \overline{N}$. This contradiction shows that $f^\vee$ is continuous.

Suppose now that K is not a Stonean space. We choose an open set G and a point x_0 according to (2.1) and define $f : K \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \overline{G}, \\ 1 & \text{if } x \in K \setminus \overline{G}. \end{cases}$$

Then f is l.s.c. The function $f^\vee$ coincides with f on $G \cup (K \setminus \overline{G})$.

Since $x_0 \in (\overline{G} \setminus G) \cap (\overline{K \setminus \overline{G}})$, every neighbourhood of x_0 contains points $x \in G$, at which $f^\vee(x) = 0$, and also points $y \in K \setminus \overline{G}$, at which $f^\vee(y) = 1$. Therefore, $f^\vee$ is not continuous at x_0. □

4 Intervals

$C(K)$ spaces have a particularly simple, though nontrivial, class of convex bodies, the intervals. By an *interval* in $C(K)$ we mean a set of the form

$$[f, g] := \{h \in C(K) : f \leq h \leq g\},$$

where $f, g : K \to \mathbb{R}$ are bounded functions. Since the functions f, g defining $[f, g]$ are in general not elements of $C(K)$, this notion of interval is more general than that of the usual order intervals in partially ordered vector spaces. We remark that every interval is closed. In fact, let $\varphi \in \overline{[f, g]}$. Then to each $\varepsilon > 0$ there is some $h \in [f, g]$ with $\|\varphi - h\|_\infty < \varepsilon$, hence with $h - \varepsilon \leq \varphi \leq h + \varepsilon$, which together with $f \leq h \leq g$ gives $f - \varepsilon \leq \varphi \leq g + \varepsilon$. Since this holds for all $\varepsilon > 0$, we conclude that $\varphi \in [f, g]$, thus $[f, g]$ is closed. Nonempty intervals can be considered as the most basic convex bodies in $C(K)$.

The representation of an interval in the form $[f, g]$ is in general not unique. But every interval has a representation $[\varphi, \psi]$ with an u.s.c. function φ and a l.s.c. function ψ. If the interval is nonempty, then $\varphi \leq \psi$, and φ and ψ are uniquely determined. To see this, we define, for any nonempty bounded set $A \subset C(K)$, the functions

$$b_A := \inf\{\alpha : \alpha \in A\}, \qquad t_A := \sup\{\alpha : \alpha \in A\}. \tag{4.1}$$

(Here b and t stand for 'bottom' and 'top', respectively.) Note that b_A is u.s.c. and t_A is l.s.c. If $\xi \in A$, then $b_A \leq \xi \leq t_A$. If $A = [f, g]$ is a nonempty interval, then $f \leq \alpha \leq g$ for all $\alpha \in A$ and hence $f \leq b_A \leq t_A \leq g$, which gives $A \subset [b_A, t_A] \subset [f, g] = A$ and thus $A = [b_A, t_A]$. Suppose that a nonempty interval A is represented as $A = [f, g]$, where f is u.s.c. and g is l.s.c. Choose a function $\xi \in A$. Since f is u.s.c., we have

$$\begin{aligned} f &= \inf\{h \in C(K) : h \geq f\} = \inf\{\min\{h, \xi\} : h \in C(K),\, h \geq f\} \\ &= \inf\{\tilde{h} : \tilde{h} \in A\} = b_A, \end{aligned}$$

and similarly $g = t_A$.

We say that $[f, g]$ is an *interval in canonical representation* if the function f is u.s.c., the function g is l.s.c., and $f \leq g$. The following useful result was first proved by Tong [28] (see also [7], p. 61, for a sketch of the proof).

Lemma 4.1. *An interval in canonical representation is not empty.*

This allows us to prove a simple but very useful formula for the vector sum of intervals in canonical representation. It is reminiscent of the Riesz decomposition property in vector lattices, but it should be kept in mind that the functions f, g, φ, ψ appearing in (4.2) are in general not elements of $C(K)$.

Theorem 4.1. *If $[f, g]$ and $[\varphi, \psi]$ are intervals in canonical representation, then*

$$[f, g] + [\varphi, \psi] = [f + \varphi, g + \psi]. \tag{4.2}$$

Proof. The relation $[f, g] + [\varphi, \psi] \subset [f + \varphi, g + \psi]$ is trivial. Assume that $h \in [f + \varphi, g + \psi]$. Let

$$u = \max\{f, h - \psi\}, \qquad v = \min\{g, h - \varphi\}.$$

Since ψ is l.s.c. and h is continuous, the function $h - \psi$ is u.s.c. Since also f is u.s.c., the function u is u.s.c. Similarly, v is l.s.c. Moreover, $f \leq g$, $f \leq h - \varphi$, $h - \psi \leq g$, $h - \psi \leq h - \varphi$, hence $u \leq v$. By Lemma 4.1, there exists a function $h_1 \in [u, v]$. Then $f \leq h_1 \leq g$, and the function $h_2 = h - h_1$ satisfies $\varphi \leq h_2 \leq \psi$, thus $h = h_1 + h_2 \in [f, g] + [\varphi, \psi]$. □

In particular, this theorem shows that the vector sum of two intervals is always an interval (and hence closed).

In [16], Proposition 2.3, formula (4.2) was proved for the case where f and g are continuous while φ and ψ are arbitrary bounded functions.

Lemma 4.2. *Let $f, g : K \to \mathbb{R}$ be bounded functions. Then $[f, g] = [f^\vee, g^\wedge]$, and*

$$[f, g] \neq \emptyset \;\Leftrightarrow\; \max\{f, \overline{f}\} \leq \min\{g, \underline{g}\}. \tag{4.3}$$

If $[f, g] \neq \emptyset$, then to each $x \in K$ and each number λ with $f^\vee(x) \leq \lambda \leq g^\wedge(x)$ there is a function $h \in [f^\vee, g^\wedge]$ with $h(x) = \lambda$.

Proof. If $h \in [f, g]$, then $h \in C(K)$ and $f \leq h \leq g$, hence $f^\vee \leq h$. Similarly, $h \leq g^\wedge$. Thus, $h \in [f, g]$ implies $h \in [f^\vee, g^\wedge]$, and the converse is trivial.

Since $[f, g] = [f^\vee, g^\wedge]$, the relation $[f, g] \neq \emptyset$ is equivalent to $f^\vee \leq g^\wedge$, which by (3.2) is equivalent to the right side of (4.3).

Suppose now that $[f, g] \neq \emptyset$ and that a point $x \in K$ and a number λ with $f^\vee(x) \leq \lambda \leq g^\wedge(x)$ are given. We define $f^\vee_\lambda(y) = f^\vee(y)$ for $y \in K \setminus \{x\}$ and $f^\vee_\lambda(x) = \lambda$, similarly $g^\wedge_\lambda(y) = g^\wedge(y)$ for $y \in K \setminus \{x\}$ and $g^\wedge_\lambda(x) = \lambda$. Then $f^\vee_\lambda \leq g^\wedge_\lambda$, the function $f^\vee_\lambda$ is u.s.c. and $g^\wedge_\lambda$ is l.s.c. By Lemma 4.1, there exists a function $h \in [f^\vee_\lambda, g^\wedge_\lambda]$. It satisfies $h \in [f^\vee, g^\wedge]$ and $h(x) = \lambda$. □

Let us consider an interval $[f, g]$ where f is l.s.c. and g is u.s.c. Then $f \leq \underline{f} \leq \overline{f}$, hence $f^\vee = \max\{f, \overline{f}\} = \overline{f}$, similarly $g^\wedge = \underline{g}$. Therefore, $f^\vee \leq g^\wedge$ is equivalent to $\overline{f} \leq \underline{g}$. This yields the assertion of Proposition 2.1 in [16].

An immediate consequence of Lemma 4.2 is that the diameter of a nonempty interval $[f, g]$ is given by

$$\operatorname{diam}[f, g] = \sup\{|g^\wedge(x) - f^\vee(x)| : x \in K\}. \tag{4.4}$$

Now we want to investigate, for later use, under which conditions the simple formula for vector sums of intervals given by Theorem 4.1 extends to non-canonical representations. We prepare this by the following lemma.

Lemma 4.3. *Let $f, g : K \to \mathbb{R}$ be bounded functions. Then*

$$(f + g)^\wedge \geq f^\wedge + g^\wedge, \tag{4.5}$$

with equality if f and g are both l.s.c., and

$$(f + g)^\vee \leq f^\vee + g^\vee, \tag{4.6}$$

with equality if f and g are both u.s.c.

Proof. There is a constant c with $f \geq c$. Let $x \in K$. Defining $q(y) = c$ for $y \in K \setminus \{x\}$ and $q(x) = f^\wedge(x)$, we have $q^\vee = q$ and $q^\vee \leq f^\wedge$. By Lemma 4.1, there exists a function $u \in [q^\vee, f^\wedge]$, hence $u \in C(K)$, $u \leq f$ and $u(x) = f^\wedge(x)$. Similarly, there is $v \in C(K)$ with $v \leq g$ and $v(x) = g^\wedge(x)$. Then $u + v \leq f + g$ and $(u + v)(x) = f^\wedge(x) + g^\wedge(x)$. It follows that

$$(f + g)^\wedge(x) = \sup\{h(x) : h \in C(K),\ h \leq f + g\} \geq f^\wedge(x) + g^\wedge(x).$$

Since $x \in K$ was arbitrary, this shows that $(f + g)^\wedge \geq f^\vee + g^\vee$.

If f and g are both l.s.c., then also $f + g$ is l.s.c., and $f^\wedge = f$, $g^\wedge = g$, $(f + g)^\wedge = f + g$, as remarked in Section 3. The remaining assertions are obtained similarly. □

Theorem 4.2. *Let $f, g, \varphi, \psi : K \to \mathbb{R}$ be bounded functions such that $[f, g] \neq \emptyset$ and $[\varphi, \psi] \neq \emptyset$. Then*

$$[f, g] + [\varphi, \psi] = [f + \varphi, g + \psi] \tag{4.7}$$

holds if and only if

$$(g + \psi)^\wedge - (f + \varphi)^\vee = g^\wedge + \psi^\wedge - f^\vee - \varphi^\vee. \tag{4.8}$$

Proof. Suppose that (4.7) holds. Writing $(f + \varphi)^\vee = u$ and $(g + \psi)^\wedge = v$, we get from Lemma 4.2 that

$$\begin{aligned} [u, v] &= [(f + \varphi)^\vee, (g + \psi)^\wedge] = [f + \varphi, g + \psi] \\ &= [f, g] + [\varphi, \psi] = [f^\vee, g^\wedge] + [\varphi^\vee, \psi^\wedge]. \end{aligned} \tag{4.9}$$

Let $x \in K$. By Lemma 4.2, there exists $h \in [u, v]$ with $h(x) = v(x)$. By (4.9), there are $h_1 \in [f^\vee, g^\wedge]$, $h_2 \in [\varphi^\vee, \psi^\wedge]$ with $h = h_1 + h_2$. This gives

$$v(x) = h(x) = h_1(x) + h_2(x) \leq g^\wedge(x) + \psi^\wedge(x) \leq (g + \psi)^\wedge(x) = v(x),$$

where (4.5) was used. Hence, $v(x) = g^\wedge(x) + \psi^\wedge(x)$. Similarly, $u(x) = f^\vee(x) + \varphi^\vee(x)$. Therefore,

$$(g + \psi)^\wedge(x) - (f + \varphi)^\vee(x) = (g^\wedge + \psi^\wedge)(x) - (f^\vee + \varphi^\vee)(x).$$

Since $x \in K$ was arbitrary, (4.8) holds.

Conversely, suppose that (4.8) is true. Then it follows from (4.5) and (4.6) that $(f + \varphi)^\vee = f^\vee + \varphi^\vee$ and $(g + \psi)^\wedge = g^\wedge + \psi^\wedge$. Therefore, from Theorem 4.1 we get

$$[f^\vee, g^\wedge] + [\varphi^\vee, \psi^\wedge] = [f^\vee + \varphi^\vee, g^\wedge + \psi^\wedge] = [(f + \varphi)^\vee, (g + \psi)^\wedge].$$

By Lemma 4.2, this means that (4.7) holds. $\square$

Having an addition of convex bodies, the question for the totality of summands of a given convex body is quite natural. The following theorem solves this problem for the case of intervals.

Theorem 4.3. *Every summand of an interval in $C(K)$ is an interval.*

Proof. Let $[\varphi, \psi] \subset C(K)$ be an interval, and let $C, D \subset C(K)$ be convex bodies with $\overline{C + D} = [\varphi, \psi]$. The inclusions

$$C \subset [b_C, t_C], \qquad D \subset [b_D, t_D] \tag{4.10}$$

are trivial. For $f \in C$ and $g \in D$ we have $f + g \in [\varphi, \psi]$, thus $\varphi \leq f + g \leq \psi$. For fixed g, this holds for all $f \in C$, hence $\varphi \leq b_C + g \leq \psi$. Since this holds for all $g \in D$, we get $\varphi \leq b_C + b_D \leq \psi$, similarly $\varphi \leq t_C + t_D \leq \psi$, thus $\varphi \leq b_C + b_D \leq t_C + t_D \leq \psi$ and hence

$$[b_C + b_D, t_C + t_D] \subset [\varphi, \psi]. \tag{4.11}$$

Suppose there is some $h_0 \in [b_C, t_C] \setminus C$. By the Hahn–Banach separation theorem, there is a linear functional $x^* \in C(K)^*$ satisfying

$$x^*(h_0) > \sup\{x^*(h) : h \in C\}.$$

Then

$$\begin{aligned}
\sup\{x^*(h) : h \in \overline{C+D}\} &= \sup\{x^*(h) : h \in C+D\} \\
&= \sup\{x^*(h) : h \in C\} + \sup\{x^*(h) : h \in D\} \\
&< x^*(h_0) + \sup\{x^*(h) : h \in D\} \\
&\leq \sup\{x^*(h) : h \in [b_C, t_C]\} + \sup\{x^*(h) : h \in [b_D, t_D]\} \\
&= \sup\{x^*(h) : h \in [b_C + b_D, t_C + t_D]\} \\
&\leq \sup\{x^*(h) : h \in [\varphi, \psi]\},
\end{aligned}$$

where (4.11) was used. The strict inequality contradicts $\overline{C+D} = [\varphi, \psi]$. Thus, $C = [b_C, t_C]$, and a similar argument gives $D = [b_D, t_D]$. □

As a supplement to Theorem 4.3, we remark that every interval different from a singleton has a non-trivial summand, that is, one which is not homothetic to it. Suppose, first, that $[f, g]$ is an interval in canonical representation, that $h \in [f, g]$ and that $[f, h]$ is homothetic to $[f, g]$. This means that there exist $\lambda \in \mathbb{R}$ and $\xi \in C(K)$ such that $[f, h] = \lambda[f, g] + \xi = [\lambda f + \xi, \lambda g + \xi]$. Since canonical representations are unique, this gives $f = \lambda f + \xi$, $h = \lambda g + \xi$ and, therefore, $h = (1-\lambda) f + \lambda g$, necessarily with $\lambda \in [0, 1]$. If $[f, g]$ contains more than one function, then with the aid of Urysohn's lemma one can construct a function $h \in [f, g]$ which is not of this form. With such a function h, the interval $[f, h]$ is not homothetic to $[f, g]$. By Theorem 4.1, $[f, h] + [0, g - h] = [f, g]$, which shows that $[f, h]$ is a non-trivial summand of $[f, g]$.

5 Intersections of balls

We recall that by 'balls' we understand closed balls with positive radius. Thus, a ball in $C(K)$ is any set of the form

$$B(f, r) = \{h \in C(K) : \|h - f\|_\infty \leq r\} = \{h \in C(K) : f - r \leq h \leq f + r\}$$

with $f \in C(K)$ (the *centre* of the ball) and $r > 0$ (the *radius* of the ball). The unit ball $B = B(0, 1)$ (where 0 denotes the constant function with value zero) can also be written as $[-1, 1]$ (where 1 denotes the constant function with value one).

Convex bodies which are intersections of balls play a central role in several questions related to convex geometry and geometry of Banach spaces. Having a manageable characterization of these sets in $C(K)$ is essential for later work. In [18], an ordered pair of bounded functions $f, g : K \to \mathbb{R}$ was called an *admissible pair* if f is l.s.c., g is u.s.c., and $\overline{f} \leq \underline{g}$.

Lemma 5.1. *A nonempty set $A \subset C(K)$ is an intersection of balls if and only if it can be represented as $A = [f, g]$ with some admissible pair f, g, equivalently, if and only if $A = [b_A^\wedge, t_A^\vee]$.*

If K is Stonean, then any nonempty intersection of balls in $C(K)$ has a representation $[f, g]$ where f and g are continuous.

Proof. The first assertion was proved in [18], Proposition 4.1 (see also [16], p. 114). Let $A = [f, g]$ with some l.s.c. function f and some u.s.c. function g. Then $f \le b_A \le t_A \le g$, by definition (4.1). Since f is the supremum of continuous functions and g is the infimum of continuous functions, we have $f \le b_A^\wedge \le b_A \le t_A \le t_A^\vee \le g$, which gives $A \subset [b_A^\wedge, t_A^\vee] \subset [f, g] = A$, hence $A = [b_A^\wedge, t_A^\vee]$. Conversely, if $A = [b_A^\wedge, t_A^\vee]$ is nonempty, then $b_A^\wedge, t_A^\vee$ form an admissible pair.

Now let K be Stonean, and let A be a nonempty intersection of balls in $C(K)$. By the first part and by Lemma 4.2, $A = [b_A^\wedge, t_A^\vee] = [b_A^{\wedge\vee}, t_A^{\vee\wedge}]$. By Theorem 3.1, the functions $b_A^{\wedge\vee}$ and $t_A^{\vee\wedge}$ are continuous. □

In view of Theorem 4.1 it is of interest to characterize also the intervals in canonical representation that are intersections of balls. For this, we observe that an u.s.c. function f satisfies

$$\limsup_{y \to x,\, y \in \mathcal{D}_f} f(y) \le \limsup_{y \to x} f(y) \le f(x) \quad \text{for } x \in K$$

(recall that $\mathcal{D}_f$ is the set of continuity points of f; for isolated points x, we use a convention corresponding to (3.1)). We say that f is *strongly u.s.c.* if

$$\limsup_{y \to x,\, y \in \mathcal{D}_f} f(y) = f(x) \quad \text{for } x \in K.$$

Similarly, a l.s.c. function g is called *strongly l.s.c.* if

$$\liminf_{y \to x,\, y \in \mathcal{D}_g} g(y) = g(x) \quad \text{for } x \in K.$$

Theorem 5.1. *An interval $[f, g]$ in canonical representation is an intersection of balls if and only if f is strongly u.s.c. and g is strongly l.s.c.*

Proof. Let $C = [b_C, t_C]$ be an interval in canonical representation. We define $D = [b_C^\wedge, t_C^\vee]$ and claim that

$$b_D(x) = \limsup_{y \to x,\, y \in \mathcal{D}_{b_C}} b_C(y) \quad \text{for } x \in K. \tag{5.1}$$

For the proof, let $x \in K$ be given. We first use that b_D is u.s.c., that $b_D \ge b_C^\wedge$ by the definitions of D and of b_D, and that $b_C^\wedge(y) = b_C(y)$ for all continuity points y of b_C, to obtain

$$\begin{aligned} b_D(x) \ge \limsup_{y \to x} b_D(y) \quad &\ge \limsup_{y \to x,\, y \in \mathcal{D}_{b_C}} b_D(y) \\ &\ge \limsup_{y \to x,\, y \in \mathcal{D}_{b_C}} b_C^\wedge(y) = \limsup_{y \to x,\, y \in \mathcal{D}_{b_C}} b_C(y) =: s. \end{aligned}$$

Let $h \in D$. If $h(x) \leq s$, then $b_D(x) \leq s$. Suppose that $h(x) > s$. Choose $0 < \varepsilon < \frac{1}{3}(h(x)-s)$. There is an open neighbourhood $U \subset K$ of x such that $b_C(y) \leq s+\varepsilon$ for all $y \in U \cap \mathcal{D}_{b_C}$ and $h(y) > h(x) - \varepsilon$ for all $y \in U$. Since $\mathcal{D}_{b_C}$ is dense in K, we then have $b_C^\wedge(y) \leq s + \varepsilon$ for all $y \in U$. By Urysohn's lemma there exists a continuous function $\xi : K \to [0,1]$ with $\xi(x) = 1$ and $\xi(y) = 0$ for $y \in K \setminus U$. The function

$$\tilde{h} := h - (h(x) - s - 2\varepsilon)\xi \in C(K)$$

then satisfies $b_C^\wedge \leq \tilde{h} \leq h$, hence $\tilde{h} \in D$, and $\tilde{h}(x) = s + 2\varepsilon$. Thus $b_D(x) \leq s + 2\varepsilon$. Letting $\varepsilon \to 0$, we conclude that $b_D(x) = s$. Since $x \in K$ was arbitrary, this finishes the proof of (5.1).

Similarly, we obtain

$$t_D(x) = \liminf_{y \to x,\, y \in \mathcal{D}_{t_C}} t_C(y) \quad \text{for } x \in K. \tag{5.2}$$

By Lemma 5.1, C is an intersection of balls if and only if $C = D$. Using (5.1) and (5.2), we obtain

$$\begin{aligned} C = D \quad &\Leftrightarrow \quad b_C = b_D \ \text{ and } \ t_C = t_D \\ &\Leftrightarrow \quad b_C(x) = \limsup_{y \to x,\, y \in \mathcal{D}_{b_C}} b_C(y) \ \text{ and } \ t_C(x) = \liminf_{y \to x,\, y \in \mathcal{D}_{t_C}} t_C(y),\ \forall x \in K. \end{aligned}$$

The latter means that b_C is strongly u.s.c. and t_C is strongly l.s.c. □

We wish to point out that the use of the continuity points in our definition of strongly u.s.c. (and similarly of strongly l.s.c.) is essential, in other words, that the assumption

$$\limsup_{y \to x} f(y) = f(x) \quad \text{for } x \in K \tag{5.3}$$

is strictly weaker than strong upper semicontinuity of f (and would, therefore, not lead to a characterization of intersections of balls). As an example, consider the space $K = [0,1]$, let $\mathcal{C} \subset K$ be the usual Cantor set, and define the function f by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C}, \\ 0 & \text{if } x \in K \setminus \mathcal{C}. \end{cases}$$

Then f is u.s.c. since $\mathcal{C}$ is closed, and (5.3) holds since $\mathcal{C}$ is dense in itself and $\mathcal{D}_f = K \setminus \mathcal{C}$ is open. On the other hand, we have

$$\limsup_{y \to x,\, y \in \mathcal{D}_f} f(y) = 0 < 1 = \limsup_{y \to x} f(y) \quad \text{for } x \in \mathcal{C}.$$

Since the sum of a strongly u.s.c. (strongly l.s.c.) function and a continuous function is always strongly u.s.c. (strongly l.s.c.), it follows from Theorems 5.1 and 4.1 that in $C(K)$ the set $\mathcal{M}$ of intersections of balls is stable under adding a ball. This was already proved in [16], in a different way. However, the sum of two strongly u.s.c. functions need not be strongly u.s.c., as simple examples show. Therefore, in $C(K)$ the set $\mathcal{M}$ is in general not stable under Minkowski addition. In fact, as already mentioned, it was proved in [16] that the stability of $\mathcal{M}$ under Minkowski addition in $C(K)$ characterizes K as Stonean.

Now we describe some special intersections of balls associated with a bounded set $A \subset C(K)$, namely its ball hull $\beta(A)$, its wide spherical hull $\eta(A)$ and its tight spherical hull $\theta(A)$, as defined in Section 2.

Let $A \subset C(K)$ be a subset of diameter $d > 0$. Let $x \in K$. For each $\varepsilon > 0$ there exist $f \in A$ with $f(x) \leq b_A(x) + \varepsilon$ and $g \in A$ with $g(x) \geq t_A(x) - \varepsilon$, hence $t_A(x) - b_A(x) \leq g(x) - f(x) + 2\varepsilon \leq d + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $t_A(x) - b_A(x) \leq d$. Thus, $0 \leq t_A - b_A \leq d$.

Lemma 5.2. *Let $A \subset C(K)$ be a subset of diameter $d > 0$. Then*

$$\begin{aligned}\beta(A) &= [b_A^\wedge, t_A^\vee],\\ \eta(A) &= [t_A - d, b_A + d] = [t_A^\vee - d, b_A^\wedge + d],\\ \theta(A) &= [b_A^\wedge, t_A^\vee],\end{aligned}$$

thus the tight spherical hull is equal to the ball hull of A.

Proof. First we consider the ball hull. Let $f \in [b_A^\wedge, t_A^\vee]$. Let $B(h, r)$ be a ball with $A \subset B(h, r)$. Then $h - r \leq \alpha \leq h + r$ for all $\alpha \in A$, hence $h - r \leq b_A \leq t_A \leq h + r$. This gives $t_A^\vee \leq h + r$, hence $f \leq h + r$. Similarly, $h - r \leq f$ and thus $f \in B(h, r)$. Since $B(h, r)$ was an arbitrary ball containing A, it follows that $f \in \beta(A)$. We have proved that $[b_A^\wedge, t_A^\vee] \subset \beta(A)$.

To prove the converse, let $f \in \beta(A)$. Let $\psi \in C(K)$ be any function with $t_A \leq \psi$. There is a number $r > 0$ with $\psi - 2r \leq b_A$, thus $B(\psi - r, r)$ is a ball containing A. This implies that $f \in B(\psi - r, r)$, hence $f \leq \psi$. Since this holds for all $\psi \in C(K)$ with $t_A \leq \psi$, we have $f \leq t_A^\vee$. Similarly, $b_A^\wedge \leq f$. Thus $f \in [b_A^\wedge, t_A^\vee]$, and hence $\beta(A) \subset [b_A^\wedge, t_A^\vee]$.

The wide spherical hull of A is given by

$$\begin{aligned}\eta(A) = \bigcap\nolimits_{\alpha \in A} B(\alpha, d) &= \{f \in C(K) : \|f - \alpha\|_\infty \leq d, \ \forall \alpha \in A\}\\ &= \{f \in C(K) : \alpha - d \leq f \leq \alpha + d, \ \forall \alpha \in A\}\\ &= \{f \in C(K) : t_A - d \leq f \leq b_A + d\}\\ &= [t_A - d, b_A + d]\,.\end{aligned}$$

The second representation of $\eta(A)$ given in the lemma follows from Lemma 4.2.

The tight spherical hull of A is given by

$$\begin{aligned}
\theta(A) &= \bigcap\nolimits_{g\in\eta(A)} B(g,d) \\
&= \{f \in C(K) : \|f-g\|_\infty \leq d,\ \ \forall g \in [t_A^\vee - d, b_A^\wedge + d]\} \\
&= \{f \in C(K) : g(x) - d \leq f(x) \leq g(x) + d,\ \ \forall x \in K,\ \ \forall g \in [t_A^\vee - d, b_A^\wedge + d]\} \\
&= \{f \in C(K) : b_A^\wedge(x) \leq f(x) \leq t_A^\vee(x),\ \ \forall x \in K\} \\
&= [b_A^\wedge, t_A^\vee].
\end{aligned}$$

Here we have used that for $f \in C(K)$ and $x \in K$ the conditions

$$g(x) - d \leq f(x) \leq g(x) + d \quad \text{for all } g \in [t_A^\vee - d, b_A^\wedge + d] \tag{5.4}$$

and

$$b_A^\wedge(x) \leq f(x) \leq t_A^\vee(x) \tag{5.5}$$

are equivalent. For the proof, suppose first that (5.4) holds. Let $x \in K$. By Lemma 4.2, there is a function $g \in [t_A^\vee - d, b_A^\wedge + d]$ with $g(x) = b_A^\wedge(x) + d$, hence $b_A^\wedge(x) = g(x) - d \leq f(x)$. This is the left inequality of (5.5), and the right one is obtained similarly.

Conversely, suppose that (5.5) holds. Let $g \in [t_A^\vee - d, b_A^\wedge + d]$ Then

$$g(x) - d \leq b_A^\wedge(x) \leq f(x) \leq t_A^\vee(x) \leq g(x) + d,$$

thus (5.4) holds. □

6 Generating sets

We are now in a position to decide which $C(K)$ spaces have a generating unit ball. Recall that a closed convex set in a Banach space is called generating if any nonempty intersection of translates of the set is a summand of the set. For sufficiently smooth strictly convex sets in a reflexive Banach space, a criterion for the sets to be generating was developed by Ivanov [10]. There seems to be no previous study about generating sets in non-reflexive Banach spaces.

Theorem 6.1. *$C(K)$ has a generating unit ball if and only if K is a Stonean space.*

Proof. First let K be a Stonean space, and let C be a nonempty intersection of translates of the unit ball. By Lemma 5.1, there is a representation $C = [f, g]$ with continuous functions f, g. Since C is an intersection of translates of the unit ball $B = [-1, 1]$, it has diameter $\operatorname{diam} C \leq 2$. Thus $g - f \leq 2$ and, therefore, $[-1 - f, 1 - g] \neq \emptyset$. From Theorem 4.1 it now follows that $[f, g] + [-1 - f, 1 - g] = [-1, 1]$, thus $[f, g]$ is a summand of the unit ball.

Now suppose that K is not a Stonean space. Choose an open set G and a point x_0 according to (2.1) and define $f, g : K \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } x \in \overline{G}, \\ 0 & \text{if } x \in K \setminus \overline{G}, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \in G, \\ 1 & \text{if } x \in K \setminus G. \end{cases}$$

Then f is l.s.c., g is u.s.c., and $f \le g$. The set of common continuity points of f and g is $G \cup (K \setminus \overline{G})$. In fact, f and g are both continuous on this set, and if $x \in \overline{G} \setminus G$, then g is not continuous at x. It follows from (4.4) that the set $[f, g]$ has diameter 1, and by Theorem 4.5 in [18] (or see Theorem 7.1 below) it is complete. Therefore, $[f, g]$ has the spherical intersection property (Eggleston [6]), that is, it is the intersection of all balls with centre in $[f, g]$ and radius 1 and thus is an intersection of translates of the unit ball. We show that it is not a summand of the unit ball, which then shows that $C(K)$ does not have a generating unit ball.

Suppose, to the contrary, that there is a convex body $D \subset C(K)$ such that $\overline{[f, g] + D} = [-1, 1]$. Since $x_0 \in (\overline{G} \setminus G) \cap \overline{(K \setminus \overline{G})}$, every neighbourhood of x_0 contains points $x \in G$, and here $g(x) = 0$, and also points $y \in K \setminus \overline{G}$, and here $f(y) = 0$. Therefore, each function $h \in [f, g]$ satisfies $h(x_0) = 0$.

Let $\varphi \in D$. Then $[f, g] + \varphi \subset [-1, 1]$. In every neighbourhood of x_0 there exists a point x at which a suitable function $\tilde{h} \in [f, g]$ attains the value -1, and since $\tilde{h} + \varphi \ge -1$, it follows that $\varphi(x) \ge 0$. Similarly, in every neighbourhood of x_0 there exists a point y at which a suitable function $\tilde{h} \in [f, g]$ attains the value 1, which gives $\varphi(y) \le 0$. It follows that $\varphi(x_0) = 0$.

Therefore, any function $h + \varphi$ with $h \in [f, g]$ and $\varphi \in D$ satisfies $(h + \varphi)(x_0) = 0$. Then also each function $\psi \in \overline{[f, g] + D}$ satisfies $\psi(x_0) = 0$. But then $[-1, 1] \not\subset \overline{[f, g] + D}$, since $[-1, 1]$ contains functions ψ with $\psi(x_0) \neq 0$. □

For reflexive Banach spaces it was proved by Karasëv [11] that the unit ball is already a generating set if any nonempty intersection of two translates of the unit ball is a summand of it. In strong contrast to this, for $C(K)$ spaces the latter property is strictly weaker, as shown by Theorem 6.1 together with the following theorem.

Theorem 6.2. *For a convex body $C \subset C(K)$, the following assertions are equivalent.*

(a) *C is a summand of the unit ball.*

(b) *$C = [f, g]$ where $f, g \in C(K)$ and $\|f - g\|_\infty \le 2$.*

(c) *C is the intersection of two translates of the unit ball.*

Proof. If (a) holds, there exists a convex body D with $\overline{C + D} = [-1, 1]$. It follows from Theorem 4.3 that C and D are intervals and hence can be written in the form $C = [b_C, t_C]$, $D = [b_D, t_D]$. We assert that

$$b_C + b_D \ge -1, \qquad t_C + t_D \le 1. \tag{6.1}$$

In fact, if, say, the first inequality does not hold (the second is treated similarly), then there is $x_0 \in K$ such that $(b_C + b_D)(x_0) < -1 - 2\varepsilon$ for some $\varepsilon > 0$. By the definition of

b_C, b_D, there are functions $f \in C$ and $g \in D$ satisfying $f(x_0) < b_C(x_0) + \varepsilon$ and $g(x_0) < b_D(x_0) + \varepsilon$ and hence $(f + g)(x_0) < -1$, which contradicts $f + g \in C + D \subset [-1, 1]$.

Using Theorem 4.1, the fact that intervals are closed, and (6.1), we get

$$[-1, 1] = \overline{[b_C, t_C] + [b_D, t_D]} = \overline{[b_C + b_D, t_C + t_D]} = [b_C + b_D, t_C + t_D] \subset [-1, 1].$$

Since equality holds here, we have $b_C + b_D = -1$ and $t_C + t_D = 1$. Since b_C, b_D are u.s.c. and their sum is continuous, they are both continuous, and similarly t_C, t_D are continuous. Moreover, it follows from $b_C, t_C \in C$ and $\operatorname{diam} C \le \operatorname{diam}[-1, 1]$ that $\|b_C - t_C\|_\infty \le 2$. Thus (b) is established.

Suppose that (b) holds, thus $C = [f, g]$ with continuous functions f, g satisfying $\|f - g\|_\infty \le 2$. Then the balls $B_f := [f, f + 2] = B(f + 1, 1)$ and $B_g := [g - 2, g] = B(g - 1, 1)$ satisfy

$$B_f \cap B_g = [\max\{f, g - 2\}, \min\{f + 2, g\}] = [f, g] = C,$$

thus C is the intersection of two translates of the unit ball, as stated in (c).

If (c) holds, then $C = [f - 1, f + 1] \cap [g - 1, g + 1]$ with suitable $f, g \in C(K)$, hence

$$C = [\max\{f - 1, g - 1\}, \min\{f + 1, g + 1\}].$$

Defining $D = [\min\{-f, -g\}, \max\{-f, -g\}]$, we have (using Theorem 4.1)

$$C + D = [\max\{f - 1, g - 1\}, \min\{f + 1, g + 1\}] + [\min\{-f, -g\}, \max\{-f, -g\}] = [-1, 1],$$

thus (a) holds. □

Leaving the field of $C(K)$ spaces, one may ask which Banach spaces are characterized by the equivalence of properties (a) and (c) in Theorem 6.2. For finite-dimensional normed spaces, we answer this question in Section 8.

7 Diametric completions

This section is devoted to (diametrically) complete sets and to completions in $C(K)$. First we state an extension of a characterization of complete sets in $C(K)$ that was proved in [18], Theorem 4.5.

Theorem 7.1. *A convex body $C \subset C(K)$ is diametrically complete of diameter $d > 0$ if and only if there are a representation $C = [f, g]$ and a dense set $S \subset K$ of common continuity points of f and g with $g(x) - f(x) = d$ for $x \in S$ and such that* (a) *or* (b) *holds:*

(a) *f is l.s.c. and g is u.s.c.,*

(b) *f is strongly u.s.c. and g is strongly l.s.c.*

Proof. With condition (a), this was proved in [18], Theorem 4.5, except that there it was assumed that S is the set of all common continuity points of f and g. It is clear that only a dense subset of this set is needed.

To prove the theorem with condition (b), assume first that C is complete. By the result already proved, there are a representation $C = [f, g]$, where f is l.s.c. and g is u.s.c., and a dense set $S \subset K$ of common continuity points of f and g such that $g(x) - f(x) = d$ for $x \in S$. By Lemma 4.2, $C = [f, g] = [f^\vee, g^\wedge]$, and the latter is the canonical representation of C. Since C is an intersection of balls, it follows from Theorem 5.1 that $f^\vee$ is strongly u.s.c. and $g^\wedge$ is strongly l.s.c. If $x \in S$, then x is also a common continuity point of $f^\vee$ and $g^\wedge$, and $f^\vee(x) = f(x)$, $g^\wedge(x) = g(x)$. Thus the condition of the theorem, using (b), is necessary.

Suppose now that the condition of the theorem, using (b), is satisfied, that is, $C = [f, g]$, where f is strongly u.s.c. and g is strongly l.s.c. and where $g(x) - f(x) = d$ for $x \in S$, for some dense set S of common continuity points of f and g. As in the proof of Theorem 4.5 in [18] it follows that C has diameter d. To show that C is complete, let $h \in C(K) \setminus C$. Then there is some $z \in K$ with $h(z) \notin [f(z), g(z)]$, say $h(z) > g(z)$. Since g is strongly l.s.c., we have

$$\liminf_{y \to z,\, y \in \mathcal{D}_g} g(y) = g(z) < h(z).$$

Since h is continuous and $S \subset \mathcal{D}_g$ is dense in K, there exists some point $x_1 \in S$ with $g(x_1) < h(x_1)$. Now the argument used *loc. cit.* shows that $\operatorname{diam}(C \cup \{h\}) > d$, which completes the proof. □

The preceding theorem gives us the opportunity to add a short remark on sets with unique completion. Results on sets with unique completion in finite-dimensional normed spaces can be found, for instance, in [12] and [25].

Corollary 7.1. *For a subset $A \subset C(K)$ of diameter $d > 0$, the following conditions* (a), (b), (c) *are equivalent.*

(a) *A has a unique completion.*

(b) *The ball hull $\beta(A)$ is complete.*

(c) *$t_A(x) - b_A(x) = d$ for every common point of continuity of b_A and t_A.*

Proof. Suppose D is the unique completion of A. Every $x \in \eta(A)$ satisfies $\sup\{\|x - y\| : y \in A\} \le d$, hence $A \cup \{x\}$ is contained in some completion of A, necessarily in D. This shows that $\eta(A) \subset D$. Since every completion of A is contained in $\eta(A)$, we have $\eta(A) = D$. Since $\eta(A)$, being complete, has the spherical intersection property, we have $\eta(A) = \theta(A)$, and by Lemma 5.2 the latter is equal to $\beta(A)$, which hence is complete. Thus (a) implies (b). Conversely, if $\beta(A)$ is complete, then it is clearly the unique completion of A. Finally, the equivalence of (b) and (c) follows from Theorem 7.1, Lemma 5.2, and the fact that every point of continuity of b_A (t_A) is a point of continuity of $b_A^\wedge$ ($t_A^\vee$). □

We point out that the equivalence of (a) and (b) in Corollary 7.1 not hold in general normed spaces. For instance, if A is an equilateral triangle in the plane $\mathbb{R}^2$ with the

Euclidean norm, then A has a unique completion, namely the corresponding Reuleaux triangle. On the other hand, $\beta(A) = A$, which is not complete.

Now we study whether Maehara sets in $C(K)$ are complete, and if not, by what they can be replaced to serve the same purpose.

Let $A \subset C(K)$ be a subset of diameter d. By Lemma 5.2, $\eta(A) = [t_A - d, b_A + d]$ and $\theta(A) = [b_A^{\wedge}, t_A^{\vee}]$, therefore the Maehara set $\mu(A)$ of A is given by

$$\begin{aligned} \mu(A) &= \frac{1}{2}\left(\eta(A) + \theta(A)\right) \\ &= \frac{1}{2}\left[t_A - d, b_A + d\right] + \frac{1}{2}[b_A^{\wedge}, t_A^{\vee}] \\ &\subset \left[\frac{1}{2}\left(t_A - d + b_A^{\wedge}\right), \frac{1}{2}\left(b_A + d + t_A^{\vee}\right)\right] =: \tilde{\mu}(A). \end{aligned} \tag{7.1}$$

While, as we shall see, the Maehara set $\mu(A)$ need not be a completion of A, the set $\tilde{\mu}(A)$ always is. In fact, it is the case $\lambda = 1/2$ of the family of completions described in the following theorem.

Theorem 7.2. *Let $0 \leq \lambda \leq 1$. For any subset $A \subset C(K)$ of diameter $d > 0$, the interval*

$$\mu_\lambda(A) = \left[(1-\lambda)(t_A - d) + \lambda b_A^{\wedge},\ \lambda(b_A + d) + (1-\lambda)t_A^{\vee}\right]$$

is a completion of A.

Proof. Each of the functions $b_A, t_A, b_A^{\wedge}, t_A^{\vee}$ is semicontinuous and hence has a residual set of continuity points. Since K is a Baire space, the intersection of finitely many residual sets is residual and hence dense. If x is a continuity point of b_A, then $b_A^{\wedge}(x) = b_A(x)$, and similarly $t_A^{\vee}(x) = t_A(x)$. Thus, there is a dense set $S \subset K$ such that at each $x \in S$ the functions $b_A, t_A, b_A^{\wedge}, t_A^{\vee}$ are continuous and $b_A(x) = b_A^{\wedge}(x)$, $t_A(x) = t_A^{\vee}(x)$, hence

$$\lambda(b_A(x) + d) + (1-\lambda)t_A^{\vee}(x) - (1-\lambda)(t_A(x) - d) - \lambda b_A^{\wedge}(x) = d \qquad \text{for } x \in S. \tag{7.2}$$

Since $(1-\lambda)(t_A - d) + \lambda b_A^{\wedge}$ is l.s.c. and $\lambda(b_A + d) + (1-\lambda)t_A^{\vee}$ is u.s.c., it now follows from Theorem 7.1 that $\mu_\lambda(A)$ is complete and of diameter d. Thus $\mu_\lambda(A)$ is a completion of A. $\square$

In general, the inclusion in (7.1) is strict. If this is the case, then the Maehara set $\mu(A)$ is a proper subset of $\tilde{\mu}(A)$. Since the latter set is a completion of A, the set $\mu(A)$ is not complete. Thus, in $C(K)$ spaces the mapping $\tilde{\mu}$ plays the role that the Maehara mapping μ plays in reflexive Banach spaces with a generating unit ball.

Theorem 7.3. *All Maehara sets in $C(K)$ are complete if and only if K is Stonean.*

Proof. Suppose that K is Stonean. Then for any set $A \subset C(K)$ of diameter $d > 0$ we have

$$\begin{aligned} 2\mu(A) &= \eta(A) + \theta(A) = [t_A - d, b_A + d] + [b_A^\wedge, t_A^\vee] \\ &= \left[t_A - d + b_A^\wedge, b_A + d + t_A^\vee\right] = 2\tilde{\mu}(A), \end{aligned}$$

where we have used that by Theorem 3.1 the functions $b_A^\wedge, t_A^\vee$ are continuous, so that Proposition 2.3 of [16] can be applied. Since $\tilde{\mu}(A)$ is a completion of A, so is $\mu(A)$.

Conversely, suppose now that all Maehara sets in $C(K)$ are complete. Assume that K is not Stonean. We choose an open set G and a point x_0 according to (2.1) and define $f, g : K \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \overline{G}, \\ -1 & \text{if } x \in K \setminus \overline{G}, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \overline{G}, \\ 1 & \text{if } x \in K \setminus \overline{G}. \end{cases}$$

Let $A = [f, g]$. Then $d := \operatorname{diam} A = 2$. Since $[f, g]$ is an interval in canonical representation, we have

$$b_A = f, \qquad t_A = g,$$

which gives

$$\begin{aligned} b_A^\wedge(x) &= f(x) \quad \text{for } x \in G \cup (K \setminus \overline{G}), \\ t_A^\vee(x) &= g(x) \quad \text{for } x \in G \cup (K \setminus \overline{G}). \end{aligned}$$

For $x \in G \cup (K \setminus \overline{G}) = K \setminus \partial G$ we obtain

$$(b_A + t_A^\vee)(x) = f(x) + g(x) = 0,$$

and for $x \in \partial G$ we get

$$(b_A + t_A^\vee)(x) = f(x) + g^\vee(x) = 0 + g^\vee(x) \geq 0.$$

This shows that the function $\tilde{h} \equiv 0$ satisfies $\tilde{h} \leq b_A + t_A^\vee$, which gives

$$(b_A + t_A^\vee)^\wedge(x_0) = \sup\{h(x_0) : h \in C(K),\, h \leq b_A + t_A^\vee\} \geq 0.$$

Similarly we obtain $(t_A + b_A^\wedge)^\vee(x_0) \leq 0$. Together this yields

$$(b_A + d + t_A^\vee)^\wedge(x_0) - (t_A - d + b_A^\wedge)^\vee(x_0) \geq 2d. \tag{7.3}$$

On the other hand, every function $h \in C(K)$ with $h \leq b_A$ satisfies $h \leq -1$ on $K \setminus \overline{G}$ and hence $h(x_0) \leq -1$. This gives $b_A^\wedge(x_0) \leq -1$. Every function $h \in C(K)$ with $h \leq t_A^\vee$ satisfies $h \leq 0$ on G and hence $h(x_0) \leq 0$, which gives $t_A^{\vee\wedge}(x_0) \leq 0$. Similarly, $t_A^\vee(x_0) \geq 1$ and $b_A^{\wedge\vee}(x_0) \geq 0$. Thus we obtain

$$(b_A + d)^\wedge(x_0) + t_A^{\vee\wedge}(x_0) - (t_A - d)^\vee(x_0) - b_A^{\wedge\vee}(x_0) \leq 2d - 2. \tag{7.4}$$

Now (7.3), (7.4) together with Theorem 4.2 show that strict inclusion holds in (7.1). Thus, the Maehara set $\mu(A)$ is a proper subset of $\tilde{\mu}(A)$ and is therefore not complete. Since we have supposed that all Maehara sets in $C(K)$ are complete, this contradiction shows that K must be Stonean. $\square$

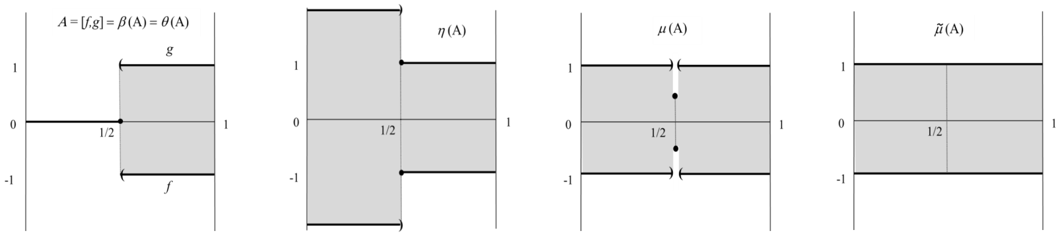

Fig.1. The figure illustrates the construction used in the previous proof, for K=[0,1] and G=[0,1/2). Notice that in this example we have $\beta(\mu(A)) = \tilde{\mu}(A)$, but it seems doubtful whether this holds generally.

For $d > 0$, we denote by D_d the set of all complete sets of diameter d in our space. For finite-dimensional normed spaces, several structural properties of D_d were investigated in [22] and [23]. For example, it was shown in [23] that in Minkowski spaces the set D_d is always contractible, but in general not starshaped ([23], Theorem 1, together with [22], Proposition 1), hence a fortiori not convex. The set $\mathcal{K}$ of convex bodies in $C(K)$ is equipped with vector addition and multiplication by nonnegative real numbers, so that starshapedness and convexity of subsets are meaningful notions; it is further endowed with the topology induced by the Hausdorff metric.

Theorem 7.4. *Let $d > 0$. In $C(K)$, the set D_d is starshaped with respect to the ball $B(0, d/2)$. In particular, D_d is pathwise connected.*

The set D_d is convex if and only if K is Stonean.

Proof. To prove that D_d is starshaped and pathwise connected, it is sufficient to treat the case $d = 2$. Let $A \in D_2$ be given. We define

$$F(\lambda) = (1-\lambda)A + \lambda B \qquad \text{for } \lambda \in [0,1].$$

Thus, if we use the canonical representation $A = [b_A, t_A]$ and put

$$b_\lambda(x) = (1-\lambda)b_A(x) - \lambda, \quad t_\lambda(x) = (1-\lambda)t_A(x) + \lambda \qquad \text{for } x \in K,\ \lambda \in [0,1],$$

then Theorem 4.1 gives

$$F(\lambda) = [b_\lambda, t_\lambda].$$

We have $F(0) = A$ and $F(1) = B$. Since A is complete, Theorem 7.1 shows that b_A is strongly u.s.c. and t_A is strongly l.s.c. (here we use that the canonical representation is unique). Moreover, there exists a dense set S of common continuity points of b_A and t_A with $t_A(x) - b_A(x) = 2$ for $x \in S$. The set S satisfies $S \subset \mathcal{D}_{b_\lambda} \cap \mathcal{D}_{t_\lambda}$ for $\lambda \in [0,1]$. For $x \in S$ and $\lambda \in [0,1]$ we have

$$\begin{aligned} t_\lambda(x) - b_\lambda(x) &= (1-\lambda)t_A(x) + \lambda - ((1-\lambda)b_A(x) - \lambda) \\ &= (1-\lambda)(t_A(x) - b_A(x)) + 2\lambda = 2. \end{aligned}$$

Now Theorem 7.1 yields that $F(A,\lambda) \in D_2$ for $\lambda \in [0,1]$. This shows that D_2 is starshaped with respect to the unit ball.

To show that D_2 is pathwise connected, it suffices to show that F is continuous. For this, we note that there is a constant k with $|b_A|, |t_A| \leq k$. For $\lambda, \mu \in [0,1]$ we obtain

$$t_\lambda = (1-\lambda)t_A + \lambda = (1-\mu)t_A + \mu + (\mu-\lambda)t_A + \lambda - \mu \leq t_\mu + |\mu-\lambda|(1+k)$$

and similarly

$$b_\lambda \geq b_\mu - |\mu-\lambda|(1+k).$$

This gives

$$F(\lambda) = [b_\lambda, t_\lambda] \subset [b_\mu - |\mu-\lambda|(1+k), t_\mu + |\mu-\lambda|(1+k)] = F(\mu) + |\mu-\lambda|(1+k)B,$$

which implies the continuity of F.

Now we assume that K is Stonean. Let $C, D \in D_d$. Being complete, C and D are intersections of balls. By Lemma 5.1, in their unique canonical representations $C = [f,g]$, $D = [\varphi,\psi]$, the functions f, g, φ, ψ are continuous. It follows from Theorem 7.1 that $g(x) - f(x) = d$ and $\psi(x) - \varphi(x) = d$ for all $x \in K$. Let $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$. Using Theorem 4.1, we get

$$\lambda C + \mu D = [\lambda f, \lambda g] + [\mu\varphi, \mu\psi] = [\lambda f + \mu\varphi, \lambda g + \mu\psi].$$

Here $\lambda f + \mu\varphi$ and $\lambda g + \mu\psi$ are continuous functions satisfying

$$(\lambda f + \mu\varphi)(x) - (\lambda g + \mu\psi)(x) = d \qquad \text{for } x \in K.$$

From Theorem 7.1 it follows that $\lambda C + \mu D$ is complete and of diameter d and thus an element of D_d. We have shown that D_d is convex.

Conversely, suppose that K is not Stonean. We choose an open set G and a point x_0 according to (2.1) and define

$$f(x) = \begin{cases} -1 & \text{if } x \in G, \\ 0 & \text{if } x \in K \setminus G, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \overline{G}, \\ 1 & \text{if } x \in K \setminus \overline{G}, \end{cases}$$

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in \overline{G}, \\ -1 & \text{if } x \in K \setminus \overline{G}, \end{cases} \qquad \psi(x) = \begin{cases} 1 & \text{if } x \in G, \\ 0 & \text{if } x \in K \setminus G. \end{cases}$$

Then $C = [f,g]$ and $D = [\varphi,\psi]$ are intervals in canonical representation; in fact, f and φ are strongly u.s.c. and g and ψ are strongly l.s.c. The set $S = G \cup (K \setminus \overline{G})$ is dense in K and consists of common continuity points of f, g, φ, ψ. For $x \in S$ we have $g(x) - f(x) = 1$ and $\psi(x) - \varphi(x) = 1$. By Theorem 7.1, C and D are complete and of diameter 1.

Again from Theorem 4.1, we have $\frac{1}{2}(C+D) = [\frac{1}{2}(f+\varphi), \frac{1}{2}(g+\psi)]$, and here

$$\frac{1}{2}(f+\varphi)(x) = -\frac{1}{2}, \qquad \frac{1}{2}(g+\psi)(x) = \frac{1}{2} \qquad \text{for } x \in G \cup (K \setminus \overline{G}).$$

Since $x_0 \in (\overline{G} \setminus G) \cap (\overline{K \setminus \overline{G}})$, we have $f(x_0) = g(x_0) = \varphi(x_0) = \psi(x_0) = 0$. It follows that $\frac{1}{2}(f+\varphi)$, though u.s.c., is not strongly u.s.c. (and similarly, $\frac{1}{2}(g+\psi)$ is not strongly l.s.c.). Now Theorem 7.1 yields that $\frac{1}{2}(C + D)$ is not complete, which shows that D_2 is not convex. □

In the first part of Theorem 7.4, the ball $B(0, d/2)$ can be replaced by any translate of this ball, as the proof shows.

8 A characterization of ℓ_∞^n

Among all n-dimensional normed spaces ($n \in \mathbb{N}$), the space ℓ_∞^n is characterized by the equivalence of conditions (a) and (c) in Theorem 6.2. This is a consequence of the following result.

Theorem 8.1. *Let $S \subset \mathbb{R}^n$ be a 0-symmetric convex body (with interior points) with the following property. For any nonempty convex body M, the conditions*

(a) *M is an intersection of two translates of S,*

(b) *M is a summand of S,*

are equivalent. Then S is a parallelepiped.

Proof. In the first part of the proof we use the assumption (b) $\Rightarrow$ (a) to show that S is the intersection of a pointed closed convex cone C with a reflected image of C. In the second part of the proof we use (a) $\Rightarrow$ (b) to show that a cross-section of C must be a simplex.

The body $M = \frac{1}{2}S$ is homothetic to S and is (trivially) a summand of S. From (b) $\Rightarrow$ (a) it follows that $M = (S + z_1) \cap (S + z_2)$ for suitable vectors z_1, z_2. Since S is 0-symmetric, the body M is symmetric with respect to $m := (z_1 + z_2)/2$. Let x be the intersection point of $\operatorname{bd} M$ with the segment $[z_1, z_2]$ such that m lies between z_1 and x. Since M is homothetic to S with dilatation factor 1/2, we have $2(x - m) = x - z_1$, which gives $x = z_2$. Thus, $z_2 \in \operatorname{bd}(S + z_1)$. Translating by $-z_1$, we obtain that the intersection $S \cap (S + z_2 - z_1)$ is homothetic to S and $z_2 - z_1 \in \operatorname{bd} S$. Now it follows from Lemma 3.1 in [1] that there exist a closed convex cone $C \subset \mathbb{R}^n$ with apex 0 and a vector p such that

$$S = (p - C) \cap (C - p).$$

Since S is bounded, the cone C is pointed, that is, it has 0 as its only apex. Since $0 \in \operatorname{int} S$, we have $p \in \operatorname{int} C$.

For the second part of the proof, we choose $\varepsilon > 0$ such that $B(0, 4\varepsilon) \subset S$ (where $B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$ and $\|\cdot\|$ denotes the norm induced by the Euclidean scalar product). For a closed convex set A and a unit vector u we denote by $H(A, u)$ the supporting hyperplane (if it exists) of A with outer normal vector u, and we write $F(A, u) = A \cap H(A, u)$ for the face of A in direction u. Since C is pointed, we can choose a unit vector u such that $F(C, u) = \{0\}$. We choose $\alpha > 0$ such that the part of C cut off by the hyperplane $H' = H(C, u) - \alpha u$ is contained in $B(0, \varepsilon)$. Let $L = C \cap H'$. Our aim is to show that L is an $(n - 1)$-dimensional simplex.

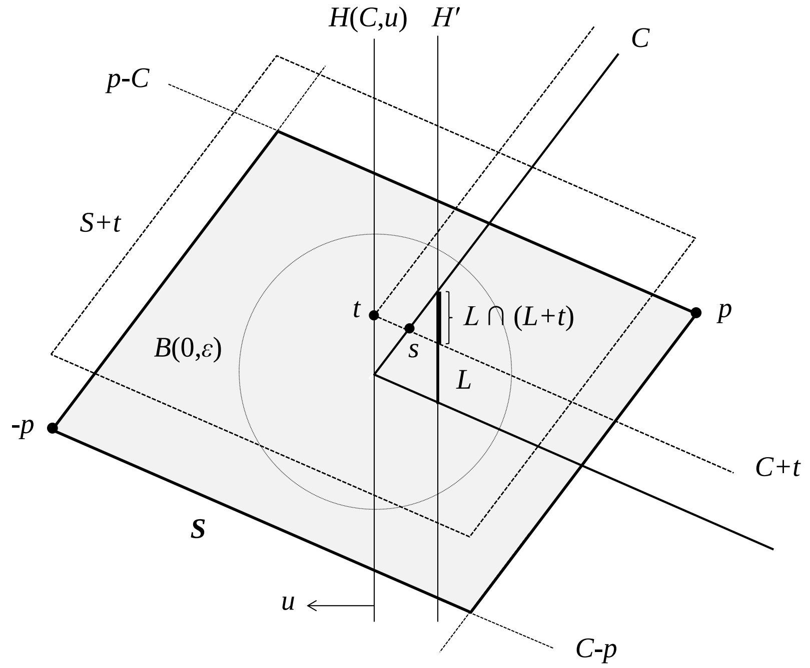

Fig.2. Note that the assumption $B(0,4\varepsilon) \subset S$ would only be shown by the figure after rescaling a part of it.

Let t be a vector such that $L \cap (L+t) \neq \emptyset$; then $t \in H(C,u)$. We want to show that $L \cap (L+t)$ is homothetic to L. Since $L \subset B(0,\varepsilon)$, we have $\|t\| \leq 2\varepsilon$. We choose a point

$$s \in F(C \cap (C+t), u).$$

Since $L \cap (L+t)$ is not empty, we have $s \in B(0,\varepsilon)$. Since C is a convex cone,

$$C + s \subset C \cap (C+t). \tag{8.1}$$

We have assumed (a) $\Rightarrow$ (b), hence the set $S \cap (S+t)$ (which is nonempty) is a summand of S. By a well-known criterion for summands (see, e.g., [27], Theorem 3.2.2), this implies the following. Since $-p$ is a boundary point of S, there exists a vector q such that

$$-p \in [S \cap (S+t)] - q \subset S. \tag{8.2}$$

This implies that $F([S \cap (S+t)] - q, u) \subset F(S,u) = F(C-p,u) = \{-p\}$. Since $S = (p-C) \cap (C-p)$, this can be written as

$$F([(p-C) \cap (C-p) \cap (p-C+t) \cap (C-p+t)] - q, u) = \{-p\}.$$

Since $-p \in \operatorname{int}(p-C)$ (which follows from $p \in \operatorname{int} C$ and hence $2p \in \operatorname{int} C$) and $-p \in \operatorname{int}(p-C+t)$ (which follows from $B(0,4\varepsilon) \subset C-p$ and hence $B(p,4\varepsilon) \subset C$, which implies $B(2p,4\varepsilon) \subset C$), we conclude that

$$F([(C-p) \cap (C-p+t) - q, u) = \{-p\}$$

and hence

$$F(C \cap (C+t), u) = \{q\}.$$

According to the choice of s, this implies $s \in \{q\}$ and thus $s = q$.

Let $a \in \mathbb{R}^n$ be a vector with $\|a\| \le 3\varepsilon$. Then

$$B(0,\varepsilon) - a \subset B(0, 4\varepsilon) \subset S \subset p - C,$$

hence $B(-p,\varepsilon) - a \subset -C$. Since $-C$ is a convex cone with $-p \in -C$, it follows that $B(-p,\varepsilon) - a - p \subset -C$ and thus $B(-p,\varepsilon) - a \subset p - C$. Therefore, $(B(-p,\varepsilon) - a) \cap S = (B(-p,\varepsilon) - a) \cap (C - p)$, equivalently

$$B(-p,\varepsilon) \cap (S + a) = B(-p,\varepsilon) \cap (C - p + a). \tag{8.3}$$

From (8.2), where $q = s$, we have

$$B(-p,\varepsilon) \cap (S - s) \cap (S + t - s) \subset B(-p,\varepsilon) \cap S.$$

Since $\|s\| \le \varepsilon$ and $\|t - s\| \le 3\varepsilon$, we can apply (8.3) and obtain

$$B(-p,\varepsilon) \cap (C - p - s) \cap (C - p + t - s) \subset B(-p,\varepsilon) \cap (C - p),$$

which gives

$$B(0,\varepsilon) \cap (C - s) \cap (C + t - s) \subset B(0,\varepsilon) \cap C.$$

Together with (8.1) this shows that

$$B(s,\varepsilon) \cap C \cap (C + t) = B(s,\varepsilon) \cap (C + s). \tag{8.4}$$

Recall that $L = H' \cap C$. Since $L \subset B(0,\varepsilon)$, we conclude from (8.4) that

$$L \cap (L + t) = (H' \cap C) \cap (H' \cap (C + t)) = H' \cap (C + s).$$

But this implies that $L \cap (L + t)$ is homothetic to L.

We have shown that any nonempty intersection $L \cap (L+t)$ is homothetic to L. This implies that L is a simplex (see, e.g., [27], p. 411). Thus, C is a cone with a simplex cross-section, which means that S is a parallelepiped. □

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