

INTERSECTION PROPERTIES OF POLYHEDRAL NORMS

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ABSTRACT. We investigate the family \mathcal{M} of intersections of balls in a finite dimensional vector space with a polyhedral norm. The spaces for which \mathcal{M} is closed under Minkowski addition are completely determined. We characterize also the polyhedral norms for which \mathcal{M} is closed under adding a ball. A subset \mathcal{P} of \mathcal{M} consists of the Mazur sets K , defined by the property that for any hyperplane H not meeting K there is a ball containing K and not meeting H . We characterize the Mazur sets in terms of their normal cones and also as summands of closed balls. As a consequence, we characterize the polyhedral spaces with only trivial Mazur sets as those whose unit ball is indecomposable.

1. INTRODUCTION

The present paper is a sequel to [15] and is part of a program initiated in recent years to investigate the basic algebraic/topological properties of the family \mathcal{M} of all intersections of closed balls in a normed space X . This family has appeared in problems from convex geometry, geometry of Banach spaces, fixed point theory, and combinatorial geometry. An early result concerns the *Mazur intersection property*, dating back to Mazur [12]. The space X has this property if \mathcal{M} is equal to the family of all closed bounded convex subsets of X . It was proved by Phelps [17] that a finite dimensional space has the Mazur intersection property if and only if the set of extreme points of the dual unit ball B^* is dense in the boundary of B^* . An infinite dimensional generalization is due to Giles, Gregory and Sims [7]. Our aim is to establish the basic facts concerning \mathcal{M} when the space lacks the Mazur intersection property. In this context, it is known that \mathcal{M} is uniformly very porous [10] and that it need not be (topologically) closed [15].

Two kinds of problems are considered in this paper, those connected with Minkowski sums and those pertaining to Mazur sets. Recall that \mathcal{M} is (i) *stable* if $\overline{C + D} \in \mathcal{M}$ whenever $C, D \in \mathcal{M}$; (ii) *ball stable* if $\overline{C + B} \in \mathcal{M}$ for every $C \in \mathcal{M}$, where B is the unit ball of the space. These properties introduced in [8] have close connections with the continuity of the ball hull mapping [15]. The closed bounded convex set K is said to be a *Mazur set* if the following separation property is satisfied: for every hyperplane H with positive distance from K , there exists a ball D with $K \subset D$ and $H \cap D = \emptyset$. The family

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of all Mazur sets in our normed space is denoted by \mathcal{P} . Observe that \mathcal{P} is contained in \mathcal{M} . The space is called a *Mazur space* if $\mathcal{P} = \mathcal{M}$. Basic results concerning Mazur sets and Mazur spaces can be found in [8], [9]. For instance, \mathcal{M} is stable in every Mazur space. In this paper, we restrict ourselves to finite dimensional polyhedral spaces, that is, finite dimensional real vector spaces with a norm whose unit ball is a polytope. In this elementary situation, several questions about the classes \mathcal{M} and \mathcal{P} have explicit answers.

It is known that \mathcal{M} is stable in ℓ_∞ and in every two-dimensional space, thus it should not come as a surprise that it is also stable in

$$X = X_1 \oplus_\infty \cdots \oplus_\infty X_k, \quad (1)$$

where each X_i is a normed space of dimension at most two. More surprising is the fact that the converse is true when the norm is polyhedral: every finite dimensional polyhedral space with stable \mathcal{M} is obtained in this way. Finite dimensional polyhedral spaces which are direct sums of the form (1) have recently come up in connection with the investigation of suns in normed spaces [4].

Next we characterize the finite dimensional polyhedral normed spaces for which \mathcal{M} is ball stable. The characterization involves relations between the normal cones of the unit ball B of the space. It should be observed that the normal cones of the unit ball B are in correspondence with the exposed faces of the dual unit ball B^* , thus this characterization can alternatively be formulated in terms of the facial structure of B^* , as in the characterization result of Phelps.

In the last section, we are concerned with spaces in which the only Mazur sets are balls and one-pointed sets. Typical examples are the spaces ℓ_1^d , for $d > 2$ [9]. We improve this result by obtaining a complete characterization of the finite dimensional polyhedral spaces satisfying this property. The notion of *indecomposability* plays a major role in this result. As a main step to achieve this characterization, the Mazur sets in finite dimensional polyhedral spaces are completely determined. Our characterization is in terms of normal fans, on the one hand, and summands of the unit ball, on the other. Using this result, we are able to characterize the finite dimensional polyhedral Mazur spaces X as those for which \mathcal{M} is stable.

2. PRELIMINARIES

Throughout this paper, X is a real normed space of dimension $1 < d < \infty$, with norm $\|\cdot\|$. Its unit ball, dual space, and dual unit ball are denoted by B , $(X^*, \|\cdot\|^*)$, and B^* , respectively. A homothet of B , that is, any set of the form $\lambda B + t$ with $\lambda > 0$ and $t \in X$, is called a *ball*. The family of all nonempty intersections of balls is denoted by \mathcal{M}_X , or briefly by \mathcal{M} , when no ambiguity arises. By \mathcal{K} we denote the set of all nonempty compact convex subsets of X .

The *support function* of $K \in \mathcal{K}$ is defined by $h(K, f) := \max_{x \in K} f(x)$ for $f \in X^*$. The functional $f \in X^* \setminus \{0\}$ *supports* K at x if $f(x) = h(K, f)$, that is, if f attains its maximum on K at the point x . We write

$$H^-(K, f) := f^{-1}(-\infty, \max f(K)] = \{x \in K : f(x) \leq h(K, f)\}$$

for the (closed) *supporting halfspace* of K determined by $f \in X^* \setminus \{0\}$, and

$$H(K, f) := \{x \in K : f(x) = h(K, f)\}$$

for the *supporting hyperplane* of K determined by f . The set $F(K, f) := K \cap H(K, f)$ is the (*exposed*) *face* of K determined by f . The dual unit sphere, that is, the boundary of B^* , is denoted by S^* . In connection with intersections of balls in Banach spaces, a certain subset of the dual unit sphere plays an important role. This is the set of semi-denting points, which in the finite-dimensional case coincide with the limits of exposed points of B^* (see [15], Remark after Corollary 3, and observe that every extreme point is a limit of exposed points, by the theorem of Straszewicz ([18], Theorem 1.4.7)). We denote by Ω the closure of the set of exposed points of B^* . The following description of the intersections of balls (see [15], (7)) will be useful.

Lemma 2.1. *Let $K \in \mathcal{K}$. Then $K \in \mathcal{M}$ if and only if $K = \bigcap_{f \in \Omega} H^-(K, f)$.*

If X has a polyhedral norm, then Ω is just the set of vertices of the dual unit ball B^* , or, equivalently, the set of all norm one functionals $f \in X^*$ for which $f^{-1}(1)$ is a facet (a face of codimension one) of the polytopal unit ball B . If $K \in \mathcal{M}$ is a polytope and $f \in S^*$ is such that $F(K, f)$ is a facet of K , then it follows from Lemma 2.1 that $f \in \Omega$.

In the following, we make essential use of support cones and normal cones, which describe the behavior of a closed convex set at one of its points. We recall briefly the definitions. Let $K \subset X$ be a closed convex set and $x \in K$. The *support cone* $S(K, x)$ of K at x is the closure of the set $\{\lambda(y - x) : y \in K, \lambda \geq 0\}$. It is a closed convex cone and can also be represented as

$$S(K, x) = \bigcap_{x \in H(K, f)} H^-(K, f) - x,$$

the intersection extending over all $f \in S^*$ for which $x \in H(K, f)$ (if x is an interior point of K , the intersection of the empty family of subsets of X is understood as X). The dual convex cone

$$N(K, x) = S(K, x)^* = \{f \in X^* : f(z) \leq 0 \ \forall z \in S(K, x)\}$$

is called the *normal cone* of K at x . Evidently,

$$N(K, x) = \{f \in X^* : f(y) \leq f(x) \ \forall y \in K\},$$

thus $f \in N(K, x)$ if and only if f supports K at x or f is the zero functional.

A closed convex cone C is called a *normal cone of K* if there exists a point x in the boundary of K such that $C = N(K, x)$. If K is a polytope, F is a face of K and x is a relatively interior point of F , then the normal cone $N(K, x)$ does not depend on the choice of x . It is denoted by $N(K, F)$ and called the *normal cone of K at F* , thus $N(K, F) = \{f \in X^* : F \subset F(K, f)\}$. Its dimension satisfies $\dim F + \dim N(K, F) = d$. The finite system of all normal cones of a polytope is called the *normal fan* of the polytope.

If x is a boundary point of the unit ball B , then

$$N(B, x) \cap S^* = \{f \in S^* : f(x) = 1\},$$

which is an exposed face of the dual unit ball B^* . Thus, the normal cones of the unit ball are precisely the positive hulls of the exposed faces of the dual unit ball. The following result will be needed.

Lemma 2.2. *Let $K, L \subset X$ be nonempty closed convex sets.*

(a) *If $x \in K$ and $y \in L$, then*

$$N(K + L, x + y) = N(K, x) \cap N(L, y).$$

(b) *If K, L are polytopes and $x \in K \cap L$, then*

$$N(K \cap L, x) = N(K, x) + N(L, x).$$

For a proof, we refer to [18], Theorem 2.2.1 (the use of a scalar product in [18] is evidently not essential for the validity of these results). However, (b) is formulated there for general convex bodies, which requires an extra condition. That this condition can be deleted in the case of polytopes, follows from the fact that for a polytope K the set $\{\lambda(y - x) : y \in K, \lambda \geq 0\}$ is already closed and hence equal to the support cone $S(K, x)$. This, in turn, implies that for polytopes K, L and $x \in K \cap L$ we have $S(K \cap L, x) = S(K, x) \cap S(L, x)$, from which (b) follows.

3. STABILITY AND BALL STABILITY OF POLYHEDRAL NORMS

The purpose of this section is to obtain characterizations for both the stability and the ball stability of \mathcal{M} in finite dimensional polyhedral spaces. We begin with a result concerning ℓ_∞ sums.

Proposition 3.1. *Let X_1, \dots, X_k be finite dimensional normed spaces, let*

$$X = X_1 \oplus_\infty \cdots \oplus_\infty X_k.$$

Then \mathcal{M}_X is stable (ball stable) if and only if each \mathcal{M}_{X_i} is stable (ball stable).

Proof. We will write the proof only for stability, since for ball stability the argument is similar. Let B_i be the unit ball of X_i ($i = 1, \dots, k$), then $B = B_1 \oplus \dots \oplus B_k$ is the unit ball of X . Let $\pi_i : X \rightarrow X_i$ denote the canonical projection. We claim that

$$K \in \mathcal{M}_X \Leftrightarrow K = K_1 \oplus \dots \oplus K_k \text{ with } K_i \in \mathcal{M}_{X_i}, \quad i = 1, \dots, k. \quad (2)$$

Indeed, if $K \in \mathcal{M}_X$, say $K = \bigcap_{\alpha \in I} (\lambda_\alpha B + t_\alpha)$, then

$$\begin{aligned} K &= \bigcap_{\alpha \in I} [(\lambda_\alpha B_1 + \pi_1(t_\alpha)) \oplus \dots \oplus (\lambda_\alpha B_k + \pi_k(t_\alpha))] \\ &= \left(\bigcap_{\alpha \in I} (\lambda_\alpha B_1 + \pi_1(t_\alpha)) \right) \oplus \dots \oplus \left(\bigcap_{\alpha \in I} (\lambda_\alpha B_k + \pi_k(t_\alpha)) \right). \end{aligned}$$

Conversely, let $K = K_1 \oplus \dots \oplus K_k$ with $K_i \in \mathcal{M}_{X_i}$, $i = 1, \dots, k$. Let $x \in X \setminus K$. There is an index j with $\pi_j(x) \notin K_j$. Hence, there are $\lambda > 0$ and $t_j \in X_j$ with $K_j \subset \lambda B_j + t_j$ and $\pi_j(x) \notin \lambda B_j + t_j$. Here λ can be chosen arbitrarily large (if t_j is suitably adapted), in particular so large that $K_i \subset \lambda B_i$ for $i \neq j$. Then it follows that $K \subset \lambda B + t$ for t with $\pi_j(t) = t_j$ and $x \notin \lambda B + t$. This proves that $K \in \mathcal{M}_X$, as desired.

For the proof of the proposition, suppose now that \mathcal{M}_{X_i} is stable for $i = 1, \dots, k$. Let $C, D \in \mathcal{M}_X$. Using (2), we can write $C = C_1 \oplus \dots \oplus C_k$ and $D = D_1 \oplus \dots \oplus D_k$ with $C_i, D_i \in \mathcal{M}_{X_i}$, $i = 1, \dots, k$, hence

$$C + D = (C_1 + D_1) \oplus \dots \oplus (C_k + D_k).$$

Since $C_i + D_i \in \mathcal{M}_{X_i}$, it follows from (2) that $C + D \in \mathcal{M}_X$. Thus, \mathcal{M}_X is stable. Conversely, suppose that \mathcal{M}_X is stable. Let $C_1, D_1 \in \mathcal{M}_{X_1}$, say. There is a family $\{\lambda_\alpha B_1 + t_\alpha\}$ of balls in X_1 with $C_1 = \bigcap_{\alpha} (\lambda_\alpha B_1 + t_\alpha)$. Put $\lambda := \inf_{\alpha} \lambda_\alpha$. Then

$$C_1 \oplus \lambda B_2 \oplus \dots \oplus \lambda B_k = \bigcap_{\alpha} [(\lambda_\alpha B_1 + t_\alpha) \oplus \lambda_\alpha B_2 \oplus \dots \oplus \lambda_\alpha B_k] \in \mathcal{M}_X.$$

Similarly, there is $\mu \geq 0$ with $D_1 \oplus \mu B_2 \oplus \dots \oplus \mu B_k \in \mathcal{M}_X$. Since \mathcal{M}_X is stable, we have

$$(C_1 + D_1) \oplus (\lambda + \mu) B_2 \oplus \dots \oplus (\lambda + \mu) B_k \in \mathcal{M}_X,$$

hence the claim (2) gives $C_1 + D_1 \in \mathcal{M}_{X_1}$. Thus \mathcal{M}_{X_1} is stable. \square

Remark. The natural question whether Proposition 3.1 has an analog for ℓ_1 -sums has a negative answer, since $\mathcal{M}_{\ell_1^3}$ is not stable [8] (yet it is ball stable, for instance by Theorem 3.3 below). Moreover, we show that the ℓ_1 -sum of two finite dimensional normed spaces, both satisfying ball stability, need not be ball stable. Consider for X the space \mathbb{R}^2 endowed with a norm whose unit ball is a hexagon, and let $Y = X \oplus_1 \mathbb{R}$. That \mathcal{M} is not ball stable in this space, can be deduced from Theorem 3.3.

For finite dimensional polyhedral spaces, we are able to give an explicit description of the spaces satisfying stability.

Theorem 3.2. *Let X be a finite dimensional space with polyhedral norm. Then \mathcal{M}_X is stable if and only if $X = X_1 \oplus_\infty \cdots \oplus_\infty X_k$, where each X_i is a polyhedral normed space of dimension at most two, or, equivalently, if the unit ball of X is a direct sum of convex polygons of dimensions one and two.*

Proof. Sufficiency is a direct consequence of Proposition 3.1 and the fact that \mathcal{M} is stable in every two dimensional space [8], so it remains to prove necessity. Recall that $\dim X = d$ and that Ω is the set of vertices of the dual unit ball B^* . Let Λ be the system of lines in X^* spanned by the pairs of antipodal points in Ω . A $(d-1)$ -dimensional subspace of X^* spanned by $d-1$ lines of Λ is called a *spanned subspace*.

Claim. If $d-1$ spanned subspaces intersect in a line, then this line belongs to Λ .

For the proof, let S_1, \dots, S_{d-1} be spanned subspaces that intersect in a line L . Let $f_L \in S^*$ be one of the two vectors (in X^*) spanning L . We may assume that $f_1, \dots, f_{d-1} \in \Omega$ are $d-1$ linearly independent vectors spanning S_1 . Then $f_L \in \text{lin}\{f_1, \dots, f_{d-1}\}$, and without loss of generality (replacing f_i by $-f_i$ where necessary) we can assume that $f_L \in \text{pos}\{f_1, \dots, f_{d-1}\}$. The functional f_i determines a unique facet $F_i = F(B, f_i) = f_i^{-1}(1)$ of the polytopal unit ball B . Choose $x_i \in \text{relint } F_i$ (the relative interior of F_i) ($i = 1, \dots, d-1$). Then

$$P_1 := \bigcap_{i=1}^{d-1} (B - x_i)$$

is a polytope and, obviously, an element of \mathcal{M} . The set $G_1 := \bigcap_{i=1}^{d-1} (F_i - x_i)$ is a face of P_1 . Since $x_i \in \text{relint } F_i$, the set G_1 has the same dimension as $\bigcap_{i=1}^{d-1} f_i^{-1}(1)$, which is 1, due to the linear independence of f_1, \dots, f_{d-1} . Thus G_1 is an edge of P_1 . The normal cone $N(P_1, G_1)$ of P_1 at G_1 is the positive hull of the vectors f_1, \dots, f_{d-1} , as follows from Lemma 2.2(b), hence $f_L \in N(P_1, G_1)$. Performing the same construction with S_2, \dots, S_{d-1} , we obtain polytopes $P_1, \dots, P_{d-1} \in \mathcal{M}$ and corresponding edges $G_i \subset P_i$ such that $f_L \in \bigcap_{i=1}^{d-1} N(P_i, G_i)$. The sum $G := G_1 + \cdots + G_{d-1}$ is a face of $P := P_1 + \cdots + P_{d-1}$ and, by Lemma 2.2(a), $N(P, G) = \bigcap_{i=1}^{d-1} N(P_i, G_i)$. Observe that $N(G_i, 0) = S_i$ and, hence

$$N(G, 0) = N(G_1 + \cdots + G_{d-1}, 0) = \bigcap_{i=1}^{d-1} S_i.$$

Since S_1, \dots, S_{d-1} intersect in a line, $\dim N(G, 0) = 1$. It follows that $\dim G = d-1$, thus G is a facet of P . From $f_L \in N(P, G)$ we see that f_L is a functional that determines G . Since \mathcal{M} is closed under Minkowski sums, we have $P \in \mathcal{M}$. Now it follows from Lemma 2.1 that $f_L \in \Omega$, which proves the claim.

The proof can now be completed by applying a result from projective geometry. The following argument was essentially used before in [19], but for greater clarity we include

it here in a simpler form. Let P^{d-1} be the $(d-1)$ -dimensional real projective space, whose points are the lines through 0 in X^* and whose k -dimensional subspaces are the $(k+1)$ -dimensional linear subspaces of X^* . Then Λ is a system of n points in P^{d-1} (where $2n$ is the number of points in Ω), not all lying in a hyperplane. By results of Motzkin [16, 4.3] and Hansen [11] (showing that Motzkin's hypothesis U_d is always satisfied), the following holds. The number of hyperplanes in P^{d-1} determined by the points of Λ is at least n , and it is exactly n if and only if (*) the points are, for odd $d-1$, on $d/2$ lines, and for even $d-1$, all but one on $(d-1)/2$ lines. Now suppose that (*) does not hold. Then the points of Λ determine at least $n+1$ hyperplanes in P^{d-1} , and these determine, by the dual of Motzkin's result, at least $n+1$ intersection points and thus at least one point not in Λ . This contradicts the claim. Thus (*) holds. For the points of Ω this means that they lie, for even d , on $d/2$ two-dimensional linear subspaces of X^* , and for odd d , on $(d-1)/2$ two-dimensional linear subspaces and a one-dimensional linear subspace. Since Ω spans X^* , it follows that B is a direct sum of two-dimensional polygons and, for odd d , a segment. \square

Remark. As noted in [15], the sets in \mathcal{M} are precisely the H -convex sets, for $H = \Omega$, the closure of the set of exposed points of B^* (in finite dimensions). Boltyanski and Martini [2] have investigated the question under which conditions the Minkowski sum of H -convex sets is again H -convex. Their Example 1 seems to contradict our Theorem 3.2. However, H. Martini has informed us that their Examples 1 and 3 are erroneous.

We will now prove a characterization of the finite dimensional polyhedral norms for which \mathcal{M} is ball stable. To this end, we define a *critical configuration* of B as a pair $(N, \{f_1, \dots, f_k\})$ consisting of a normal cone N of B of dimension at least two and a set $\{f_1, \dots, f_k\} \subset \Omega$ of vertices of B^* such that

$$\dim(N \cap \text{pos}\{f_1, \dots, f_k\}) = 1.$$

Theorem 3.3. *Let X be a finite dimensional polyhedral normed space. Then \mathcal{M}_X is ball stable if and only if B does not have a critical configuration.*

Proof. Suppose that $(N, \{f_1, \dots, f_k\})$ is a critical configuration of B . As in the proof of Theorem 3.2, we choose $x_i \in \text{relint } F(B, f_i)$ ($i = 1, \dots, k$) and put

$$P := \bigcap_{i=1}^k (B - x_i),$$

then $P \in \mathcal{M}_X$ and, by Lemma 2.2, $N(P, 0) = \text{pos}\{f_1, \dots, f_k\}$. Let F be the face of B with $N(B, F) = N$ and choose $x \in \text{relint } F$. Again by Lemma 2.2, $N(P + B, x) = N(P, 0) \cap N(B, x)$. Since $x \in \text{relint } F$, it follows that $N(P + B, x) = N(B, F) \cap N(P, 0) = N \cap \text{pos}\{f_1, \dots, f_k\}$. By assumption, this cone has dimension one, hence the unique $g \in S^* \cap N(P + B, x)$ determines a facet of $P + B$. If we had $P + B \in \mathcal{M}_X$, then Lemma

2.1 would imply that $g \in \Omega$. Since $g \in N(B, F)$, the face F would be a facet and thus $\dim N(B, F) = 1$, a contradiction. Thus, \mathcal{M}_X is not ball stable.

Conversely, suppose that \mathcal{M}_X is not ball stable. Then there exists a set $P \in \mathcal{M}_X$, necessarily a polytope, such that $P + B \notin \mathcal{M}_X$. By Lemma 2.1,

$$P + B \neq \bigcap_{f \in \Omega} H^-(P + B, f).$$

Therefore, the polytope $P + B$ has a facet F_1 whose determining functional, say $g \in S^*$, is not an element of Ω . Let

$$F := F(B, g), \quad G := F(P, g),$$

then $F_1 = F + G$. Choosing a point $x_1 \in \text{relint } F_1$, we can represent it as $x_1 = x + y$ with $x \in \text{relint } F$ and $y \in \text{relint } G$. Therefore, by Lemma 2.2,

$$N(P + B, F) = N(P + B, x_1) = N(P, y) \cap N(B, x) = N(P, G) \cap N(B, F).$$

By Lemma 2.1, the polytope $P \in \mathcal{M}_X$ can be represented as $P = \bigcap_{f \in \Omega} H^-(P, f)$. The face G is an intersection of facets of P . Hence, there are $f_1, \dots, f_k \in \Omega$ such that

$$N(P, G) = \text{pos}\{f_1, \dots, f_k\}.$$

Since F_1 is a facet of $P + B$, we have

$$1 = \dim N(P + B, F_1) = \dim(N(B, F) \cap \text{pos}\{f_1, \dots, f_k\}).$$

Since $F = F(B, g)$ and $g \notin \Omega$, the face F is not a facet, hence $\dim N(B, F) \geq 2$. Thus, $(N(B, F), \{f_1, \dots, f_k\})$ is a critical configuration. \square

As mentioned earlier, ℓ_1^d is an example of a space in which \mathcal{M} is ball stable, but not stable.

4. MAZUR SETS IN POLYHEDRAL SPACES

The family \mathcal{M} of all intersections of balls has two nice subfamilies, both in the core of Minkowski's convexity theory: convex bodies of constant width and diametrically maximal sets [5]. In addition to these classical examples, the class \mathcal{P} of Mazur sets (see the definition in the introduction) is another interesting subfamily of \mathcal{M} . It is known that sets of constant width are always Mazur sets while this is not the case for diametrically maximal sets [14]. The space is called a *Mazur space* if $\mathcal{P} = \mathcal{M}$.

It was shown in [9] that the only Mazur sets in ℓ_1^d are one-pointed sets and balls. The main purpose of this section is to give a characterization of the finite dimensional

polyhedral spaces satisfying this curious property. To this end, the Mazur sets in a finite dimensional polyhedral space are determined in the following theorem.

Let $P, Q \in \mathcal{K}$ be polytopes. We say that the normal fan of P is *coarser* than the normal fan of Q if the normal fan of Q is a refinement of the normal fan of P , that is, every normal cone of P is a union of normal cones of Q . A *summand* of a polytope P is a compact convex set K of positive dimension for which there exists a compact convex set M of positive dimension such that $P = K + M$. Summands of polytopes are necessarily polytopes ([18], p. 143).

Theorem 4.1. *For a convex body $K \in \mathcal{K}$ of positive dimension in a polyhedral space X , the following conditions are equivalent:*

- (a) K is a Mazur set.
- (b) K is a polytope whose normal fan is coarser than the normal fan of B .
- (c) K is homothetic to a summand of B .

Proof. We begin with an elementary preparation.

Claim. Let $C \subset X$ be a closed convex cone, let $f \in X^*$ be a functional such that $C \cap f^{-1}(0) = \{0\}$. Let $K \in \mathcal{K}$ be a convex body such that to every $\varepsilon > 0$ there exists a vector $t_\varepsilon \in X$ with

$$K + t_\varepsilon \subset C \quad \text{and} \quad (K + t_\varepsilon) \cap f^{-1}(-\varepsilon) \neq \emptyset. \quad (3)$$

If $x \in K$ is such that $f(x) = \max f(K)$, then $0 \in K - x \subset C$.

For the proof, we observe that $C \subset f^{-1}(-\infty, 0)$. If t_ε satisfies (3), then $x + t_\varepsilon \in C \cap f^{-1}(-\varepsilon, 0)$. Choosing a sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ with $\varepsilon_i \rightarrow 0$, we see from

$$\bigcap_{i \in \mathbb{N}} [C \cap f^{-1}(-\varepsilon_i, 0)] = \{0\}$$

that $t_{\varepsilon_i} \rightarrow -x$ and, hence, $0 \in K - x \subset C$, which proves the claim.

Now let $K \in \mathcal{K}$ be a Mazur set. Let $N(K, x)$ be a normal cone of K , and let $f \in N(K, x) \cap S^*$; then $x \in F(K, f)$. The support set $F = F(B, f)$ is a face of the polytope B . Let $\varepsilon > 0$ be given. The hyperplane $f^{-1}(h(K, f) + \varepsilon)$ does not meet K . Since K is a Mazur set, there exists a ball $B_\varepsilon = \lambda_\varepsilon B + t_\varepsilon$, $\lambda_\varepsilon > 0$, with $K \subset B_\varepsilon \subset f^{-1}(-\infty, h(K, f) + \varepsilon)$.

Choose $y \in \text{relint } F$ and let $y_\varepsilon := \lambda_\varepsilon y + t_\varepsilon$. Then $K - y_\varepsilon \subset S(B_\varepsilon, y_\varepsilon) = S(B, y)$, the latter by an obvious property of support cones. Moreover, $f(x) = h(K, f) \leq f(y_\varepsilon) \leq h(K, f) + \varepsilon$ and thus $-\varepsilon \leq f(x - y_\varepsilon) \leq 0$. In particular, $K - y_\varepsilon \cap f^{-1}(-\varepsilon) \neq \emptyset$.

Since $y \in \text{relint } F$, the support cone $S(B, y)$ is a direct sum, $S(B, y) = X_1 \oplus C$, where the subspace X_1 is a translate of the affine hull of F , and C is a pointed convex cone in a subspace X_2 complementary to X_1 . Let π_2 be the canonical projection to X_2 . Writing

$-y_\varepsilon = t_{1,\varepsilon} + t_{2,\varepsilon}$ with $t_{i,\varepsilon} \in X_i$, we get from the claim (with X replaced by X_2 and K replaced by $\pi_2(K)$) that $0 \in \pi_2(K) - \pi_2(x) \subset \pi_2(S(B, y))$. Since $K + t \subset S(B, y)$ for all $t \in X_1$, this implies $0 \in K - x \subset S(B, y)$. From this, we get $S(K, x) = S(K - x, 0) \subset S(B, y)$. By duality, it follows that $N(B, y) \subset N(K, x)$; moreover, $f \in N(B, y)$. Since $N(K, x)$ was an arbitrary normal cone of K , this shows that every normal cone of K is a union of normal cones of B . In particular, K has only finitely many normal cones and hence is a polytope. The normal fan of this polytope is coarser than the normal fan of B . Thus, (b) holds.

The equivalence of (b) and (c) may be ‘folklore’, but in default of a suitable reference we give a proof of the necessity. Suppose that (b) holds. Let $f \in S^*$ be such that the face $F = F(K, f)$ is an edge of K . Since the normal fan of K is coarser than the normal fan of B , there exists a normal cone N of B such that $f \in N \subset N(K, F)$. There exists a face G of B with $N = N(B, G)$. Since $N(B, G) \subset N(K, F)$, the face G contains a line segment parallel to the edge F . Since there are only finitely many such pairs (F, G) , we can choose a number $\lambda > 0$ with the following property: whenever $F(\lambda K, f)$ ($f \in S^*$) is an edge of λK , then the support set $F(B, f)$ contains a translate of $F(\lambda K, f)$. Now it follows from [18, Theorem 3.2.8] that λK is a summand of B . Thus (c) holds.

Suppose that (c) holds. Then there are a number $\lambda > 0$ and a convex body M , necessarily a polytope, such that $B = \lambda K + M$. Then, $K + \lambda^{-1}M = \lambda^{-1}B$. Let $f \in X^*$ and $\mu \in \mathbb{R}$ be such that $\sup f(K) < \mu$. We choose $y \in \lambda^{-1}M$ satisfying $f(y) = \sup f(\lambda^{-1}M)$ and consider the ball $C = -y + \lambda^{-1}B$. Since $0 \in \lambda^{-1}M - y$ and

$$K + (\lambda^{-1}M - y) = -y + \lambda^{-1}B$$

we see that $K \subset C$. Finally,

$$\begin{aligned} \sup f(C) &= f(-y) + \sup f(\lambda^{-1}M) + \sup f(K) \\ &= \sup f(K) < \mu \end{aligned}$$

which, since f and μ were arbitrary, proves that K is a Mazur set. \square

Corollary 4.2. *A finite dimensional polyhedral space X is a Mazur space if and only if \mathcal{M}_X is stable.*

Proof. A useful property of \mathcal{P} is that it is stable under Minkowski sums in any normed space [8]. Hence, in every Mazur space X , the set \mathcal{M}_X is stable. Conversely, assume that X is a finite dimensional polyhedral space and \mathcal{M}_X is stable. Then it follows from Theorem 3.2 that X is of the form $X = X_1 \oplus_\infty \cdots \oplus_\infty X_k$ with $\dim X_i \leq 2$ and each X_i polyhedral. If $K \in \mathcal{M}_X$, then $K = K_1 \oplus \cdots \oplus K_k$ with $K_i \in \mathcal{M}_{X_i}$, by Proposition 3.1. Each X_i is a Mazur space ([8], Theorem 6.5), hence by Theorem 4.1, K_i is homothetic to a summand of B_i (the unit ball of X_i). This means that there exist a number $\lambda_i > 0$ and a convex body $M_i \subset X_i$ with $B_i = \lambda_i K_i + M_i$ ($i = 1, \dots, k$). Choosing a number λ with

$0 < \lambda < 1$, we have $B_i = \lambda K_i + (\lambda_i - \lambda)K_i + M_i$ and hence

$$B_1 \oplus \cdots \oplus B_k = \lambda(K_1 \oplus \cdots \oplus K_k) + [((\lambda_1 - \lambda)K_1 + M_1) \oplus \cdots \oplus ((\lambda_k - \lambda)K_k + M_k)].$$

Thus, K is homothetic to a summand of $B = B_1 \oplus \cdots \oplus B_k$ and is, therefore, a Mazur set. This shows that X is a Mazur space. \square

Our final result shows that the property that balls and one-pointed sets are the only Mazur sets is not so rare. Here the notion of indecomposable polytopes plays a role. A convex body $K \in \mathcal{K}$ is *indecomposable* if a representation $K = M + L$ with convex bodies $M, L \in \mathcal{K}$ is only possible if M and L are either one-pointed or homothetic to K .

Corollary 4.3. *For a polyhedral space X , the following conditions are equivalent:*

- (a) *The only Mazur sets in X are balls and one-pointed sets.*
- (b) *The unit ball B is indecomposable.*

Proof. Suppose that (b) holds and that K is a Mazur set. By Theorem 4.1, K is homothetic to a summand of B . Since B is indecomposable, K can only be a trivial summand, that is, either one-pointed or homothetic to B .

Conversely, suppose that (b) does not hold. Then there are two convex bodies K, M , neither one-pointed nor homothetic to B , such that $B = K + M$. These bodies are polytopes. It follows from Theorem 4.1 that K and M are Mazur sets. Thus (a) does not hold. \square

If we assume that $d \geq 3$ and endow the set of all isometry classes of d -dimensional normed spaces with the Banach-Mazur metric, then the subset of classes of normed spaces with only trivial Mazur sets is dense. This follows from Corollary 4.3, together with the facts that every simplicial polytope of dimension at least three is indecomposable ([18, Corollary 3.2.13]) and that every convex body can be approximated arbitrarily closely by simplicial polytopes.

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