

HÖLDER CONTINUITY OF NORMAL CYCLES AND OF SUPPORT MEASURES OF CONVEX BODIES

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ABSTRACT. The normal cycle T_K associated with a convex body $K \subset \mathbb{R}^n$ is a current which in principle contains complete information about K . It is known that if a sequence of convex bodies K_i , $i \in \mathbb{N}$, converges to a convex body K in the Hausdorff metric, then the associated normal cycles T_{K_i} converge to T_K in the dual flat seminorm. We give a quantitative improvement of this convergence result by providing an estimate of the distance (in the dual flat seminorm) of the normal cycles of convex bodies with given Hausdorff distance. The support measures of a convex body K arise from a local Steiner formula or, alternatively, by evaluating suitable differential forms at the normal cycle of K . Complementing the estimate for the normal cycles, we establish an upper bound for the distance (in the bounded Lipschitz metric) of the support measures of two convex bodies in terms of the Hausdorff distance of these bodies. A special case of these estimates yields reverse forms of known stability results for area measures.

1. INTRODUCTION

In 1986, Martina Zähle [20] presented a current representation of Federer's curvature measures. For $k = 0, \dots, n-1$, she defined a differential form φ_k of degree $n-1$ on the Euclidean space \mathbb{R}^{2n} such that, for each compact set K of positive reach in \mathbb{R}^n and for each Borel set β in \mathbb{R}^n , the evaluation of $\mathbf{1}_\beta \varphi_k$ at the normal cycle of K yields the k th curvature measure of K , evaluated at β . The curvature measures had previously been introduced by Federer [6] in a different way. The approach using currents has later been investigated, applied and considerably extended in work of Zähle [21, 22], Rataj and Zähle [16, 17, 18], Fu [8], Pokorný and Rataj [15], and others. The normal cycle has thus become an important tool for the treatment of curvature properties of very general classes of sets, and it has found many applications in integral geometry. In this note, we restrict ourselves to convex bodies. The normal cycle T_K of a convex body K in \mathbb{R}^n (we recall the definition in Section 2) has a useful continuity property. If K_i , $i \in \mathbb{N}$, and K are convex bodies in \mathbb{R}^n and $K_i \rightarrow K$ in the Hausdorff metric, as $i \rightarrow \infty$, then $T_{K_i} \rightarrow T_K$ in the dual flat seminorm for currents. This was stated without proof in [22, p. 251] and was proved in [16, Thm. 3.1]; see also [9, Thm. 3.1]. The continuity property has been used in the theory of valuations on manifolds (see, for instance, [1]). It is also a crucial ingredient in [13], in the course of the proof for a classification theorem for local tensor valuations on the space of convex bodies.

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The purpose of this note is to obtain a quantitative improvement of the preceding continuity result, in the form of a Hölder estimate. To formulate it, we denote by \mathcal{K}^n the space of convex bodies (nonempty compact convex subsets) in Euclidean space \mathbb{R}^n , as usual equipped with the Hausdorff metric d_H . We write B^n and \mathbb{S}^{n-1} for the unit ball and the unit sphere, respectively, in \mathbb{R}^n , and \mathcal{H}^k for the k -dimensional Hausdorff measure. For $\rho > 0$, $K_\rho := K + \rho B^n$ is the parallel body of the convex body K at distance ρ . Below, we identify $\mathbb{R}^n \times \mathbb{R}^n$ with \mathbb{R}^{2n} and denote by $\mathcal{E}^{n-1}(\mathbb{R}^{2n}) = \mathcal{E}(\mathbb{R}^{2n}, \wedge^{n-1} \mathbb{R}^{2n})$ the vector space of all differential forms of degree $n-1$ on \mathbb{R}^{2n} with real coefficients and of class C^∞ .

Theorem 1.1. *Let $K, L \in \mathcal{K}^n$, and let $M \subset \mathbb{R}^{2n}$ be a compact convex set containing $K_1 \times \mathbb{S}^{n-1}$ and $L_1 \times \mathbb{S}^{n-1}$. Then, for each $\varphi \in \mathcal{E}^{n-1}(\mathbb{R}^{2n})$,*

$$|T_K(\varphi) - T_L(\varphi)| \leq C(M, \varphi) d_H(K, L)^{\frac{1}{2n+1}},$$

where $C(M, \varphi)$ is a constant which depends (for given dimension) on M and on the Lipschitz constant and the sup-norm of φ on M .

According to the definition of the dual flat seminorm, this result can be interpreted as local Hölder continuity of the normal cycles of convex bodies with respect to the Hausdorff metric and the dual flat seminorm. A similar, but essentially different quantitative result is obtained in [3, Thm. 2]. It refers to more general sets and is, therefore, less explicit. On the other hand, its restriction to convex bodies does not yield the present result, since at least one of the sets in [3] has to be bounded by a submanifold of class C^2 .

As mentioned, normal cycles are useful for introducing curvature measures for quite general classes of sets. In the theory of convex bodies, they have been used to introduce a generalization of curvature measures, the support measures. For \mathbb{R}^n , these are Borel measures on the product of \mathbb{R}^n and the unit sphere \mathbb{S}^{n-1} , with the property that their marginal measures are the curvature measures on one hand and the surface area measures on the other hand. On the space of convex bodies with the Hausdorff metric, the support measures are weakly continuous. We improve this statement by showing that the support measures are locally Hölder continuous with respect to the bounded Lipschitz metric d_{bL} (we recall its definition in Section 4). Let $\Lambda_i(K, \cdot)$ denote the i th support measure of $K \in \mathcal{K}^n$, normalized as explained in Section 4.

Theorem 1.2. *Let $K, L \in \mathcal{K}^n$ be convex bodies, and let R be the radius of a ball containing K_2 and L_2 . Then*

$$d_{bL}(\Lambda_i(K, \cdot), \Lambda_i(L, \cdot)) \leq C(R) d_H(K, L)^{1/2}$$

for $i \in \{0, \dots, n-1\}$, where $C(R)$ is a constant which (for given dimension) depends only on R .

An estimate of this type, though with a smaller exponent of the Hausdorff distance, could be derived directly from Theorem 1.1. We shall obtain the stronger result of Theorem 1.2 by adapting an approach due to Chazal, Cohen-Steiner and Mérigot [2].

A special case of Theorem 1.2 concerns the area measure $S_{n-1}(K, \cdot)$. If $\omega \subset \mathbb{S}^{n-1}$ is a Borel set, then $S_{n-1}(K, \omega) = 2\Lambda_{n-1}(K, \mathbb{R}_n \times \omega)$. From Theorem 1.2 it follows under the same assumptions on K and L that

$$(1.1) \quad d_{bL}(S_{n-1}(K, \cdot), S_{n-1}(L, \cdot)) \leq C'(R) d_H(K, L)^{1/2}.$$

We want to present some motivation for proving such an inequality.

The area measure is the subject of a famous existence and uniqueness theorem due to Minkowski (see, e.g., [19, Sec. 7.1]). The uniqueness assertion has been improved by some stability results. One of these (going back to Diskant; see [4], [19, Thm. 7.2.2]) says that for convex bodies $K, L \in \mathcal{K}^n$ one has

$$(1.2) \quad d_H(K, L') \leq \gamma \|S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot)\|_{\text{TV}}^{1/n}$$

for a suitable translate L' of L , where $\|\cdot\|_{\text{TV}}$ denotes the total variation norm. Here $\gamma > 0$ is a constant depending only on the dimension and on a-priori bounds for the inradius and circumradius of K and L .

The stability result (1.2) has the blemish that its assumption is too strong: the left side can be small even if the right side is large. For example, if K is a unit cube and L is a rotated image of K , arbitrarily close to K but not a translate of it, then $\|S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot)\|_{\text{TV}} \geq 1$. It seems, therefore, more meaningful to replace the right-hand side in (1.2) by an expression involving a metric for measures that metrizes the weak convergence. For the Lévy–Prokhorov metric, such a stability result was proved in [12]. It was deduced from a corresponding stability result for the bounded Lipschitz metric (which is implicit in the proof, though it was not stated explicitly), namely

$$(1.3) \quad d_H(K, L') \leq \gamma d_{bL}(S_{n-1}(K, \cdot), S_{n-1}(L, \cdot))^{1/n}$$

for a suitable translate L' of L , with a constant γ as above. It appears that the Hölder continuity (1.1) is, in principle, a more elementary fact than its reverse, the stability estimate (1.3), and should therefore have preceded it.

2. NOTATION AND PRELIMINARIES

We have to use several definitions and results from geometric measure theory, therefore we choose most of our notation as in Federer's [7] book, in order to facilitate the comparison. For example, we denote the scalar product in \mathbb{R}^n by \bullet and the induced norm by $|\cdot|$. The same notation is used also for other Euclidean spaces which will come up in the following. We identify \mathbb{R}^n and its dual space via the given scalar product.

Next we recall some notation and basic facts from multilinear algebra. Let V be finite-dimensional real vector spaces. Then $\bigwedge_m V$, for $m \in \mathbb{N}_0$, denotes the vector space of m -vectors of V , and $\bigwedge^m V$ is the vector space of all m -linear alternating maps from V^m to \mathbb{R} , whose elements are called m -covectors. The map $\bigwedge^m V \rightarrow \text{Hom}(\bigwedge_m V, \mathbb{R})$, which assigns to $f \in \bigwedge^m V$ the homomorphism

$$v_1 \wedge \dots \wedge v_m \mapsto f(v_1, \dots, v_m),$$

allows us to identify $\bigwedge^m V$ and $\text{Hom}(\bigwedge_m V, \mathbb{R})$. By this identification, the dual pairing of elements $a \in \bigwedge_m V$ and $\varphi \in \bigwedge^m(V, \mathbb{R})$ can be defined by $\langle a, \varphi \rangle := \varphi(a)$. If V' is another finite-dimensional vector space and $f : V \rightarrow V'$ is a linear map, then a linear map $\bigwedge_m f : \bigwedge_m V \rightarrow \bigwedge_m V'$ is determined by

$$(\bigwedge_m f)(v_1 \wedge \dots \wedge v_m) = f(v_1) \wedge \dots \wedge f(v_m),$$

for all $v_1, \dots, v_m \in V$.

Given an inner product space (V, \bullet) with norm $|\cdot|$ we obtain an inner product on $\bigwedge_m V$. For $\xi, \eta \in \bigwedge_m V$ with $\xi = v_1 \wedge \dots \wedge v_m$ and $\eta = w_1 \wedge \dots \wedge w_m$, where

$v_i, w_j \in V$, we define

$$\xi \bullet \eta = \det \left(\langle v_i, w_j \rangle_{i,j=1}^m \right).$$

This is independent of the particular representation of ξ, η . For general $\xi, \eta \in \bigwedge_m V$ the inner product is defined by linear extension, and then we put $|\xi| := \sqrt{\xi \bullet \xi}$ for $\xi \in \bigwedge_m V$. If (b_1, \dots, b_n) is an orthonormal basis of V , then the m -vectors $b_{i_1} \wedge \dots \wedge b_{i_m}$ with $1 \leq i_1 < \dots < i_m \leq n$ form an orthonormal basis of $\bigwedge_m V$. Moreover, if $\xi \in \bigwedge_p V$ or $\eta \in \bigwedge_q V$ is simple, then

$$(2.1) \quad |\xi \wedge \eta| \leq |\xi| |\eta|.$$

Let (b_1, \dots, b_n) be an orthonormal basis of V , and let (b_1^*, \dots, b_n^*) be the dual basis in $V^* = \bigwedge^1 V$. We endow $\bigwedge^m V$ with the inner product for which the vectors $b_{i_1}^* \wedge \dots \wedge b_{i_m}^*$, for $1 \leq i_1 < \dots < i_m \leq n$, are an orthonormal basis. Then

$$(2.2) \quad |\langle \xi, \Phi \rangle| \leq |\xi| |\Phi|$$

for $\xi \in \bigwedge_m V$ and $\Phi \in \bigwedge^m V$.

The preceding facts are essentially taken from [7, Section 1.7].

Let V be an n -dimensional inner product space. Then *comass* and *mass* are defined as in [7, Section 1.8]. In particular, for $\Phi \in \bigwedge^m V$ the comass $\|\Phi\|$ of Φ satisfies $\|\Phi\| = |\Phi|$ if Φ is simple. Moreover, for $\xi \in \bigwedge_m V$ the mass $\|\xi\|$ of ξ satisfies $\|\xi\| = |\xi|$ if ξ is simple.

Now we turn to convex bodies. For notions from the theory of convex bodies which are not explained here, we refer to [19]. Let $K \in \mathcal{K}^n$. The metric projection, which is denoted by $p(K, \cdot)$, maps \mathbb{R}^n to K , and

$$u(K, x) := |x - p(K, x)|^{-1} (x - p(K, x))$$

is defined for $x \in \mathbb{R}^n \setminus K$. Let ∂K denote the topological boundary of K . The map $F : \partial K \rightarrow \mathbb{R}^n \times \mathbb{S}^{n-1}$ given by $F(x) := (p(K, x), u(K, x))$ is bi-Lipschitz, and the image is the *normal bundle* $\text{Nor } K$ of K , which is an $(n-1)$ rectifiable subset of \mathbb{R}^{2n} . Hence, for \mathcal{H}^{n-1} -almost all $(x, u) \in \text{Nor } K$, the set of $(\mathcal{H}^{n-1} \llcorner \text{Nor } K, n-1)$ approximate tangent vectors at (x, u) is an $(n-1)$ -dimensional linear subspace of \mathbb{R}^{2n} , which is denoted by $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \text{Nor } K, (x, u))$. Let $\Pi_1 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(x, u) \mapsto x$, and $\Pi_2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(x, u) \mapsto u$, be projection maps and Ω_n the volume form for which $\Omega_n(e_1, \dots, e_n) = 1$ for the standard basis (e_1, \dots, e_n) of \mathbb{R}^n . The following statements hold for \mathcal{H}^{n-1} -almost all $(x, u) \in \text{Nor } K$. First, we can choose an orthonormal basis $(a_1(x, u), \dots, a_{n-1}(x, u))$ of $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \text{Nor } K, (x, u))$ such that the $(n-1)$ -vector $a_K(x, u) := a_1(x, u) \wedge \dots \wedge a_{n-1}(x, u)$ satisfies

$$(2.3) \quad \langle \bigwedge_{n-1} (\Pi_1 + \varrho \Pi_2) a_K(x, u) \wedge u, \Omega_n \rangle > 0$$

for all $\varrho > 0$ and thus determines an orientation of $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \text{Nor}(K), (x, u))$. Then the *normal cycle* associated with the convex body K is the $(n-1)$ -dimensional current in \mathbb{R}^{2n} which is defined by

$$T_K := (\mathcal{H}^{n-1} \llcorner \text{Nor } K) \wedge a_K.$$

More generally, we can define

$$T_K(\varphi) = \int_{\text{Nor } K} \langle a_K(x, u), \varphi(x, u) \rangle \mathcal{H}^{n-1}(d(x, u))$$

for all $\mathcal{H}^{n-1} \llcorner \text{Nor } K$ -integrable functions $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \bigwedge^{n-1} \mathbb{R}^{2n}$. Here we use that T_K is a rectifiable current, which has compact support, and thus T_K can be defined for a larger class of functions than just for smooth differential forms.

Sometimes it is convenient to work with the orthonormal basis of the approximate tangent space of $\text{Nor } K$ at (x, u) that is given by

$$a_i(x, u) := \left(\alpha_i(x, u) b_i(x, u), \sqrt{1 - \alpha_i(x, u)^2} b_i(x, u) \right),$$

where $(b_1(x, u), \dots, b_{n-1}(x, u))$ is a suitably chosen orthonormal basis of u^\perp (the orthogonal complement of the linear subspace spanned by u) such that the orthonormal basis (b_1, \dots, b_{n-1}, u) has the same orientation as the standard basis (e_1, \dots, e_n) of \mathbb{R}^n , and $\alpha_i(x, u) \in [0, 1]$ for $i = 1, \dots, n-1$. Note that the dependence of a_i, b_i, α_i on K is not made explicit by our notation. The data $b_i, \alpha_i, i = 1, \dots, n-1$, are essentially uniquely determined (cf. [18, Proposition 3 and Lemma 2]). Moreover, we can assume that $b_i(x + \varepsilon u, u) = b_i(x, u)$, independent of $\varepsilon > 0$, where $(x, u) \in \text{Nor } K$ and $(x + \varepsilon u, u) \in \text{Nor } K_\varepsilon$ with $K_\varepsilon := K + \varepsilon B^n$. However, in general $\alpha_i(x + \varepsilon u, u)$ is not independent of ε and will be positive for $\varepsilon > 0$. See [20, 16, 18, 10, 11] for a geometric description of the numbers $\alpha_i(x, u)$ in terms of generalized curvatures and for arguments establishing the facts stated here.

3. PROOF OF THEOREM 1.1

In order to obtain an upper bound for $|T_K - T_L|$, we first establish an upper bound for $|T_{A_\varepsilon} - T_A|$, for $A \in \{K, L\}$ and $\varepsilon \in [0, 1]$, which is done in Lemma 3.1. Then we derive an upper bound for $|T_{K_\varepsilon} - T_{L_\varepsilon}|$ under the assumption that the Hausdorff distance of K and L is sufficiently small. This bound is provided in Lemma 3.6, which in turn is based on four preparatory lemmas.

Lemma 3.1. *Let $K \in \mathcal{K}^n$ and $\varepsilon \in [0, 1]$. Let $\varphi \in \mathcal{E}^{n-1}(\mathbb{R}^{2n})$. Then*

$$|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \leq C(K, \varphi) \varepsilon,$$

where $C(K, \varphi)$ is a real constant, which depends on the maximum and the Lipschitz constant of φ on $K_1 \times \mathbb{S}^{n-1}$ and on $\mathcal{H}^{n-1}(\partial K_1)$.

Proof. We consider the bi-Lipschitz map

$$F_\varepsilon : \text{Nor } K \rightarrow \text{Nor } K_\varepsilon, \quad (x, u) \mapsto (x + \varepsilon u, u).$$

The extension of F_ε to all $(x, u) \in \mathbb{R}^{2n}$ by $F_\varepsilon(x, u) := (x + \varepsilon u, u)$ is differentiable for all $(x, u) \in \mathbb{R}^{2n}$. By [7, Theorem 3.2.22 (1)], for \mathcal{H}^{n-1} -almost all $(x, u) \in \text{Nor } K$ the approximate Jacobian of F_ε satisfies

$$(3.1) \quad \text{ap } J_{n-1} F_\varepsilon(x, u) = \left\| \bigwedge_{n-1} \text{ap } DF_\varepsilon(x, u) a_K(x, u) \right\| > 0,$$

and the simple orienting $(n-1)$ -vectors $a_K(x, u)$ and $a_{K_\varepsilon}(x + \varepsilon u, u)$ are related by

$$(3.2) \quad a_{K_\varepsilon}(x + \varepsilon u, u) = \frac{\bigwedge_{n-1} \text{ap } DF_\varepsilon(x, u) a_K(x, u)}{\left\| \bigwedge_{n-1} \text{ap } DF_\varepsilon(x, u) a_K(x, u) \right\|}.$$

It follows from (2.3) that the orientations coincide. Thus, first using the coarea theorem [7, Theorem 3.2.22] and then (3.1) and (3.2), we get

$$\begin{aligned} T_{K_\varepsilon}(\varphi) &= \int_{\text{Nor } K_\varepsilon} \langle a_{K_\varepsilon}, \varphi \rangle d\mathcal{H}^{n-1} \\ &= \int_{\text{Nor } K} \langle a_{K_\varepsilon} \circ F_\varepsilon(x, u), \varphi \circ F_\varepsilon(x, u) \rangle \text{ap } J_{n-1} F_\varepsilon(x, u) \mathcal{H}^{n-1}(d(x, u)) \\ &= \int_{\text{Nor } K} \langle \wedge_{n-1} \text{ap } DF_\varepsilon(x, u) a_K(x, u), \varphi \circ F_\varepsilon(x, u) \rangle \mathcal{H}^{n-1}(d(x, u)). \end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned} &|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \\ &\leq \int_{\text{Nor } K} \left\{ \left| \langle (\wedge_{n-1} \text{ap } DF_\varepsilon(x, u) - \wedge_{n-1} \text{id}) a_K(x, u), \varphi \circ F_\varepsilon(x, u) \rangle \right| \right. \\ &\quad \left. + |\langle a_K(x, u), \varphi(x + \varepsilon u, u) - \varphi(x, u) \rangle| \right\} \mathcal{H}^{n-1}(d(x, u)). \end{aligned}$$

We have

$$\begin{aligned} &\left| \langle (\wedge_{n-1} \text{ap } DF_\varepsilon(x, u) - \wedge_{n-1} \text{id}) a_K(x, u), \varphi \circ F_\varepsilon(x, u) \rangle \right| \\ &\leq |\varphi(x + \varepsilon u, u)| \left| \langle (\wedge_{n-1} \text{ap } DF_\varepsilon(x, u) - \wedge_{n-1} \text{id}) a_K(x, u), \varphi \circ F_\varepsilon(x, u) \rangle \right|, \end{aligned}$$

where we used (2.2). Now $a_K(x, u)$ is of the form $\wedge_{i=1}^{n-1} (v_i, w_i)$ with suitable $(v_i, w_i) \in \mathbb{R}^{2n}$ and $|v_i|^2 + |w_i|^2 = 1$. Moreover, we have $DF_\varepsilon(x, u)(v, w) = (v + \varepsilon w, w)$, for all $(v, w) \in \mathbb{R}^{2n}$. Writing $z_i^0 := v_i$, $z_i^1 := w_i$, we have

$$\begin{aligned} &\left| \langle (\wedge_{n-1} \text{ap } DF_\varepsilon(x, u) - \wedge_{n-1} \text{id}) a_K(x, u), \varphi \circ F_\varepsilon(x, u) \rangle \right| \\ &= \left| \wedge_{i=1}^{n-1} (v_i + \varepsilon w_i, w_i) - \wedge_{i=1}^{n-1} (v_i, w_i) \right| \\ &= \left| \sum_{\alpha_i \in \{0,1\}} \varepsilon^{\sum \alpha_i} \wedge_{i=1}^{n-1} (z_i^{\alpha_i}, w_i) - \wedge_{i=1}^{n-1} (z_i^0, w_i) \right| \\ &\leq \varepsilon \sum_{\alpha_i \in \{0,1\}, \sum \alpha_i \geq 1} \left| \wedge_{i=1}^{n-1} (z_i^{\alpha_i}, w_i) \right| \\ &\leq c(n)\varepsilon, \end{aligned}$$

where we used (2.1) and the fact that $|(v_i, w_i)| = 1$ and $|(w_i, w_i)| \leq 2$. We deduce that

$$|\varphi(x + \varepsilon u, u)| \left| \langle (\wedge_{n-1} \text{ap } DF_\varepsilon(x, u) - \wedge_{n-1} \text{id}) a_K(x, u), \varphi \circ F_\varepsilon(x, u) \rangle \right| \leq C_1(K, \varphi)\varepsilon.$$

Furthermore, again by (2.2) we get

$$\begin{aligned} |\langle a_K(x, u), \varphi(x + \varepsilon u, u) - \varphi(x, u) \rangle| &\leq |\varphi(x + \varepsilon u, u) - \varphi(x, u)| \\ &\leq C_2(K, \varphi)\varepsilon. \end{aligned}$$

Thus we conclude that

$$|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \leq C_3(K, \varphi)\varepsilon \mathcal{H}^{n-1}(\text{Nor } K).$$

Since $F : \partial K_1 \rightarrow \text{Nor } K$, $z \mapsto (p(K, z), z - p(K, z))$, is Lipschitz with Lipschitz constant bounded from above by 3, the assertion follows. \square

A convex body $K \in \mathcal{K}^n$ is said to be ε -smooth (for some $\varepsilon > 0$), if $K = K' + \varepsilon B^n$ for some $K' \in \mathcal{K}^n$. For a nonempty set $A \subset \mathbb{R}^n$, we define the distance from A to $x \in \mathbb{R}^n$ by $d(A, x) := \inf\{|a - x| : a \in A\}$. The signed distance is defined by $d^*(A, x) := d(A, x) - d(\mathbb{R}^n \setminus A, x)$, $x \in \mathbb{R}^n$, if $A, \mathbb{R}^n \setminus A \neq \emptyset$. If K is ε -smooth, then ∂K has positive reach. More precisely, if $x \in \mathbb{R}^n$ satisfies $d(\partial K, x) < \varepsilon$, then there is a unique point $p(\partial K, x) \in \partial K$ such that $d(\partial K, x) = |p(\partial K, x) - x|$.

Lemma 3.2. *Let $\varepsilon \in (0, 1)$ and $\delta \in (0, \varepsilon/2)$. Let $K, L \in \mathcal{K}^n$ be ε -smooth and assume that $d_H(K, L) \leq \delta$. Then*

$$p : \partial K \rightarrow \partial L, \quad x \mapsto p(\partial L, x),$$

is well-defined, bijective, bi-Lipschitz with $\text{Lip}(p) \leq \varepsilon/(\varepsilon - \delta)$, and $|p(x) - x| \leq \delta$ for all $x \in \partial K$.

Proof. Since $d_H(K, L) \leq \delta$, we have $K \subset L + \delta B^n$, $L \subset K + \delta B^n$, and a separation argument yields that

$$(3.3) \quad \{x \in L : d(\partial L, x) \geq \delta\} \subset K.$$

This shows that $\partial K \subset \{z \in \mathbb{R}^n : d(\partial L, z) \leq \delta\}$ and therefore the map p is well-defined on ∂K and $|p(x) - x| \leq \delta$ for all $x \in \partial K$. By [6, Theorem 4.8 (8)] it follows that $\text{Lip}(p) \leq \varepsilon/(\varepsilon - \delta)$. Since L is ε -smooth, for $y \in \partial L$ there is a unique exterior unit normal of L at y , which we denote by $u = u(L, y) =: u_L(y)$. Put $y_0 := y - \varepsilon u$ and note that $y_0 + (\varepsilon - \delta)B^n \subset K \cap L$ by (3.3). Then $x \in \partial K$ is uniquely determined by the condition $\{x\} = (y_0 + [0, \infty)u) \cap \partial K$ and satisfies $p(x) = y$. This shows that p is surjective.

Now let $x_1, x_2 \in \partial K$ satisfy $p(x_1) = p(x_2) =: p_0 \in \partial L$. Since there is a ball B of radius ε with $p_0 \in B \subset L$, the points $x_1, x_2 \in \partial K$ are on the line through p_0 and the center of B . By (3.3), they cannot be on different sides of p_0 , hence $x_1 = x_2$. This shows that the map p is also injective. If $d^*(\partial K, \cdot) : \mathbb{R}^n \rightarrow \partial K$ denotes the signed distance function of ∂K , then $q : \partial L \rightarrow \partial K$, $z \mapsto z - d^*(\partial K, z)u_L(z)$, is the inverse of p . Since the signed distance function is Lipschitz, Lemma 3.3 shows that q is Lipschitz as well. \square

The following lemma provides a simple argument for the fact that the spherical image map of an ε -smooth convex body is Lipschitz with Lipschitz constant at most $1/\varepsilon$. A less explicit assertion is contained in [14, Hilfssatz 1].

Lemma 3.3. *Let $K \in \mathcal{K}^n$ be ε -smooth, $\varepsilon > 0$. Then the spherical image map u_K is Lipschitz with Lipschitz constant $1/\varepsilon$.*

Proof. Let $x, y \in \partial K$, and define $u := u_K(x)$, $v := u_K(y)$. Then

$$x - \varepsilon u + \varepsilon v \in x - \varepsilon u + \varepsilon B^n \subset K,$$

and hence $(x - \varepsilon u + \varepsilon v - y) \bullet v \leq 0$. This yields

$$(3.4) \quad \varepsilon(v - u) \bullet v \leq (y - x) \bullet v.$$

By symmetry, we also have $\varepsilon(u - v) \bullet u \leq (x - y) \bullet u$, and therefore

$$(3.5) \quad \varepsilon(v - u) \bullet (-u) \leq (y - x) \bullet (-u).$$

Addition of (3.4) and (3.5) yields

$$\varepsilon |v - u|^2 \leq (y - x) \bullet (v - u) \leq |y - x| |v - u|,$$

which implies the assertion. \square

Lemma 3.4. *Let $\varepsilon \in (0, 1)$ and $\delta \in (0, \varepsilon/2)$. Let $K, L \in \mathcal{K}^n$ be ε -smooth and assume that $d_H(K, L) \leq \delta$. Put $p(x) := p(\partial L, x)$ for $x \in \partial K$. Then*

$$G : \text{Nor } K \rightarrow \text{Nor } L, \quad (x, u) \mapsto (p(x), u_L(p(x))),$$

is bijective, bi-Lipschitz with $\text{Lip}(G) \leq 2/(\varepsilon - \delta) \leq 4/\varepsilon$, and

$$|G(x, u) - (x, u)| \leq \delta + 2\sqrt{\delta/\varepsilon}$$

for all $(x, u) \in \text{Nor } K$.

Proof. It follows from Lemma 3.2 that G is bijective. Then, for $(x, u), (y, v) \in \text{Nor } K$ we get

$$\begin{aligned} |G(x, u) - G(y, v)| &\leq |p(x) - p(y)| + |u_L(p(x)) - u_L(p(y))| \\ &\leq \frac{\varepsilon}{\varepsilon - \delta} |x - y| + \frac{1}{\varepsilon} \frac{\varepsilon}{\varepsilon - \delta} |x - y| \\ &\leq \frac{\varepsilon + 1}{\varepsilon - \delta} |x - y| \\ &\leq \frac{2}{\varepsilon - \delta} |(x, u) - (y, v)|, \end{aligned}$$

where we have used again Lemma 3.2 and Lemma 3.3. Let $x \in \partial K$ and $z := p(x) \in \partial L$. We want to bound $u_L(z) \bullet u_K(x)$ from below. If $x \notin L$, then

$$\text{conv}(\{x\} \cup (z - \varepsilon u_L(z) + (\varepsilon - \delta)B^n)) \subset K,$$

and therefore

$$u_L(z) \bullet u_K(x) \geq \frac{\varepsilon - \delta}{\varepsilon + \delta} \geq 1 - \frac{2\delta}{\varepsilon}.$$

If $x \in L$, then in a similar way we obtain

$$u_L(z) \bullet u_K(x) \geq \frac{\varepsilon - \delta}{\varepsilon} \geq 1 - \frac{\delta}{\varepsilon},$$

hence

$$(3.6) \quad u_L(z) \bullet u_K(x) \geq 1 - \frac{2\delta}{\varepsilon}$$

holds for all $x \in \partial K$. Thus

$$|u_L(z) - u_K(x)| \leq 2\sqrt{\delta/\varepsilon},$$

which finally implies that, for all $(x, u) \in \text{Nor } K$,

$$\begin{aligned} |G(x, u) - (x, u)| &\leq |p(x) - x| + |u_L(p(x)) - u_K(x)| \\ &\leq \delta + 2\sqrt{\delta/\varepsilon}. \end{aligned}$$

Since $G^{-1} : \text{Nor } L \rightarrow \text{Nor } K$ is given by $G^{-1}(z, u) = (q(z), u_K(q(z)))$, it follows that also G^{-1} is Lipschitz. \square

Next we show that, under the assumptions of Lemma 3.5, $\bigwedge_{n-1} DG(x, u)$ is an orientation preserving map from the approximate tangent space of $\text{Nor } K$ to the approximate tangent space of $\text{Nor } L$. It seems that a corresponding fact is not provided in the proofs of related assertions in the literature.

Lemma 3.5. *Let $\varepsilon \in (0, 1)$ and $\delta \in (0, \varepsilon/(4n))$. Let $K, L \in \mathcal{K}^n$ be ε -smooth and assume that $d_H(K, L) \leq \delta$. Then, for \mathcal{H}^{n-1} -almost all $(x, u) \in \text{Nor } K$, the $(n-1)$ -vector $\bigwedge_{n-1} DG(x, u)a_K(x, u) \in \text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \text{Nor } L, G(x, u))$ has the same orientation as $a_L(G(x, u))$.*

Proof. Let $x \in \partial K$, $u := u_K(x)$, and $\bar{x} := p(x)$, hence $d(\partial L, x) = |x - \bar{x}|$. The orientation of $\text{Tan}^{n-1}(\partial K, x)$ is determined by an arbitrary orthonormal basis $b_1(x), \dots, b_{n-1}(x)$ of u^\perp with $\Omega_n(b_1(x), \dots, b_{n-1}(x), u) = 1$. Similarly, any orthonormal basis $\bar{b}_1(\bar{x}), \dots, \bar{b}_{n-1}(\bar{x}), \bar{u}$ with $\bar{u} := u_L(p(x))$ determines the orientation of $\text{Tan}^{n-1}(\partial L, p(x))$. Since G is bi-Lipschitz, we can assume that $(x, u) \in \text{Nor } K$ is such that all differentials exist that are encountered in the proof. Moreover, we can also assume that $\bigwedge_{n-1} DG(x, u) a_K(x, u)$ spans $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \sqsubset \text{Nor } L, G(x, u))$, where we write again G for a Lipschitz extension of the given map G to \mathbb{R}^{2n} . In the following, we put $b_i := b_i(x)$ and $\bar{b}_i := \bar{b}_i(\bar{x})$ for $i = 1, \dots, n-1$.

The differentials of the maps $\text{Nor } K \rightarrow \partial K$, $(x, u) \mapsto x$, and $\partial L \rightarrow \text{Nor } L$, $z \mapsto (z, u_L(z))$, are orientation preserving, which follows for instance from the discussion at the end of Section 2. Hence, it remains to be shown that the differential of $p : \partial K \rightarrow \partial L$, $x \mapsto p(x)$, is orientation preserving, that is,

$$\Delta := \Omega_n(Dp(x)(b_1), \dots, Dp(x)(b_{n-1}), \bar{u}) > 0.$$

First, we assume that $x \neq \bar{x}$, that is, $x \notin \partial L$. Since $Dp(x)(\bar{u}) = o$, we get

$$Dp(x)(b_i) = \sum_{j=1}^{n-1} b_i \bullet \bar{b}_j Dp(x)(\bar{b}_j),$$

and thus

$$\Delta = \det(B) \Omega_n(Dp(x)(\bar{b}_1), \dots, Dp(x)(\bar{b}_{n-1}), \bar{u}),$$

where $B = (B_{ij})$ with $B_{ij} := b_i \bullet \bar{b}_j$ for $i, j \in \{1, \dots, n-1\}$. We choose $\bar{b}_1, \dots, \bar{b}_{n-1}$ as principal directions of curvature of ∂L at $\bar{x} = p(x)$. Then $Dp(x)(\bar{b}_i) = \tau_i \bar{b}_i$ with

$$\tau_i := 1 - d(\partial L, x) k_i \left(\partial L, \bar{x}, \frac{x - \bar{x}}{|x - \bar{x}|} \right) > 0,$$

for $i = 1, \dots, n-1$. Here we use that L is ε -smooth, hence ∂L has positive reach, $d(\partial L, x) < \varepsilon$ and

$$\left| k_i \left(\partial L, \bar{x}, \frac{x - \bar{x}}{|x - \bar{x}|} \right) \right| \leq 1/\varepsilon.$$

Hence it follows that $\Delta > 0$ if we can show that $\det(B) > 0$. Let $\tilde{B} = (\tilde{B}_{ij})$ be defined by $\tilde{B}_{ij} := b_{ij}$, $\tilde{B}_{in} := b_i \bullet \bar{u}$, $\tilde{B}_{nj} := u \bullet \bar{b}_j$, and $\tilde{B}_{nn} := u \bullet \bar{u}$, for $i, j \in \{1, \dots, n-1\}$. Then

$$\begin{aligned} 1 &= \Omega_n(b_1, \dots, b_{n-1}, u) \Omega_n(\bar{b}_1, \dots, \bar{b}_{n-1}, \bar{u}) = \det(\tilde{B}) \\ &\leq u \bullet \bar{u} \det(B) + \sum_{i=1}^{n-1} |b_i \bullet \bar{u}| \cdot 1 \\ &\leq u \bullet \bar{u} \det(B) + \sqrt{n-1} \sqrt{1 - (u \bullet \bar{u})^2}. \end{aligned}$$

From (3.6) and our assumptions, we get $u \bullet \bar{u} \geq 1 - (2\delta)/\varepsilon \geq 1 - 1/(2n)$, and therefore

$$\sqrt{1 - (u \bullet \bar{u})^2} \leq \sqrt{1/n}.$$

Thus

$$1 < u \bullet \bar{u} \det(B) + 1,$$

which implies that $\det(B) > 0$.

Finally, we have to consider the case where $x \in \partial L$. For \mathcal{H}^{n-1} -almost all $x \in \partial K \cap \partial L$, we have $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner (\partial K \cap \partial L), x) = u^\perp$ and $Dp(x) = \text{id}_{u^\perp}$, since $p(z) = z$ for all $z \in \partial K \cap \partial L$. Hence, $\Delta = \Omega_n(b_1, \dots, b_{n-1}, \bar{u}) = u \bullet \bar{u} > 0$. \square

Lemma 3.6. *Let $\varepsilon \in (0, 1)$ and $\delta \in (0, \varepsilon/(4n))$. Let $K, L \in \mathcal{K}^n$ be ε -smooth and assume that $d_H(K, L) \leq \delta$. Let $M \subset \mathbb{R}^{2n}$ be a compact convex set containing $K_{1-\varepsilon} \times \mathbb{S}^{n-1}$ and $L_{1-\varepsilon} \times \mathbb{S}^{n-1}$ in its interior. Then*

$$|T_K(\varphi) - T_L(\varphi)| \leq C(M, \varphi) \left(\frac{4}{\varepsilon}\right)^{n-1} \left(\delta + 2\sqrt{\delta/\varepsilon}\right)$$

for $\varphi \in \mathcal{E}^{n-1}(\mathbb{R}^{2n})$, where $C(M, \varphi)$ is a constant which depends on the sup-norm and the Lipschitz constant of φ on M , and on $\mathcal{H}^{n-1}(\partial K_1)$.

Proof. Let G be as in Lemma 3.4 (or a Lipschitz extension to the whole space with the same Lipschitz constant). Then [7, Theorem 4.1.30] implies that

$$T_L = G_\# T_K,$$

since $\bigwedge_{n-1} DG$ preserves the orientation of the orienting $(n-1)$ -vectors, by Lemma 3.5. (In [16] a corresponding fact is stated without further comment.) Recall the definitions of the dual flat metric \mathbf{F}_M from [7, 4.1.12] and of the mass \mathbf{M} (of a current) from [7, p. 358]. Using [7, 4.1.14], $\partial T_K = 0$, the fact that T_K has compact support contained in the interior of M and Lemma 3.4, we get

$$\begin{aligned} \mathbf{F}_M(T_L - T_K) &= \mathbf{F}_M(G_\# T_K - T_K) \\ &\leq \mathbf{M}(T_K) \cdot \|G - \text{id}\|_{\text{Nor } K, \infty} \cdot \left(\frac{4}{\varepsilon}\right)^{n-1} \\ &\leq \mathcal{H}^{n-1}(\partial K_1) \left(\frac{4}{\varepsilon}\right)^{n-1} \left(\delta + 2\sqrt{\delta/\varepsilon}\right), \end{aligned}$$

where $\|G - \text{id}\|_{\text{Nor } K, \infty} := \sup\{|G(x, u) - (x, u)| : (x, u) \in \text{Nor } K\}$. The assertion now follows from the definition of \mathbf{F}_M , since $\|d\varphi\|$ can be bounded in terms of the sup-norm and the Lipschitz constant of φ on M . \square

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\varphi \in \mathcal{E}^{n-1}(\mathbb{R}^{2n})$. Let $\delta := d_H(K, L) > 0$ and $\varepsilon := \delta^{\frac{1}{2n+1}}$. Assume that $\delta < (4n)^{-\frac{2n+1}{2n}}$. Then $\delta < \varepsilon/(4n)$. Lemma 3.1 implies that

$$\begin{aligned} |T_K(\varphi) - T_{K_\varepsilon}(\varphi)| &\leq C(M, \varphi) \varepsilon, \\ |T_L(\varphi) - T_{L_\varepsilon}(\varphi)| &\leq C(M, \varphi) \varepsilon. \end{aligned}$$

Since K_ε and L_ε are ε -smooth, $d_H(K_\varepsilon, L_\varepsilon) = \delta$, $(K_\varepsilon)_{1-\varepsilon} = K$ and $(L_\varepsilon)_{1-\varepsilon} = L$, Lemma 3.6 shows that

$$|T_{K_\varepsilon}(\varphi) - T_{L_\varepsilon}(\varphi)| \leq C(M, \varphi) \left(\frac{4}{\varepsilon}\right)^{n-1} \left(\delta + 2\sqrt{\delta/\varepsilon}\right).$$

The triangle inequality then yields

$$\begin{aligned} |T_K(\varphi) - T_L(\varphi)| &\leq C_4(M, \varphi) \left(\varepsilon + \frac{\delta}{\varepsilon^{n-1}} + \frac{1}{\varepsilon^{n-1}} \sqrt{\frac{\delta}{\varepsilon}}\right) \\ &\leq C_5(M, \varphi) \delta^{\frac{1}{2n+1}}. \end{aligned}$$

If $\delta \geq (4n)^{-\frac{2n+1}{2n}}$, we simply adjust the constant. \square

4. PROOF OF THEOREM 1.2

In the theory of convex bodies, Federer's curvature measures are supplemented by the area measures, which are finite Borel measures on the unit sphere (and were, in fact, introduced more than 20 years earlier). The two series of measures are generalized by the support measures. We briefly recall their definition (see [19, Chap. 4]).

For $K \in \mathcal{K}^n$, we have already used the metric projection $p(K, \cdot)$ and the vector function $u(K, x) := (x - p(K, x))/d(K, x)$, where $d(K, x) := |x - p(K, x)|$ denotes the distance of the point x from K . We also write $p(K, \cdot) =: p_K$, $u(K, \cdot) =: u_K$ and $d(K, \cdot) =: d_K$. For $\rho > 0$, we set $K^\rho := K_\rho \setminus K$, where $K_\rho = K + \rho B^n$ is the already defined parallel body of K at distance ρ . The product space $\mathbb{R}^n \times \mathbb{S}^{n-1}$ is denoted by Σ . For given K , the map $f_\rho : K^\rho \rightarrow \Sigma$ is defined by $f_\rho(x) := (p_K(x), u_K(x))$ for $x \in K^\rho$, and $\mu_{K, \rho} = \mu_\rho(K, \cdot)$ is the image measure of $\mathcal{H}^n \llcorner K^\rho$ under f_ρ . This is a finite Borel measure on Σ , concentrated on $\text{Nor } K$. The *support measures* $\Lambda_0(K, \cdot), \dots, \Lambda_{n-1}(K, \cdot)$ of K can be defined by

$$(4.1) \quad \mu_{K, \rho} = \sum_{i=0}^{n-1} \rho^{n-i} \kappa_{n-i} \Lambda_i(K, \cdot).$$

Thus, the normalization is different from [19, (4.2.4)]; the connection is given by $n\kappa_{n-i}\Lambda_i(K, \cdot) = \binom{n}{i}\Theta_i(K, \cdot)$. The support measures have the property of weak continuity: if a sequence $(K_j)_{j \in \mathbb{N}}$ of convex bodies converges to a convex body K in the Hausdorff metric, then the sequence $(\Lambda_i(K_j, \cdot))_{j \in \mathbb{N}}$ converges weakly to $\Lambda_i(K, \cdot)$. The topology of weak convergence can be metrized by the bounded Lipschitz metric d_{bL} or the Lévy–Prokhorov metric d_{LP} (see, e.g., Dudley [5, Sec. 11.3]). Therefore, the question arises whether the weak continuity of the support measures can be improved to Hölder continuity with respect to one of these metrics.

For bounded real functions f on Σ we define

$$\|f\|_L := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad \|f\|_\infty := \sup_x |f(x)|.$$

For finite Borel measures μ, ν on Σ , their *bounded Lipschitz distance* is defined by

$$d_{bL}(\mu, \nu) := \sup \left\{ \left| \int_\Sigma f \, d\mu - \int_\Sigma f \, d\nu \right| : f : \Sigma \rightarrow \mathbb{R}, \|f\|_L \leq 1, \|f\|_\infty \leq 1 \right\}.$$

The following lemma is modeled after Proposition 4.1 of Chazal, Cohen–Steiner and Mérigot [2]. Under the restriction to convex bodies, it extends the latter to the measures $\mu_{K, \rho}$.

Lemma 4.1. *If $K, L \in \mathcal{K}^n$ are convex bodies and $\rho > 0$, then*

$$d_{bL}(\mu_{K, \rho}, \mu_{L, \rho}) \leq \int_{K^\rho \cap L^\rho} |p_K - p_L| \, d\mathcal{H}^n + \int_{K^\rho \cap L^\rho} |u_K - u_L| \, d\mathcal{H}^n + \mathcal{H}^n(K^\rho \Delta L^\rho),$$

where Δ denotes the symmetric difference.

Proof. Let $f : \Sigma \rightarrow \mathbb{R}$ be a function with $\|f\|_L \leq 1$ and $\|f\|_\infty \leq 1$. Using the transformation formula for integrals and the properties of f , we obtain

$$\begin{aligned}
& \left| \int_\Sigma f \, d\mu_{K,\rho} - \int_\Sigma f \, d\mu_{L,\rho} \right| \\
&= \left| \int_{K^\rho} f \circ (p_K, u_K) \, d\mathcal{H}^n - \int_{L^\rho} f \circ (p_L, u_L) \, d\mathcal{H}^n \right| \\
&\leq \int_{K^\rho \cap L^\rho} |f \circ (p_K, u_K) - f \circ (p_L, u_L)| \, d\mathcal{H}^n \\
&\quad + \int_{K^\rho \setminus L^\rho} |f \circ (p_K, u_K)| \, d\mathcal{H}^n + \int_{L^\rho \setminus K^\rho} |f \circ (p_L, u_L)| \, d\mathcal{H}^n \\
&\leq \int_{K^\rho \cap L^\rho} |(p_K, u_K) - (p_L, u_L)| \, d\mathcal{H}^n + \int_{K^\rho \setminus L^\rho} 1 \, d\mathcal{H}^n + \int_{L^\rho \setminus K^\rho} 1 \, d\mathcal{H}^n \\
&\leq \int_{K^\rho \cap L^\rho} (|p_K - p_L| + |u_K - u_L|) \, d\mathcal{H}^n + \mathcal{H}^n(K^\rho \Delta L^\rho),
\end{aligned}$$

from which the assertion follows. \square

Proof of Theorem 1.2. We assume that $K, L \in \mathcal{K}^n$ and $d_H(K, L) =: \delta < 1$. Let R be the radius of a ball containing K_2 and L_2 . For $0 < \rho \leq 1$ we use Lemma 4.1 (where for convex bodies, the estimation of the first and the third term on the right-hand side is easier than for the case of general compact sets considered in [2]). First, from Lemma 1.8.9 in [19] we get

$$(4.2) \quad \int_{K^\rho \cap L^\rho} |p_K - p_L| \, d\mathcal{H}^n \leq \sqrt{5D} \mathcal{H}^n(K^\rho \cap L^\rho) \sqrt{\delta} \leq C_1(R) \sqrt{\delta},$$

where $D = \text{diam}(K_\rho \cup L_\rho)$ and the constant $C_1(R)$ depends only on R .

About the distance function d_K , it is well known that

$$\sup_{x \in \mathbb{R}^n} |d_K(x) - d_L(x)| = d_H(K, L) = \delta$$

and that

$$\nabla d_K = u_K \quad \text{on } \mathbb{R}^n \setminus K.$$

Therefore, it follows immediately from Theorem 3.5 of Chazal, Cohen–Steiner and Mérigot [2] (applied to $E = \text{int}(K^\rho \cap L^\rho)$) that

$$(4.3) \quad \int_{K^\rho \cap L^\rho} |u_K - u_L| \, d\mathcal{H}^n \leq C_2(R) \sqrt{\delta}.$$

For the estimation of $\mathcal{H}^n(K^\rho \Delta L^\rho)$, let $x \in K^\rho \setminus L^\rho$; then $x \in K_\rho \setminus K$ and $x \notin L_\rho \setminus L$. If $x \in L$, then $d(K, x) \leq \delta$, hence $x \in K_\delta \setminus K$. If $x \notin L$, then $x \notin L_\rho$ but $x \in K_\rho$, $K_\rho \subset (L_\delta)_\rho = L_{\rho+\delta}$, and hence $x \in L_{\rho+\delta} \setminus L_\rho$. It follows that

$$K^\rho \setminus L^\rho \subset (K_\delta \setminus K) \cup (L_{\rho+\delta} \setminus L_\rho)$$

and hence

$$\begin{aligned}
\mathcal{H}^n(K^\rho \setminus L^\rho) &\leq \mathcal{H}^n(K_\delta) - \mathcal{H}^n(K) + \mathcal{H}^n(L_{\rho+\delta}) - \mathcal{H}^n(L_\rho) \\
&\leq C_3(R) \delta \leq C_3(R) \sqrt{\delta}.
\end{aligned}$$

Here K and L can be interchanged, and together with (4.2), (4.3) and Lemma 4.1 this gives

$$(4.4) \quad d_{bL}(\mu_{K,\rho}, \mu_{L,\rho}) \leq C_4(R) \sqrt{\delta}.$$

To deduce an estimate for the support measures, we apply the usual procedure (e.g., [19], p. 202) and choose in (4.1) for ρ each of the n fixed values $\rho_j = j/n$, $j = 1, \dots, n$, and solve the resulting system of linear equations (which has a non-zero Vandermonde determinant), to obtain representations

$$\Lambda_i(K, \cdot) = \sum_{j=1}^n a_{i,j} \mu_{K, \rho_j}, \quad i = 0, \dots, n-1,$$

with constants $a_{i,j}$ depending only on i, j . Using the definition of the bounded Lipschitz metric, we deduce that

$$(4.5) \quad d_{bL}(\Lambda_i(K, \cdot), \Lambda_i(L, \cdot)) \leq \sum_{j=1}^n |a_{i,j}| d_{bL}(\mu_{K, \rho_j}, \mu_{L, \rho_j}) \leq C(R) \sqrt{\delta}.$$

This completes the proof of Theorem 1.2. □

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