# On integral geometry in projective Finsler spaces

By

Rolf Schneider

**Abstract**

We prove integral geometric formulae of Crofton type for Holmes-Thompson areas of rectifiable Borel sets in smooth projective Finsler spaces.

## 1 Introduction

A classical result of integral geometry in Euclidean spaces, usually associated with the name of Crofton (although he obtained only the first very special cases), expresses the area of a submanifold as an integral of the number of intersection points with the affine subspaces of complementary dimension. More precisely, let $M$ be a $k$-dimensional $C^1$ submanifold of Euclidean space $\mathbb{R}^n$ ($n \geq 2$, $k \in \{1, \dots, n-1\}$). Let $\lambda_k$ denote the $k$-dimensional differential-geometric surface area, and let $\mu_j$ be a rigid motion invariant measure on the affine Grassmannian $A(n,j)$ of $j$-flats ($j$-dimensional affine subspaces) of $\mathbb{R}^n$. Then

$$\int_{A(n,n-k)} \operatorname{card}(M \cap E)\, d\mu_{n-k}(E) = a_{nk}\lambda_k(M), \tag{1}$$

with a constant $a_{nk}$ depending on the normalization of the measure $\mu_{n-k}$. More generally, if $j \in \{n-k, \dots, n-1\}$, then

$$\int_{A(n,j)} \lambda_{k+j-n}(M \cap E)\, d\mu_j(E) = a_{nkj}\lambda_k(M), \tag{2}$$

with a constant $a_{nkj}$. For proofs and further references, we refer to the books of Santaló [30] (p. 245, (14.69)) and of Sulanke and Wintgen [36] (p. 252, (5)).

In the present paper, we extend formula (2) to Holmes-Thompson areas and rectifiable Borel sets in smooth projective Finsler spaces. The role of the Haar measures $\mu_j$ on the affine Grassmannians $A(n,j)$ is then played by suitable signed measures. (For $j = 1$, these are positive measures, and they exist also in general, not necessarily smooth, projective Finsler spaces, see Schneider [34].) Before stating the main result, we want to put these investigations in a wider context and explain some background.

The beauty of formula (1) is an invitation for various generalizations in different directions. Starting from the left side of (1) as a definition, for more general sets $M$, one is led to the notion of integral geometric (or Favard) measures; see, e.g., Mattila [26], Section 5.14, and the references given there. In the following, we look at (1) from an opposite point of view: suppose some notion of $k$-dimensional area is given instead of $\lambda_k$; does there exist a measure or signed measure on $A(n, n-k)$, replacing $\mu_{n-k}$, so that (1) holds for a reasonably large class of sets $M$? One may also think of replacing the affine Grassmannians by more general systems of sets with similar properties. We list some work that can be subsumed under this general program. There is a clear distinction between the possibilities and results in dimension two and in higher dimensions.

In dimension two, the program concerns notions of length and measures on the system $A(2, 1)$ of lines or on similar curve systems. A very satisfactory result has been obtained in the course of the solution of Hilbert's fourth problem in the plane by Pogorelov [28], Ambartzumian [7], Alexander [1]. Let $\mu$ be a Borel measure on $A(2, 1)$. For $p, q \in \mathbb{R}^2$, let $d(p, q)$ be the $\mu$ measure of the set of lines weakly separating $p$ and $q$, and suppose that $d(p, p) = 0$ and $0 < d(p, q) < \infty$ for $p \neq q$. Then $d$ is a continuous metric on $\mathbb{R}^2$ which is additive along lines, and the induced curve length has the property that the lines are geodesics. The quoted papers establish, through different approaches, the converse: every metric with these properties, and hence every notion of curve length for which the lines are geodesics, is obtained in the described way from a measure. A related investigation of Ambartzumian [8] replaces the lines by certain axiomatically defined systems of curves. For sufficiently smooth two-dimensional Finsler or Riemannian manifolds, densities on sets of geodesics leading to Crofton formulae were considered by Blaschke [12], Santaló [29], Owens [27] (who was apparently unaware of Blaschke's work). An elementary treatment of a Crofton formula in Minkowski planes was given by Chakerian [15]. For the classical Crofton formula in the Euclidean plane, an elementary proof for rectifiable curves can be found in a paper by Ayari and Dubuc [9].

About dimensions greater than two, we mention first that (1) holds also in spaces of constant curvature, with flats replaced by totally geodesic submanifolds, see Santaló [30]. The investigation of general versions of Crofton formulae began with Busemann [13], [14]. Generalizing Hilbert's fourth problem, he suggested to study axiomatically defined $k$-dimensional areas in affine spaces for which flats minimize area. Closely related is the question about the validity of Crofton formulae with positive measures, and then the consideration of Crofton formulae involving signed measures is a natural generalization. Concrete Crofton formulae were obtained for Minkowski spaces (finite-dimensional real normed spaces), in special cases by El-Ekhtiar [18] and more systematically by Schneider and Wieacker [35]. The latter paper contains a version of (2) for Holmes-Thompson areas of rectifiable Borel sets in hypermetric Minkowski spaces, with suitable positive measures on $A(n, j)$. In Minkowski geometry, there are different notions of area, see Thompson [37], but only the Holmes-Thompson area seems generally suitable for this type of integral geometric formulae. This was made clear in [32] (Theorem 1) and [33]. The mentioned results of [35] do not require any smoothness assumptions. On the other hand, under smoothness assumptions, there are quite general investigations about Crofton type results for densities, due to Gelfand and Smirnov [22] and to Álvarez, Gelfand and Smirnov [6]. The work of Álvarez and Fernandes [3], [4], [5] and of Fernandes [21] combines a tool from this theory, double fibrations and the Gelfand transform, with other methods, in part from symplectic geometry, to obtain Crofton formulae for Holmes-Thompson areas of smooth submanifolds of smooth projective Finsler spaces. In particular, [3] and [21] extend (1) to this situation, as well as (2) for the special case $k = n-1$. In Section 4 below, we generalize this to all $k \in \{1, \dots, n-1\}$ and to $(\mathcal{H}^k, k)$-rectifiable Borel sets. It turns out that the methods used in [35] for the special case of hypermetric Minkowski spaces are sufficiently general to be adaptable to smooth projective Finsler spaces, thus yielding the following theorem (explanations and precise definitions are given in Section 2).

**Theorem 1.** *Let $(\mathbb{R}^n, F)$ be a smooth projective Finsler space, and let $\mathrm{vol}_k$ denote the corresponding k-dimensional Holmes-Thompson area. For $j \in \{1, \dots, n-1\}$, there exists a signed measure $\eta_j$ on the affine Grassmannian $A(n, j)$ such that, for $k \in \{n-j, \dots, n\}$ and for every $(\mathcal{H}^k, k)$-rectifiable Borel set $M \subset \mathbb{R}^n$,*

$$\int_{A(n,j)} \mathrm{vol}_{k+j-n}(M \cap E)\, d\eta_j(E) = a_{nkj} \mathrm{vol}_k(M) \tag{3}$$

*with a constant $a_{nkj}$.*

Theorem 1 can be considered as giving, for projective Finsler spaces, a positive answer to the first of the three open problems formulated by Chakerian [16] (p. 50). The second of his problems was solved in [34], and the third one in [33] (Theorem 1).

The standard classical examples of projective Finsler spaces are the Minkowski spaces and the Hilbert geometries. In the latter case, the Finsler metric is not defined on all of $\mathbb{R}^n$, but on the interior of a convex body in $\mathbb{R}^n$. In both cases, a single convex body determines the whole geometry, and the smoothness properties of the induced Finsler metric depend on the smoothness of that convex body. In order that arbitrary convex bodies can be admitted, one has to consider general Finsler metrics, which satisfy the usual convexity and continuity, but no smoothness assumptions. In [34], a version of (1), with Holmes-Thompson areas, was obtained for general Finsler metrics $F$ on $\mathbb{R}^n$ such that $(\mathbb{R}^n, F)$ is a hypermetric projective Finsler space. For $k = n-1$, the assumption 'hypermetric' can be deleted ([34], Theorem 2). The corresponding (positive) measure on the space $A(n, 1)$ of lines was obtained by approximation and was, therefore, not described in any explicit way. The existence of this line measure can also be proved for Hilbert geometries. For Hilbert geometries in planar polygons, the line measure is known explicitly, see Alexander [1] and Alexander, Berg and Foote [2]. We mention here that for the special case of the Hilbert geometry in an $n$-dimensional simplex, an explicit description of the line measure can be obtained, using the fact, established by de la Harpe [17], that this metric space is isometric to a certain Minkowski space. We hope to treat the line measure in a polytopal Hilbert geometry somewhere else.

# 2 Finsler spaces and areas

We restrict ourselves here to Finsler metrics on $\mathbb{R}^n$; the case of an open convex subset instead of $\mathbb{R}^n$ requires only obvious modifications. For convenience, we always assume that $\mathbb{R}^n$ is equipped with its standard scalar product $\langle \cdot, \cdot \rangle$, for $n \geq 2$. One reason for this is that it allows us to talk of Lipschitz mappings $f : \mathbb{R}^k \to \mathbb{R}^n$, of the $k$-dimensional Hausdorff measure $\mathcal{H}^k$ on $\mathbb{R}^n$ (for $k \geq 0$), and of $(\mathcal{H}^k, k)$-rectifiable sets. A set $M \subset \mathbb{R}^n$ is called $(\mathcal{H}^k, k)$-rectifiable (for $k \in \{1, \dots, n\}$) if $\mathcal{H}^k(M) < \infty$ and there exist Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$, $i \in \mathbb{N}$, such that $\mathcal{H}^k(K \setminus \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^k)) = 0$. The notion of Lipschitz map, the classes of sets of zero or finite Hausdorff measure, and the notion of $(\mathcal{H}^k, k)$-rectifiable sets do not depend on the choice of the Euclidean metric.

We canonically identify the tangent space $T_x\mathbb{R}^n$ of $\mathbb{R}^n$ at $x$ with $\mathbb{R}^n$. By a *Finsler metric* on $\mathbb{R}^n$ we understand here a continuous function $F : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ such that $F(x, \cdot)$ is a norm on $\mathbb{R}^n$, for each fixed $x \in \mathbb{R}^n$. The Finsler metric $F$ is said to be *smooth* if $F$ is of class $C^\infty$ on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. (The additional assumption made in the differential geometry of Finsler spaces, that $F^2(x, \cdot)$ has positive definite Hessian on $\mathbb{R}^n \setminus \{0\}$, is not needed in the following.) If $F$ is a (smooth) Finsler metric on $\mathbb{R}^n$, we say that $(\mathbb{R}^n, F)$ is a (smooth) Finsler space. In such a space, the length of a parameterized $C^1$ curve $\gamma : [a, b] \to \mathbb{R}^n$ is defined by $\int_a^b F(\gamma(t), \gamma'(t))\, dt$. The Finsler space $(\mathbb{R}^n, F)$ is called *projective* if line segments are shortest curves (not necessarily the only ones) connecting their endpoints. The metric $d_F$ *induced by* $F$ is defined by letting $d_F(p, q)$ be the infimum of the lengths of the piecewise $C^1$ curves connecting the points $p, q \in \mathbb{R}^n$. If $(\mathbb{R}^n, F)$ is projective, the segment $[p, q]$ with endpoints $p, q$ has length $d_F(p, q)$.

We assume that a Finsler metric $F$ on $\mathbb{R}^n$ is given. For $x \in \mathbb{R}^n$, we write

$$F(x, \cdot) =: \|\cdot\|_x$$

and

$$B_x := \{\xi \in \mathbb{R}^n : \|\xi\|_x \leq 1\}.$$

This convex body, the unit ball of the Minkowski space $(T_x\mathbb{R}^n, \|\cdot\|_x)$, is called the *indicatrix* of the Finsler metric $F$ at $x$. Since we have identified $T_x\mathbb{R}^n$ with $\mathbb{R}^n$, each $\|\cdot\|_x$ is a norm on $\mathbb{R}^n$, and $B_x$ is a convex body in $\mathbb{R}^n$ which is centrally symmetric with respect to the origin.

Let $B_x^o$ be the polar body of $B_x$ with respect to the chosen scalar product, thus

$$B_x^o = \{\eta \in \mathbb{R}^n : \langle \xi, \eta \rangle \leq 1 \text{ for all } \xi \in B_x\}.$$

This body is called the *figuratrix* of the Finsler metric $F$ at $x$ (for its role in the calculus of variations, see Blaschke [11]).

The metric $d_F$ induces, in the usual way, an $s$-dimensional Hausdorff measure $\mathcal{H}_F^s$, for each $s \geq 0$. We recall its definition. Let $\operatorname{diam}_F$ denote the diameter in terms of $d_F$. For a subset $A \subset \mathbb{R}^n$ and for $\delta > 0$, let

$$\Omega_\delta(A) := \{(C_i)_{i\in\mathbb{N}} : C_i \subset \mathbb{R}^n,\ \operatorname{diam}_F C_i < \delta \text{ for all } i,\ A \subset \textstyle\bigcup_{i\in\mathbb{N}} C_i\ \}$$

and

$$\mathcal{H}_F^s(A) := \frac{\pi^{s/2}}{2^s \Gamma(1 + s/2)} \sup_{\delta>0} \inf_{(C_i)\in\Omega_\delta(A)} \sum_{i\in\mathbb{N}} (\operatorname{diam}_F C_i)^s.$$

This yields a metric outer measure $\mathcal{H}_F^s$ on $\mathbb{R}^n$, and its restriction to the Borel sets is a measure.

Similarly, for each $x \in \mathbb{R}^n$, an $s$-dimensional Hausdorff measure on $\mathbb{R}^n$ is defined with respect to the metric induced by the norm $\|\cdot\|_x$. This Hausdorff measure is denoted by $\mathcal{H}^s_{F,x}$. In particular, $\mathcal{H}^n_{F,x}$ is the translation invariant Haar measure satisfying $\mathcal{H}^n_{F,x}(B_x) = \kappa_n$, where $\kappa_n$ denotes the volume of the $n$-dimensional Euclidean unit ball (cf. [35], p. 235).

Recall that the $s$-dimensional Hausdorff measure on $\mathbb{R}^n$ that is induced by the auxiliary Euclidean metric, coming from the scalar product $\langle \cdot, \cdot \rangle$, is denoted by $\mathcal{H}^s$. In particular, $\mathcal{H}^n$ coincides with the Lebesgue (outer) measure. It is easy to see, using the continuity of the Finsler metric, that the outer measures $\mathcal{H}^s_F, \mathcal{H}^s_{F,x}, \mathcal{H}^s$ all have the same classes of null sets and of measurable sets.

For a $(\mathcal{H}^k, k)$-rectifiable Borel set $M$ in $\mathbb{R}^n$ (where $k \in \{1, \dots, n\}$), the *Busemann $k$-area* of $M$ is defined as the Hausdorff measure $\mathcal{H}^k_F(M)$. The *Holmes-Thompson $k$-area* of $M$ can be defined by

$$\mathrm{vol}_k(M) = \frac{1}{\kappa_k^2} \int_M vp(B_x \cap T_x M)\, d\mathcal{H}^k_F(x). \tag{4}$$

Here $T_x M$ is the approximate tangent space of $M$ at $x$ (a linear subspace of $\mathbb{R}^n$, since $T_x\mathbb{R}^n$ was identified with $\mathbb{R}^n$); it exists and is unique for $\mathcal{H}^k_F$-almost all $x \in M$ and is measurable in dependence on $x$. The functional $vp$ is the volume product, that is, $vp(K)$ is the product of the (Euclidean) volumes of $K$ and $K^o$; this definition does not depend on the choice of the scalar product.

The definitions of Busemann and Holmes-Thompson area as given here for rectifiable Borel sets in Finsler spaces are the natural extensions of these notions for smooth submanifolds of Minkowski spaces. In a sense which can be made precise, these two area notions are dual to each other. Areas in Minkowski spaces are thoroughly discussed in the book of Thompson [37]. The Holmes-Thompson area appears also in a natural way as a symplectic volume; see Álvarez and Fernandes [3].

The auxiliary Euclidean structure on $\mathbb{R}^n$ has been introduced for two additional reasons. First, the introduction below of signed measures on the affine Grassmannians $A(n, j)$, which replace the motion invariant measures in Euclidean spaces and yield Crofton formulae for the Holmes-Thompson areas, rely on results of Pogorelov, which are conveniently formulated in Euclidean terms. Second, we will have to use results from the Euclidean geometry of convex bodies. For notions from this theory which are used below without explanation, we refer to the book [31].

We introduce some Euclidean terminology referring to the scalar product $\langle \cdot, \cdot \rangle$. The unit sphere is given by

$$S^{n-1} := \{u \in \mathbb{R}^n : \langle u, u \rangle = 1\},$$

and its spherical Lebesgue measure is denoted by $\sigma$. If $(\cdot)|E$ denotes orthogonal projection from $\mathbb{R}^n$ to the linear subspace $E$ of $\mathbb{R}^n$, then $(B_x \cap E)^o = B_x^o|E$. We will show in Section 3 that in a projective Finsler space (4) can be replaced by

$$\mathrm{vol}_k(M) = \frac{1}{\kappa_k} \int_M \mathcal{H}^k(B_x^o|T_x M)\, d\mathcal{H}^k(x). \tag{5}$$

(Note that $\mathcal{H}^k$ and the orthogonal projection depend on the auxiliary scalar product; the integral, however, is independent of the choice of the Euclidean structure.)

Analogously, the Busemann $k$-area can be represented by

$$\mathcal{H}_F^k(M) = \kappa_k \int_M \frac{1}{\mathcal{H}^k(B_x \cap T_x M)} d\mathcal{H}^k(x). \tag{6}$$

Special cases of (6) for general Finsler spaces are contained in Theorem 4.1 of Belletini, Paolini and Venturini [10].

Defining the 'local scaling function' of the Holmes-Thompson $k$-area (with respect to the chosen auxiliary Euclidean structure) by

$$\sigma_k(x, E) := \frac{1}{\kappa_k} \mathcal{H}^k(B_x^o | E) \quad \text{for } x \in \mathbb{R}^n \text{ and } E \in G(n,k) \tag{7}$$

($G(n,k)$ is the Grassmannian of $k$-dimensional linear subspaces of $\mathbb{R}^n$), we write (5) in the form

$$\mathrm{vol}_k(M) = \int_M \sigma_k(x, T_x M)\, d\mathcal{H}^k(x). \tag{8}$$

Now we assume that $(\mathbb{R}^n, F)$ is a smooth projective Finsler space. It follows from the work of Pogorelov [28] (see [34] for a brief sketch of the relevant parts) that there exists a continuous function $g : S^{n-1} \times \mathbb{R} \to \mathbb{R}$ such that, for each $x \in \mathbb{R}^n$, the support function $h(B_x^o, \cdot)$ of the figuratrix can be represented by

$$h(B_x^o, \xi) = \int_{S^{n-1}} |\langle \xi, u \rangle| g(u, \langle x, u \rangle)\, d\sigma(u) \tag{9}$$

for $\xi \in \mathbb{R}^n$. Since the integral depends only on the even part of the function $u \mapsto g(u, \langle x, u \rangle)$, one can assume that $g(u,t) = g(-u,-t)$ for $(u,t) \in S^{n-1} \times \mathbb{R}$. Parameterizing hyperplanes of $\mathbb{R}^n$ by

$$H_{u,t} := \{y \in \mathbb{R}^n : \langle y, u \rangle = t\}$$

with $u \in S^{n-1}$ and $t \in \mathbb{R}$, we can consider the function $g$ as a function on the space of hyperplanes, via $g(H_{u,t}) = g(u,t)$. If this function is considered as a density with respect to the (Euclidean) Haar measure on $A(n, n-1)$, it defines a signed measure $\eta$ on $A(n, n-1)$. This signed measure is given by

$$\int_{A(n,n-1)} f\, d\eta = \int_{S^{n-1}} \int_{\mathbb{R}} f(H_{u,t}) g(u,t)\, dt\, d\sigma(u) \tag{10}$$

for nonnegative measurable functions $f$ on the space $A(n, n-1)$ of hyperplanes.

Let $k \in \{1, \dots, n\}$. The signed measure $\eta$ induces a signed measure $\eta_{n-k}$ on the space $A(n, n-k)$ of $(n-k)$-flats by means of

$$\int_{A(n,n-k)} f\, d\eta_{n-k} = c_k \int_{A(n,n-1)} \cdots \int_{A(n,n-1)} f(H_1 \cap \ldots \cap H_k)\, d\eta(H_1) \cdots d\eta(H_k) \tag{11}$$

for nonnegative measurable functions $f$ on $A(n, n-k)$; here

$$c_k := \frac{2^k}{k! \kappa_k}$$

is a convenient normalizing factor. (Observe that $H_1 \cap \ldots \cap H_k \in A(n, n-k)$ for $\eta^{\otimes k}$-almost all $k$-tuples $(H_1, \ldots, H_k) \in A(n, n-1)^k$.) In terms of hyperplane parameters, this reads

$$\begin{aligned}\int_{A(n,n-k)} f\, d\eta_{n-k} &= c_k \int_{S^{n-1}} \cdots \int_{S^{n-1}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f\left(H_{u_1,t_1} \cap \ldots \cap H_{u_k,t_k}\right) \\ &\qquad g(u_1, t_1) \cdots g(u_k, t_k)\, dt_1 \cdots dt_k\, d\sigma(u_1) \cdots d\sigma(u_k). \qquad (12)\end{aligned}$$

The measures $\eta_j$ thus defined appear in the Crofton formulae of Theorem 1. In the Euclidean case, where $F(x, \xi) = \langle \xi, \xi \rangle$, they coincide with the Haar measures $\mu_j$ in the classical formula (1). This construction of the measures $\eta_{n-k}$ on $A(n, n-k)$ appeared first, for the case of Minkowski spaces, in [35], Theorem 7.1.

For the proof of Theorem 1 in Section 4, we need the following preparations. For each $x \in \mathbb{R}^n$, we define a signed measure $\rho_x$ on $S^{n-1}$ by

$$\rho_x(A) := \int_A g(u, \langle x, u \rangle)\, d\sigma(u) \qquad (13)$$

for Borel sets $A \subset S^{n-1}$. Then we can write (9) as

$$h(B_x^o, \xi) = \int_{S^{n-1}} |\langle \xi, u \rangle|\, d\rho_x(u). \qquad (14)$$

It is known from the theory of generalized zonoids that this formula, which can be considered as giving (half of) the Euclidean lengths of the one-dimensional orthogonal projections of $B_x^o$, extends to higher-dimensional projections. For affine subspaces $E, L \in A(n, k)$, let $[E, L^\perp] = |\langle E, L \rangle|$ be the absolute value of the determinant (in dimension $k$) of the orthogonal projection from $E$ to $L$. By $L(u_1, \ldots, u_k)$ and $[u_1, \ldots, u_k]$ we denote, respectively, the linear subspace spanned by the vectors $u_1, \ldots, u_k$ and the $k$-dimensional volume of the parallelepiped spanned by them. Let $k \in \{1, \ldots, n\}$ and $E \in A(n, k)$. Then, for $x \in \mathbb{R}^n$,

$$\begin{aligned}&\mathcal{H}^k(B_x^o | E) \\ &= \frac{2^k}{k!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} \left[E, L(u_1, \ldots, u_k)^\perp\right] [u_1, \ldots, u_k]\, d\rho_x(u_1) \cdots d\rho_x(u_k); \quad (15)\end{aligned}$$

see Weil [38], p. 176. It is important to notice that this follows from (14) even if $\rho_x$ is only a signed measure. Defining the signed measure $\rho_x^{(k)}$ on $G(n, k)$ by

$$\rho_x^{(k)}(A) := c_k \int_{S^{n-1}} \cdots \int_{S^{n-1}} \mathbf{1}_A(L(u_1, \ldots, u_k))[u_1, \ldots, u_k]\, d\rho_x(u_1) \cdots d\rho_x(u_k) \qquad (16)$$

for Borel sets $A \subset G(n, k)$, we can write (7) and (15) in the form

$$\sigma_k(x, E) = \int_{G(n,k)} [E, L^\perp]\, d\rho_x^{(k)}(L) \quad \text{for } x \in \mathbb{R}^n, E \in G(n, k). \qquad (17)$$

(Essentially, the definition (16) goes back to Matheron [25], p. 101; later uses of this ‘projection generating measure’, as it has been called, begin with Goodey and Weil [23].) With these notations, we have

$$\int_{G(n,n-j)} \sigma_{k+j-n}(x, E \cap L^\perp)[E, L^\perp]\, d\rho_x^{(n-j)}(L) = \frac{c_{k+j-n}c_{n-j}}{c_k}\sigma_k(x, E) \tag{18}$$

for $k, j \in \{1, \dots, n-1\}$ with $k + j - n \geq 0$ and for every $E \in G(n, k)$. The proof given in [35], Lemma 7.2, for this equation in the case of a measure $\rho$ carries over without change to the signed measure $\rho_x$, for every $x \in \mathbb{R}^n$.

# 3 The area formula for projective Finsler spaces

For the proof of (5), we need an extension of Federer’s area formula ([20], p. 243), which holds for Lipschitz mappings from $\mathbb{R}^k$ to $\mathbb{R}^n$ ($k \leq n$), to Lipschitz mappings into a projective Finsler space $(\mathbb{R}^n, F)$. This could be deduced from a general version for metric spaces (Kirchheim [24], Corollary 8), but for convenience we give here an elementary proof, by just adapting the proof in [35] for Minkowski spaces to the present situation.

In the following, $E^k$ denotes the Euclidean unit ball of $\mathbb{R}^k$. For a Lipschitz mapping $f : \mathbb{R}^k \to \mathbb{R}^n$, the differential of $f$ at $z \in \mathbb{R}^k$ exists for $\mathcal{H}^k$-almost all $z$ and is denoted by $Df_z$.

**Theorem 2.** *Let $(\mathbb{R}^n, F)$ be a projective Finsler space. Let $k \in \{1, \dots, n\}$, and let $f : \mathbb{R}^k \to \mathbb{R}^n$ be a Lipschitz map. Then*

$$\kappa_k \int_{\mathbb{R}^n} \operatorname{card}(A \cap f^{-1}(\{x\}))\, d\mathcal{H}_F^k(x) = \int_A \mathcal{H}^k_{F,f(z)}(Df_z(E^k))\, d\mathcal{H}^k(z)$$

*for every $\mathcal{H}^k$-measurable subset $A$ of $\mathbb{R}^k$.*

*If $h$ is a nonnegative $\mathcal{H}^k$-measurable function on $\mathbb{R}^k$, then*

$$\kappa_k \int_{\mathbb{R}^n} \sum_{y \in f^{-1}(\{x\})} h(y)\, d\mathcal{H}_F^k(x) = \int_{\mathbb{R}^k} h(z)\mathcal{H}^k_{F,f(z)}(Df_z(E^k))\, d\mathcal{H}^k(z).$$

The metric $d_F$ in a Finsler space $(\mathbb{R}^n, F)$ was introduced in Section 2. For a norm $N$ on $\mathbb{R}^k$ and functions $f : \mathbb{R}^k \to \mathbb{R}^n$ and $g : C \to \mathbb{R}^k$ with $C \subset \mathbb{R}^n$ we use the notation

$$\operatorname{Lip}(N, d_F, f) := \sup_{x \neq y} \frac{d_F(f(x), f(y))}{N(x - y)}$$

and

$$\operatorname{Lip}(d_F, N, g) := \sup_{x \neq y} \frac{N(g(x) - g(y))}{d_F(x, y)}.$$

By $[x, y]$ we denote the closed segment in $\mathbb{R}^n$ with endpoints $x, y$.

Now suppose that a projective Finsler space $(\mathbb{R}^n, F)$, a positive integer $k \leq n$ and a Lipschitz map $f : \mathbb{R}^k \to \mathbb{R}^n$ are given, as in Theorem 2. The proof of Theorem 2 requires the following lemma.

**Lemma.** *Let $A$ be a Borel subset of* $\{x \in \mathbb{R}^k : f$ is differentiable at $x$ and $Df_x$ is injective$\}$, *and let $t > 1$. Then there is a countable Borel covering $\mathcal{C}$ of $A$ such that, for each $C \in \mathcal{C}$, the restriction $f|_C$ is injective and there is a norm $N$ (depending on $C$) on $\mathbb{R}^k$ satisfying*

$$\mathrm{Lip}(N, d_F, f|_C) \leq t, \qquad \mathrm{Lip}(d_F, N, (f|_C)^{-1}) \leq t$$

*and*

$$t^{-k}\mu_N(E^k) \leq \mathcal{H}^k_{F,f(x)}(Df_x(E^k)) \leq t^k\mu_N(E^k) \qquad \text{for } x \in C,$$

*where $\mu_N$ is the $k$-dimensional Hausdorff measure induced on $\mathbb{R}^k$ by the norm $N$.*

*Proof.* We extend the proof of Lemma 5.1 in [35]. Choose $\epsilon > 0$ such that $t^{-1} + \epsilon < 1 < t - \epsilon$, further a countable, dense subset $D$ of $\mathbb{R}^k$ and a countable family $\mathcal{N}$ of norms on $\mathbb{R}^k$ such that, for each norm $N'$ on $\mathbb{R}^k$, there is a norm $N \in \mathcal{N}$ satisfying

$$(t^{-1} + \epsilon)N \leq N' \leq (t - \epsilon)N.$$

For $z \in D$, $N \in \mathcal{N}$ and $i \in \mathbb{N}$ let $C(z, N, i)$ be the set of all $b \in E(z, i^{-1})$ (where $E(z, r)$ is the Euclidean ball in $\mathbb{R}^k$ with centre $z$ and radius $r$) such that, for all $a \in E(z, i^{-1})$ and all $p \in [f(a), f(b)]$,

$$(t^{-1} + \epsilon)N \leq \|Df_b(\cdot)\|_p \leq (t - \epsilon)N, \tag{19}$$

$$\|f(a) - f(b) - Df_b(a - b)\|_p \leq \epsilon N(a - b). \tag{20}$$

For $a, b \in C(z, N, i)$ we infer from (19) and (20) that

$$t^{-1}N(a - b) \leq \|f(a) - f(b)\|_p \leq tN(a - b) \qquad \text{for all } p \in [f(a), f(b)]. \tag{21}$$

In particular, $f|_{C(z,N,i)}$ is injective. We assert that

$$\mathrm{Lip}(N, d_F, f|_{C(z,N,i)}) \leq t, \tag{22}$$

$$\mathrm{Lip}(d_F, N, (f|_{C(z,N,i)})^{-1}) \leq t. \tag{23}$$

For the proof, let $a, b \in C(z, N, i)$. Since the Finsler space $(\mathbb{R}^n, F)$ is projective, the distance $d_F(f(a), f(b))$ is given by the Finsler length of the segment $[f(a), f(b)]$, thus

$$d_F(f(a), f(b)) = \int_0^1 \|f(b) - f(a)\|_{(1-\tau)f(a)+\tau f(b)}\, d\tau \leq tN(a - b)$$

by (21). This gives (22), and (23) is obtained similarly (only here we use the fact that the Finsler space is projective).

The inequalities (19) for the norms $(t^{-1} + \epsilon)N$, $\|Df_b(\cdot)\|_p$, $(t - \epsilon)N$ imply for the induced $k$-dimensional Hausdorff measures the estimates

$$t^{-k}\mu_N(E^k) \leq (t^{-1} + \epsilon)^k\mu_N(E^k) \leq \mathcal{H}^k_{F,f(b)}(Df_b(E^k)) \leq (t - \epsilon)^k\mu_N(E^k) \leq t^k\mu_N(E^k).$$

We show that $\{C(z,N,i) : z \in D,\ N \in \mathcal{N},\ i \in \mathbb{N}\}$ is a covering of $A$. Let $b \in A$. For $c \in E(b,2)$, consider the norm $N_{b,c} := \|Df_b(\cdot)\|_{f(c)}$. Since $F$ is continuous, we can choose a number $i \in \mathbb{N}$ and a norm $N \in \mathcal{N}$ so that

$$(t^{-1}+\epsilon)N \le N_{b,c} \le (t-\epsilon)N \qquad \text{for all } c \in E(b, 2i^{-1}).$$

Also by continuity of $F$, and by compactness of $E(b,2)$, there is a number $M$ so that $\|\cdot\|_{f(c)} \le M\|\cdot\|_{f(b)}$ for all $c \in E(b,2)$. Since $f$ is differentiable at $b$, we can further choose $i$ so small that, for all $a \in E(b, 2i^{-1})$,

$$\|f(a) - f(b) - Df_b(a-b)\|_{f(b)} \le M^{-1}\epsilon N(a-b).$$

This implies

$$\|f(a) - f(b) - Df_b(a-b)\|_{f(c)} \le \epsilon N(a-b) \qquad \text{for all } a, c \in E(b, 2i^{-1}).$$

Now we choose $z \in D$ with $z \in E(b, i^{-1})$. Then (20) is satisfied for all $a \in E(z, i^{-1})$ and all $p \in [f(a), f(b)]$, hence $b \in C(z,N,i)$.

Finally, we choose $\{x_j : j \in \mathbb{N}\}$ dense in $\mathbb{R}^k$ and $\{\tau_j : j \in \mathbb{N}\}$ dense in $[0,1]$ and put

$$A_{j,m} := \{b \in E(z,i^{-1}) : (t^{-1}+\epsilon)N(x_j) \le \|Df_b(x_j)\|_{(1-\tau_m)f(x_j)+\tau_m f(b)} \le (t-\epsilon)N(x_j)\}$$

and

$$B_{j,m} := \{b \in E(z,i^{-1}) : \|f(x_j) - f(b) - Df_b(x_j - b))\|_{(1-\tau_m)f(x_j)+\tau_m f(b)} \le \epsilon N(x_j - b)\}.$$

Then

$$C(z,N,i) = \bigcap_{j,m\in\mathbb{N}} A_{j,m} \cap \bigcap_{\substack{j,m\in\mathbb{N} \\ x_j \in E(z,i^{-1})}} B_{j,m},$$

which shows that $C(z,N,i)$ is a Borel set. ■

*Proof of Theorem 2.* This is now a straightforward generalization of the proof of Theorem 5.2 (and of (32), corrected) in [35]: one has merely to replace $\mu^k(f(G))$ in that proof by $\mathcal{H}^k_F(f(G))$ and $\mu^k(Df_z(E^k))$ in the integrands by $\mathcal{H}^k_{F,f(z)}(Df_z(E^k))$. ■

For the envisaged application, recall that $\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure induced on $\mathbb{R}^n$ by the auxiliary Euclidean structure. Let $L \subset \mathbb{R}^n$ be a $k$-dimensional linear subspace. Let $x \in \mathbb{R}^n$. The restriction of the norm $\|\cdot\|_x$ to $L$ has unit ball $B_x \cap L$; the corresponding $k$-dimensional Hausdorff measure on $L$ is $\mathcal{H}^k_{F,x} \llcorner L$, and we have $\mathcal{H}^k_{F,x}(B_x \cap L) = \kappa_k$. Since both, $\mathcal{H}^k_{F,x} \llcorner L$ and $\mathcal{H}^k \llcorner L$ are Haar measures on $L$, they are proportional, thus

$$\frac{\mathcal{H}^k_{F,x}(\cdot \cap L)}{\kappa_k} = \frac{\mathcal{H}^k(\cdot \cap L)}{\mathcal{H}^k(B_x \cap L)}. \tag{24}$$

Now suppose that $f : A \to \mathbb{R}^n$ is an injective Lipschitz map, $A \subset \mathbb{R}^k$ is a bounded Borel set, and $f(A) = M$ is a Borel set. Let $g : M \to \mathbb{R}$ be a nonnegative $\mathcal{H}^k$-measurable function. If we apply Theorem 2, equation (24), and then Theorem 2 to the Euclidean metric, we get

$$\begin{aligned}\int_M g(x)\, d\mathcal{H}^k_F(x) &= \int_A g(f(z)) \frac{\mathcal{H}^k_{F,f(z)}(Df_z(E^k))}{\kappa_k}\, d\mathcal{H}^k(z) \\ &= \int_A g(f(z)) \frac{\mathcal{H}^k(Df_z(E^k))}{\mathcal{H}^k(B_{f(z)} \cap Df_z(\mathbb{R}^k))}\, d\mathcal{H}^k(z) \\ &= \kappa_k \int_M g(x) \frac{1}{\mathcal{H}^k(B_x \cap T_x M)}\, d\mathcal{H}^k(x).\end{aligned}$$

Now the choice $g = 1$ on $M$ gives (6), and the choice

$$g(x) = vp(B_x \cap T_x M)/\kappa_k^2 = \mathcal{H}^k(B_x^o | T_x M)\mathcal{H}^k(B_x \cap T_x M)/\kappa_k^2$$

gives (5), under our special assumptions on $M$. However, to this special case the proof for a general $(\mathcal{H}^k, k)$-rectifiable Borel set $M$ can be reduced; see Theorem 3.2.29 in Federer [20]. The same special assumptions on $M$ can be made in the proof of Theorem 1 in the next section. Here one has to observe that both sides of (3) are zero if $\mathcal{H}^k_F(M) = 0$. This is true in the Euclidean case, as follows from Federer [19]; in a smooth projective Finsler space it then follows by observing that the measure $\eta_j$ has a density with respect to the Euclidean invariant measure $\mu_j$ and that in compact subsets of $\mathbb{R}^n$, the Hausdorff measure $\mathcal{H}^s_F$ can be estimated from above by a constant multiple of $\mathcal{H}^s$.

# 4 A general Crofton formula

Now we can prove Theorem 1. We assume that $(\mathbb{R}^n, F)$ is a smooth projective Finsler space, and the signed measure $\eta_j$ on $A(n,j)$ is defined as in Section 2. Let $k \in \{1, \dots, n\}$, $j \in \{1, \dots, n-1\}$ with $q := k + j - n \geq 0$, and put $m := n - j$. Let $M \subset \mathbb{R}^n$ be a $(\mathcal{H}^k, k)$-rectifiable Borel set. We may assume that $M$ is of the special form as assumed at the end of the last section. From (12), we have

$$\int_{A(n,j)} \mathrm{vol}_q(F \cap M)\, d\eta_j(F) = c_m \int_{(S^{n-1})^m} I(u_1, \dots, u_m)\, d\sigma^{\otimes m}(u_1, \dots, u_m)$$

with

$$\begin{aligned}& I(u_1, \dots, u_m) \\ &:= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathrm{vol}_q(H_{u_1,t_1} \cap \ldots \cap H_{u_m,t_m} \cap M) g(u_1, t_1) \cdots g(u_m, t_m)\, dt_1 \cdots dt_m.\end{aligned}$$

For $i = 1, \dots, m$, let $C_i$ be an $(n-1)$-dimensional unit cube in $u_i^\perp := \{x \in \mathbb{R}^n : \langle x, u_i \rangle = 0\}$. As in [35], formula (54) (where no invariance property of $\tau_q$ is needed), we get

$$\begin{aligned}& \mathrm{vol}_q(H_{u_1,t_1} \cap \ldots \cap H_{u_m,t_m} \cap M) \\ &= \int_{u_1^\perp} \cdots \int_{u_m^\perp} \mathrm{vol}_q(D_q(y,t))\, d\lambda_{n-1}(y_1) \cdots d\lambda_{n-1}(y_m),\end{aligned}$$

where $\lambda_{n-1}$ denotes the $(n-1)$-dimensional Lebesgue measure and where

$$D_q(y,t) := (C_1 + t_1 u_1 + y_1) \cap \ldots \cap (C_m + t_m u_m + y_m) \cap M.$$

By (8), this gives

$$\begin{aligned}
&I(u_1,\ldots,u_m)\\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{u_1^\perp} \cdots \int_{u_m^\perp} \int_{D_q(y,t)} \sigma_q(x, T_x D_q(y,t))\, d\mathcal{H}^q(x)\\
&\quad d\lambda_{n-1}(y_1) \cdots d\lambda_{n-1}(y_m) g(u_1,t_1) \cdots g(u_m,t_m)\, dt_1 \cdots dt_m\\
&= \int_{u_1^\perp} \cdots \int_{u_m^\perp} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{D_q(y,t)} \sigma_q(x, T_x D_q(y,t)) g(u_1, \langle x, u_1\rangle) \cdots g(u_m, \langle x, u_m\rangle)\\
&\quad d\mathcal{H}^q(x) dt_1 \cdots dt_m\, d\lambda_{n-1}(y_1) \cdots d\lambda_{n-1}(y_m).
\end{aligned}$$

Here we have applied Fubini's theorem and then made use of the fact that $x \in D_q(y,t)$ satisfies $x \in u_i^\perp + t_i u_i = H_{u_i,t_i}$, hence $\langle x, u_i\rangle = t_i$, for $i = 1,\ldots,m$. Thus we obtain

$$\begin{aligned}
&I(u_1,\ldots,u_m)\\
&= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \int_{D_q'(z)} \sigma_q(x, T_x D_q'(z)) g(u_1, \langle x, u_1\rangle) \cdots g(u_m, \langle x, u_m\rangle)\\
&\quad d\mathcal{H}^q(x) d\lambda_n(z_1) \cdots d\lambda_n(z_m)
\end{aligned}$$

with

$$D_q'(z) := (C_1 + z_1) \cap \ldots \cap (C_m + z_m) \cap M.$$

Writing

$$f(x) := \sigma_q\left(x, L(u_1,\ldots,u_m)^\perp \cap T_x M\right) g(u_1, \langle x, u_1\rangle) \cdots g(u_m, \langle x, u_m\rangle),$$

we get

$$I(u_1,\ldots,u_m) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \int_{(C_1+z_1)\cap\ldots\cap(C_m+z_m)\cap M} f(x)\, d\mathcal{H}^q(x)\, d\lambda_n(z_1) \cdots d\lambda_n(z_m).$$

Now we use Lemma 6.1 of [35], where we put $p = m$, $M_0 = M$, $M_i = C_i$ for $i = 1,\ldots,m$. We obtain

$$\begin{aligned}
&I(u_1,\ldots,u_m)\\
&= \int_{C_1} \cdots \int_{C_m} \int_M f(x_0) [T_{x_0}^\perp M, T_{x_1}^\perp C_1, \ldots, T_{x_m}^\perp C_m]\, d\mathcal{H}^k(x_0) d\mathcal{H}^{n-1}(x_1) \cdots d\mathcal{H}^{n-1}(x_m)\\
&= [u_1,\ldots,u_m] \int_M f(x) [L(u_1,\ldots,u_m)^\perp, T_x M]\, d\mathcal{H}^k(x).
\end{aligned}$$

Inserting this and using (13), (16) and (18) we conclude that

$$\int_{A(n,j)} \mathrm{vol}_q(F \cap M)\, d\eta_j(F)$$

$$= c_m \int_{(S^{n-1})^m} \int_M [u_1, \ldots, u_m][L(u_1, \ldots, u_m)^\perp, T_x M] \sigma_q(x, L(u_1, \ldots, u_m)^\perp \cap T_x M)$$

$$g(u_1, \langle x, u_1 \rangle) \cdots g(u_m, \langle x, u_m \rangle) d\mathcal{H}^k(x)\, d\sigma^{\otimes m}(u_1, \ldots, u_m)$$

$$= c_m \int_M \int_{(S^{n-1})^m} \sigma_q(x, L(u_1, \ldots, u_m)^\perp \cap T_x M)[L(u_1, \ldots, u_m)^\perp, T_x M]$$

$$[u_1, \ldots, u_m] g(u_1, \langle x, u_1 \rangle) \cdots g(u_m, \langle x, u_m \rangle)\, d\sigma^{\otimes m}(u_1, \ldots, u_m)\, d\mathcal{H}^k(x)$$

$$= \int_M \int_{G(n,m)} \sigma_q(x, L^\perp \cap T_x M)[L^\perp, T_x M]\, d\rho_x^{(m)}(L)\, d\mathcal{H}^k(x)$$

$$= \frac{c_q c_m}{c_k} \int_M \sigma_k(x, T_x M)\, d\mathcal{H}^k(x)$$

$$= \frac{c_{k+j-n} c_{n-j}}{c_k} \mathrm{vol}_k(M).$$

This completes the proof of Theorem 1. ∎

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Rolf Schneider
Mathematisches Institut
Albert-Ludwigs-Universität
Eckerstraße 1
D-79104 Freiburg i.Br.
Germany

rolf.schneider@math.uni-freiburg.de