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Kinematic and Crofton formulae of integral geometry: recent variants and extensions

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Abstract

The principal kinematic formula and the closely related Crofton formula are central themes of integral geometry in the sense of Blaschke and Santaló. There have been various generalizations, variants, and analogues of these formulae, in part motivated by applications. We give a survey of recent investigations in the spirit of the kinematic and Crofton formulae, concentrating essentially on developments during the last decade.

In the early days of integral geometry, the later illustrious geometers S.S. Chern, H. Hadwiger, L.A. Santaló were attracted by Wilhelm Blaschke's geometric school and all spent some time with him in Hamburg. There, the young Santaló wrote his work (Santaló 1936) on the kinematic measure in space, studying various mean values connected with the interaction of fixed and moving geometric objects and applying them to different questions about geometric probabilities. Fourty years later, when Santaló's (1976) comprehensive book on integral geometry appeared, the principal kinematic formula, which is now associated with the names of Blaschke, Santaló and Chern, was still a central theme of integral geometry, together with its generalizations and analogues. At about the same time, the old connections of integral geometry with geometric probabilities were deepened through the use that was made of kinematic formulae, Crofton formulae and integral geometric transformations in stochastic geometry, for example in the theoretical foundations of stereology under invariance assumptions. To get an impression of this, the reader is referred to the books of Matheron (1975), Schneider and Weil (1992, 2000). Integral geometry has also begun to play a role in statistical physics, see Mecke (1994, 1998). Motivated by demands from applications, but also for their inherent geometric beauty, kinematic formulae of integral geometry and their ramifications have continuously remained an object of investigation. In the following, we give a survey of recent progress. We concentrate roughly on the period since 1990, since much of the earlier development is covered by the survey articles of Weil (1979) and Schneider and Wieacker (1993). To the bibliographies of these articles and of Schneider and Weil (1992) we refer for the earlier literature.

Notation

By $\mathbb{E}^n$ we denote the n-dimensional Euclidean vector space, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Its unit ball and unit sphere are $B^n := \{x \in \mathbb{E}^n : \|x\| \leq 1\}$ and $S^{n-1} := \{x \in \mathbb{E}^n : \|x\| = 1\}$, respectively. Lebesgue measure on $\mathbb{E}^n$ is denoted by λ_n, and spherical Lebesgue measure on S^{n-1} by σ_{n-1}. Then $\kappa_n := \lambda_n(B^n) = \pi^{n/2}/\Gamma(1+n/2)$ and $\omega_n := \sigma_{n-1}(S^{n-1}) = n\kappa_n = 2\pi^{n/2}/\Gamma(n/2)$.

G_n is the group of rigid motions of $\mathbb{E}^n$, and μ is the invariant (or Haar) measure on G_n, normalized so that $\mu(\{g \in G_n : gx \in B^n\}) = \kappa_n$ for $x \in \mathbb{E}^n$. The rotation group of $\mathbb{E}^n$ is denoted by SO_n, its invariant probability measure by ν. By $\mathcal{L}_q^n$ we denote the Grassmannian of q-dimensional linear subspaces of $\mathbb{E}^n$, for $q \in \{0, \dots, n\}$, its rotation invariant probability measure is ν_q. Similarly, $\mathcal{E}_q^n$ is the space of q-flats in $\mathbb{E}^n$, and μ_q is its motion invariant measure, normalized so that $\mu_q(\{E \in \mathcal{E}_q^n : E \cap B^n \neq \emptyset\}) = \kappa_{n-q}$.

By a convex body we understand a non-empty compact convex subset of $\mathbb{E}^n$. The space $\mathcal{K}^n$ of convex bodies in $\mathbb{E}^n$ is equipped with the Hausdorff metric. A function φ on $\mathcal{K}^n$ with values in some abelian group is called *additive* or a *valuation* if $\varphi(K \cup K') + \varphi(K \cap K') = \varphi(K) + \varphi(K')$ whenever $K, K', K \cup K' \in \mathcal{K}^n$. For such a function, one extends the definition by $\varphi(\emptyset) := 0$.

For a topological space X, the σ-algebra of Borel sets in X is denoted by $\mathcal{B}(X)$.

1 The classical kinematic and Crofton formulae

For the purpose of introduction, we begin with the simplest version of the principal kinematic formula in Euclidean space $\mathbb{E}^n$, namely

$$\int_{G_n} \chi(K \cap gK')\mu(dg) = \sum_{k=0}^{n} \alpha_{n0k} V_k(K) V_{n-k}(K') \tag{1}$$

for convex bodies $K, K' \in \mathcal{K}^n$. Here χ is the Euler characteristic, that is, $\chi(K) = 1$ for $K \in \mathcal{K}^n$ and $\chi(\emptyset) = 0$. We put

$$\alpha_{njk} := \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n+j-k+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} = \frac{k!\kappa_k (n+j-k)!\kappa_{n+j-k}}{j!\kappa_j n!\kappa_n}.$$

The functionals $V_0, \dots, V_n$ appearing on the right-hand side of (1) are the *intrinsic volumes.* They can be represented by

$$\int_{\mathcal{E}_q^n} \chi(K \cap E)\mu_q(dE) = \alpha_{n0q} V_{n-q}(K) \tag{2}$$

for $q = 0, \dots, n$. In particular, $V_0 = \chi$, and V_n is the volume. For $K' = \epsilon B^n$ with $\epsilon > 0$, (1) reduces to

$$\lambda_n(K_\epsilon) = \sum_{k=0}^{n} \epsilon^{n-k} \kappa_{n-k} V_k(K), \tag{3}$$

where K_ϵ is the set of points having distance at most ϵ from K. The existence of such a polynomial expansion can be proved directly, and then (3) can be used for defining the intrinsic volumes. The *Steiner formula* (3) has a natural local version. For $K \in \mathcal{K}^n$ and $x \in \mathbb{E}^n$, let $p(K, x)$ be the (unique) point in K nearest to x. For $\epsilon > 0$ and $\beta \in \mathcal{B}(\mathbb{E}^n)$, a local parallel set is defined by $A_\epsilon(K, \beta) := \{x \in K_\epsilon : p(K, x) \in \beta\}$. Then one has a polynomial expansion

$$\lambda_n(A_\epsilon(K, \beta)) = \sum_{k=0}^{n} \epsilon^{n-k} \kappa_{n-k} \Phi_k(K, \beta)$$

with finite Borel measures $\Phi_k(K, \cdot)$, $k \in \{0, \dots, n\}$, the *curvature measures* of K (where $\Phi_n(K, \beta) = \lambda_n(K \cap \beta)$). The total measures are the intrinsic volumes, $\Phi_k(K, \mathbb{E}^n) = V_k(K)$.

These curvature measures now appear in the *general kinematic formula*

$$\int_{G_n} \Phi_j(K \cap gK', \beta \cap g\beta')\mu(dg) = \sum_{k=j}^{n} \alpha_{njk}\Phi_k(K,\beta)\Phi_{n+j-k}(K',\beta') \tag{4}$$

for $\beta, \beta' \in \mathcal{B}(\mathbb{E}^n)$, and in the *general Crofton formula*

$$\int_{\mathcal{E}_q^n} \Phi_j(K \cap E, \beta \cap E)\mu_q(dE) = \alpha_{njq}\Phi_{n+j-q}(K,\beta) \tag{5}$$

for $q \in \{0,\dots,n\}$ and $j \in \{0,\dots,q\}$.

The validity of these formulae goes far beyond convexity: the curvature measures can be defined, and (4) and (5) are true, if K and M are sets with positive reach. This general result, due to Federer, comprises also the case where K and M are regular, sufficiently smooth submanifolds. In that case, Weyl's tube formula provides the additional information that the curvature measures are intrinsic, that is, depend only on the inner Riemannian metrics of the submanifolds.

If K and K' are compact smooth submanifolds, of dimensions k and j, respectively, where $k + j \geq n$, then the case j replaced by $k + j - n$, $\beta = K$, $\beta' = K'$ of (4) reduces to the equation

$$\int_{G_n} \mathcal{H}^{k+j-n}(K \cap gK')\mu(dg) = \alpha_{n(k+j-n)k}\mathcal{H}^k(K)\mathcal{H}^j(K'), \tag{6}$$

where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure. Similarly, (5) gives

$$\int_{\mathcal{E}_j^n} \mathcal{H}^{k+j-n}(K \cap E)\mu_j(dE) = \alpha_{n(k+j-n)j}\mathcal{H}^k(K). \tag{7}$$

Federer (1954) has shown that these formulae, which no longer involve curvatures, hold in great generality, namely for analytic sets K and K' such that K is Hausdorff k rectifiable and K' is j rectifiable.

Looking at the prototype (1) of a kinematic formula, we see that the left side involves a transformation group and its invariant measure, a fixed and a moving set, here both convex, the operation of intersection, and a geometric functional, here the Euler characteristic (which is trivial only as long as the sets involved are convex bodies). Each of these ingredients may be altered. The following survey describes various instances where this has been done successfully.

2 Extended kinematic formulae

In this section, the underlying group will be the group of rigid motions of Euclidean space, respectively the group of rotations in some cases. Integrations are always with respect to motion or rotation invariant measures.

2.1 Curvature measures for more general sets

First we mention investigations in which the definition of curvature measures and the classes of sets for which corresponding kinematic formulae can be proved has been widened considerably. We do this only briefly, since our main concern will be other variants and extensions in different directions, where the involved sets will mostly remain convex bodies.

In order to establish (1), (4) and (5) for various classes of sets with possibly severe singularities, it is useful to associate with suitable subsets $X \subset \mathbb{E}^n$ an $(n-1)$-dimensional integral current $N(X)$ in $T\mathbb{E}^n \cong T^*\mathbb{E}^n$ (or in the corresponding unit tangent sphere bundle $S\mathbb{E}^n$), which encodes the information about X relevant for the purposes of integral geometry. Such a current is a special linear functional on the space $\mathcal{D}^{n-1}(T\mathbb{E}^n)$ of smooth differential forms of degree $n-1$ on $T\mathbb{E}^n$ with coefficients in $\bigwedge^{n-1} T\mathbb{E}^n$. For convex sets X (and similarly for sets with positive reach) and $\psi \in \mathcal{D}^{n-1}(T\mathbb{E}^n)$ the value $N(X)(\psi)$ is obtained by integrating ψ against an orienting $(n-1)$-vector field of the generalized normal bundle $\operatorname{Nor} X$ (see below) with respect to the appropriate Hausdorff measure $\mathcal{H}^{n-1}$. It follows that $N(X)$ is a cycle which annihilates the contact form α and the two-form $d\alpha$. Moreover, substitution of specially chosen differential forms ψ_j into $N(X)$ leads to the curvature measures $\Phi_j(X,\cdot)$ of X. Thus the general kinematic formula (4) can be written as an equation involving the integral currents of K, K' and $K \cap gK'$ evaluated at such differential forms.

Starting from these observations, which originate in the work of Wintgen and Martina Zähle, Joseph Fu deduced by means of abstract bundle theoretic constructions a very general algebraic form of a kinematic integral formula involving normal currents (Fu 1990). Despite the generality of his result, which yields, e.g., Shifrin's kinematic formula in complex projective space or the kinematic formula for isotropic spaces (including the case of space forms) and sets with positive reach, still considerable efforts are required to derive specific cases from the general result. Having described $N(X)$ for convex sets, we are led to ask for which $X \subset \mathbb{E}^n$ a current $N(X)$ exists, enjoying some basic properties similar to the above, to investigate how it can be constructed, and to explore whether it is uniquely determined by these properties. These are precisely the questions considered in Fu (1994), although this paper is primarily concerned with integral geometric results for subanalytic sets.

An essential new idea in Fu (1994) is to work with compact sets $X \subset \mathbb{E}^n$ for which there is a nondegenerate Monge-Ampère function (called *aura*) $f : \mathbb{E}^n \to [0,\infty)$ such that $X = f^{-1}(\{0\})$. Results about Monge-Ampère functions have been provided by the author in two preceding papers. From these results, and by means of his abstract kinematic formula and a general version of the Chern-Gauss-Bonnet theorem, Fu derives an extension of formula (1) for sets which have an aura and satisfy a weak finiteness condition. In the subanalytic framework, it is also shown that the Gauss curvature measure is defined intrinsically. A survey of these and related results is given by Fu (1993).

This subject has been studied further in Bröcker and Kuppe (2000), by a different approach. The class of sets considered by these authors are compact tame Whitney-stratified sets. Examples of such sets admitting natural tame stratifications are semi-algebraic sets, subanalytic sets, sets belonging to an o-minimal system or to an $\mathfrak{X}$-system (with increasing degree of generality), especially smooth manifolds and Riemannian polyhedra. Sets from these classes may have various kinds of singularities, but convex sets and sets with positive reach do in general not fall into this category. The curvature measures of such a tame set Y can be defined as coefficients of a polynomial which is obtained by integration of an index function over tubular neighbourhoods of Y. Instead of working with normal cycles of the sets considered, Bröcker and Kuppe use stratified Morse theory as an essential tool. Thus they prove a Gauss-Bonnet formula, a local kinematic and a Crofton formula for compact sets from an $\mathfrak{X}$-system. Moreover they show that the curvature measures of sets from such a system are (in a reasonable sense) defined intrinsically. In part, these results are based on the approximation of tame sets from outside and from inside, respectively, by smooth sets, for which the corresponding results are basically known. In fact, under various additional assumptions, a kinematic formula is also established for more general compact tame Whitney-stratified sets.

Bernig and Bröcker (2002) use Fu's kinematic formula for subanalytic sets in space forms M_κ, $\kappa \in \{-1,0,1\}$, to extend this formula by approximation to subsets of M_κ which are *definable* with respect to a given analytic-geometric category in the sense of van den Dries and Miller. To some extent, Bröcker and Bernig (2002) aim at a synthesis with Fu (1994), by associating with a definable set Y from a given o-minimal system ω a normal cycle $N(Y)$ and by transferring thus to ω the flat topology from the space of $(n-1)$-dimensional flat chains in $S\mathbb{E}^n$. Existence of $N(Y)$ is derived from the existence of a sequence of smooth manifolds Y_r approximating Y and by means of a compactness theorem for currents. Uniqueness is achieved with the help of a uniqueness result proved in Fu (1994).

Curvature measures for certain unions of sets with positive reach and related kinematic formulae will be discussed in subsection 3.1.

2.2 Kinematic formulae for other integrands

In the following, we describe variants of the kinematic formulae (4) and the Crofton formulae (5) where the curvature measures are replaced by other measures or functionals. The sets to which such formulae apply will again be convex bodies. We mention, however, that all the formulae of this subsection can be extended to finite unions of convex bodies. This is due to the fact that the involved functions on the space of convex bodies are additive and have additive extensions to the convex ring, the set of finite unions of convex bodies, which are also known as polyconvex sets.

For a convex body $K \in \mathcal{K}^n$, the curvature measures $\Phi_0(K,\cdot),\dots,\Phi_{n-1}(K,\cdot)$ can be considered as measures on Borel sets of boundary points of K. Counterparts of these measures are defined on sets of normal vectors, and a common generalization of both types of measures involves support elements. A support element of the convex body K is a pair (x,u), where x is a boundary point of K and u is an outer unit normal vector of K at x. We write $\Sigma := \mathbb{E}^n \times S^{n-1}$. The motion group G_n operates on Σ by $g(x,u) := (gx, g_0u)$, where g_0 is the rotation part of g.

The generalized normal bundle $\operatorname{Nor} K \subset \Sigma$ of K is the set of all support elements of K (and $\operatorname{Nor}\emptyset := \emptyset$). Let $x \in \mathbb{E}^n \setminus K$. In addition to the nearest point $p(K,x)$, we consider the unit vector $u(K,x) := (x - p(K,x))/\|x - p(K,x)\|$; then $(p(K,x),u(K,x)) \in \operatorname{Nor} K$. For $\epsilon > 0$ and a Borel set $\eta \subset \Sigma$, a generalized local parallel set is defined by

$$M_\epsilon(K,\eta) := \{x \in K_\epsilon \setminus K : (p(K,x),u(K,x)) \in \eta\}.$$

Again one has a polynomial expansion,

$$\lambda_n(M_\epsilon(K,\eta)) = \sum_{k=0}^{n-1} \epsilon^{n-k}\kappa_{n-k}\Lambda_k(K,\eta),$$

and this defines the *generalized curvature measures* or *support measures* $\Lambda_0(K,\cdot),\dots,\Lambda_{n-1}(K,\cdot)$ of K. The curvature measure $\Phi_j(K,\cdot)$ is the image measure of $\Lambda_j(K,\cdot)$ under the projection $(x,u) \mapsto x$.

An extension of the formulae (4) and (5) to support measures is possible if one introduces a suitable law of composition for subsets of Σ, which is adapted to intersections of convex bodies. For $\eta,\eta' \subset \Sigma$, let

$$\begin{aligned} \eta \wedge \eta' \quad := \quad & \{(x,u) \in \Sigma : \text{ there are } u_1,u_2 \in S^{n-1} \text{ with} \\ & (x,u_1) \in \eta, (x,u_2) \in \eta', u \in \operatorname{pos}\{u_1,u_2\}\}, \end{aligned}$$

where $\operatorname{pos}\{u_1,u_2\} := \{\lambda_1u_1 + \lambda_2u_2 : \lambda_1,\lambda_2 \ge 0\}$ is the positive hull of $\{u_1,u_2\}$. For a q-flat $E \in \mathcal{E}^n_q$, $q \in \{1,\dots,n-1\}$, one defines

$$\begin{aligned} \eta \wedge E \quad := \quad & \{(x,u) \in \Sigma : \text{ there are } u_1,u_2 \in S^{n-1} \text{ with} \\ & (x,u_1) \in \eta, x \in E, u_2 \in E^\perp, u \in \operatorname{pos}\{u_1,u_2\}\}, \end{aligned}$$

where $E^\perp$ is the linear subspace totally orthogonal to E.

Now suppose that $K, K' \in \mathcal{K}^n$ are convex bodies, $\eta \subset \operatorname{Nor} K$ and $\eta' \subset \operatorname{Nor} K'$ are Borel sets, and $j \in \{0, \dots, n-2\}$. Then

$$\int_{G_n} \Lambda_j(K \cap gK', \eta \wedge g\eta')\mu(dg) = \sum_{k=j+1}^{n-1} \alpha_{njk}\Lambda_k(K,\eta)\Lambda_{n+j-k}(K',\eta') \tag{8}$$

(for $j = n-1$, both sides would give 0). This result is due to Glasauer (1997), under an additional assumption. A common boundary point x of the convex bodies K, K' is said to be exceptional if the linear hulls of the normal cones of K and K' at x have a non-zero intersection. Glasauer assumed that the set of rigid motions g for which K and gK' have some exceptional common boundary point is of Haar measure zero, and he conjectured that this assumption is always satisfied. This was proved by Schneider (1999).

The Crofton formula (5) has the following counterpart, also proved by Glasauer (1997). Let $K \subset \mathbb{E}^n$ be a convex body, $q \in \{1, \dots, n-1\}$, $j \in \{0, \dots, q-1\}$, and let $\eta \subset \operatorname{Nor} K$ be a Borel set. Then

$$\int_{\mathcal{E}_q^n} \Lambda_j(K \cap E, \eta \wedge E)\mu_q(dE) = \alpha_{njq}\Lambda_{n+j-q}(K,\eta). \tag{9}$$

We leave the curvature measures and turn to Hadwiger's general integral geometric theorem. Let $\varphi : \mathcal{K}^n \to \mathbb{R}$ be a continuous, additive function. Then

$$\int_{G_n} \varphi(K \cap gK')\mu(dg) = \sum_{k=0}^{n} \varphi_k(K)V_{n-k}(K') \tag{10}$$

for $K, K' \in \mathcal{K}^n$, where the coefficients are given by

$$\varphi_k(K) = \int_{\mathcal{E}_{n-k}^n} \varphi(K \cap E)\mu_{n-k}(dE). \tag{11}$$

This beautiful theorem, connecting kinematic and Crofton type integrals, and applicable to functions φ which need not satisfy any invariance property, is due to Hadwiger (1957) (p. 241, with different notation). We mention it here, although it is an old result, for three reasons. First, the only known proof uses Hadwiger's axiomatic characterization of the linear combinations of intrinsic volumes. For this theorem, Klain (1995) has given a short proof, so that also Hadwiger's integral geometric theorem is now more accessible. Second, the result (10) allows one to obtain kinematic formulae for additive, continuous functions φ once the corresponding Crofton integrals (11) can be evaluated. As one application, one may remark that a kinematic formula for projection functions that was proved by Goodey and Weil (1992), can alternatively be derived from the Crofton type formulae for projection functions, which these authors have also obtained. Other applications will be described below. Third, we want to point out that (4) and (5) can be extended in a similar way as (1) and (2) are extended by (10) and (11). This abstract version of (4) reads as follows.

Let $\Xi : \mathcal{K}^n \times \mathcal{B}(\mathbb{E}^n) \to \mathbb{R}$ be a mapping with the following properties:

(a) $\Xi(K,\cdot)$ is a finite positive measure concentrated on K, for all $K \in \mathcal{K}^n$.

(b) The map $K \mapsto \Xi(K,\cdot)$ is additive and weakly continuous.

(c) If $K, K' \in \mathcal{K}^n$, $\beta \subset \mathbb{E}^n$ is open and $K \cap \beta = K' \cap \beta$, then $\Xi(K,\alpha) = \Xi(K',\alpha)$ for all Borel sets $\alpha \subset \beta$.

Then, for $K, K' \in \mathcal{K}^n$, $\beta, \beta' \in \mathcal{B}(\mathbb{E}^n)$ and $j \in \{0, \dots, n\}$, the formula

$$\int_{G_n} \Xi(K \cap gK', \beta \cap g\beta')\mu(dg) = \sum_{k=0}^{n} \Xi_k(K,\beta)\Phi_{n-k}(K',\beta') \tag{12}$$

(with $\Xi(\emptyset, \cdot) := 0$) holds, where

$$\Xi_k(K, \beta) = \int_{\mathcal{E}^n_{n-k}} \Xi(K \cap E, \beta)\mu_{n-k}(dE). \tag{13}$$

This was proved by Schneider (1994). An analogue in spherical space and a simpler proof in Euclidean space were given by Glasauer (1995). Examples of mappings Ξ satisfying the above properties are the relative curvature measures introduced in Schneider (1994). Also (8) admits an abstract generalization in the spirit of Hadwiger's general integral geometric theorem; see Glasauer (1999), Theorem 7.

So far, all the considered integrands were real-valued functions or measures. The intrinsic volumes, which play an essential role in (1) and (2), arise from the notion of volume, through the Steiner formula (3). Replacement of the volume by vectorial or higher rank tensorial moments leads to a new class of problems. To explain these, we denote by $\mathbb{T}^p$ the vector space of symmetric tensors of rank p over $\mathbb{E}^n$ (we identify $\mathbb{E}^n$ with its dual space, using the scalar product, so that no distinction between covariant and contravariant tensors is necessary). If $p \in \mathbb{N}$ and $x \in \mathbb{E}^n$, we write x^p for the p-fold tensor product $x \otimes \cdots \otimes x$, and we put $x^0 := 1$. For symmetric tensors a and b, their symmetric product is denoted by ab. For $K \in \mathcal{K}^n$ and $p \in \mathbb{N}_0$, let

$$\Psi_p(K) := \frac{1}{p!} \int_K x^p \lambda_n(dx).$$

The Steiner formula (3) extends to a polynomial expansion

$$\Psi_p(K_\epsilon) = \sum_{k=0}^{n+p} \epsilon^{n+p-k} \kappa_{n+p-k} V_k^{(p)}(K) \tag{14}$$

with $V_k^{(p)}(K) \in \mathbb{T}^p$. Each function $V_k^{(p)} : \mathcal{K}^n \to \mathbb{T}^p$ is additive, continuous and isometry covariant (which means that $V_k^{(p)}(\rho K) = \rho V_k^{(p)}(K)$ for every rotation ρ and that $V_k^{(p)}(K + t)$ is a (tensor) polynomial in $t \in \mathbb{E}^n$ of degree p). The known facts in the case $p = 0$ suggest the following questions: (I) Do the coefficients $V_k^{(p)}$ satisfy kinematic and Crofton formulae? (II) Is an additive, continuous, isometry covariant function $f : \mathcal{K}^n \to \mathbb{T}^p$ necessarily a linear combination of $V_0^{(p)}, \ldots, V_{n+p}^{(p)}$? For $p = 0$, positive answers are given by (4) and (5) (for $\beta = \beta' = \mathbb{E}^n$) and by Hadwiger's characterization theorem. For $p = 1$, both questions were answered affirmatively by Hadwiger and Schneider (1971). Alesker (1999b) (based on Alesker (1999a)) has extended Hadwiger's characterization theorem to $p \geq 2$, but here the functionals $V_0^{(p)}, \ldots, V_{n+p}^{(p)}$ are no longer sufficient. More generally, one has to consider

$$\Phi_{m,r,s}(K) := \frac{1}{r!s!} \frac{\omega_{n-m}}{\omega_{n-m+s}} \int_\Sigma x^r u^s \Lambda_m(K, d(x, u))$$

for $K \in \mathcal{K}^n$ and integers $r, s \geq 0$, $0 \leq m \leq n - 1$ (the factors before the integral turn out to be convenient). Besides these tensor functions $\Phi_{m,r,s} : \mathcal{K}^n \to \mathbb{T}^{r+s}$ one also needs the metric tensor $G \in \mathbb{T}^2$ of $\mathbb{E}^n$. Now Alesker's characterization theorem reads as follows. If $p \in \mathbb{N}_0$ and if $f : \mathcal{K}^n \to \mathbb{T}^p$ is an additive, continuous, isometry covariant function, then f is a linear combination of the functions $G^q \Phi_{m,r,s}$ (with $2q + r + s = p$) and the functions $G^q \Psi_r$ (with $2q + r = p$).

In particular, the coefficient $V_k^{(p)}$ appearing in (14) is a sum of functions $\Phi_{k-p+s,p-s,s}$. Question (I) above should, therefore, be modified to ask whether the functions $\Phi_{m,r,s}$ satisfy kinematic and Crofton formulae. For dimension two and ranks one and two, kinematic formulae were already obtained by Müller (1953) (except for $\Phi_{0,1,1}$, in our notation), who took up a suggestion of Blaschke. An investigation for all dimensions and ranks was begun by Schneider (2000a) and continued by Schneider and Schuster (2002), but the question is not yet settled completely. Kinematic and Crofton formulae for $\Phi_{m,r,0}$ follow from the equations (4) and (5) for curvature measures. For $\Phi_{m,r,s}$ in general, it is sufficient to derive Crofton formulae, since then Hadwiger's general integral geometric theorem, which in the case of tensor functions can be applied coordinate-wise, immediately yields kinematic formulae. The following special cases of Crofton formulae were proved in Schneider and Schuster (2002):

$$\int_{\mathcal{E}^n_{n-1}} \Phi_{n-1,r,s}(K \cap E)\mu_{n-1}(dE) = \delta(n,s)G^{s/2}\Phi_{n,r,0}(K),$$

$$\int_{\mathcal{E}^n_{n-1}} \Phi_{n-2,r,s}(K \cap E)\mu_{n-1}(dE) = \sum_{m=0}^{\lfloor s/2 \rfloor} \alpha(n,s,m)G^m\Phi_{n-1,r,s-2m}(K),$$

$$\int_{\mathcal{E}^n_{n-2}} \Phi_{n-2,r,s}(K \cap E)\mu_{n-2}(dE) = \beta(n,s)G^{s/2}\Phi_{n,r,0}(K),$$

with explicit (though complicated) coefficients δ, α, β. Together with some linear relations between the functions $\Phi_{m,r,s}$, these results are sufficient for establishing a complete set of Crofton and kinematic formulae for all functions $\Phi_{m,r,s}$ in dimensions two and three. We give only one example in $\mathbb{E}^3$:

$$\begin{aligned}&\int_{G_3} \Phi_{0,1,1}(K \cap gK')\mu(dg)\\ &= \Phi_{0,1,1}(K)V_3(K') + \left[\frac{1}{16}GV_2(K) - \frac{\pi}{4}\Phi_{2,0,2}(K)\right]V_2(K') + \frac{1}{6\pi}GV_3(K)V_1(K').\end{aligned}$$

2.3 Kinematic formulae for non-intersecting sets

All the integral geometric results discussed so far concern the intersection of a fixed and a moving set. For convex sets, some other operations instead of intersection have appeared in integral geometric formulae. We refer to the survey article by Schneider and Wieacker (1993), in particular Section 3, where Minkowski addition and projection are the relevant operations, and to Section 4 as well as to the survey by Weil (1979) for results involving distances between fixed and moving convex sets. It appears that these investigations are rather complete and have not been continued during the last decade.

A completely new type of kinematic formulae, involving the convex hull of a fixed and a moving convex body, was found by Glasauer (1998). Since convex hulls with a freely moving convex body are not uniformly bounded, the results can only be of the type of weighted limits. Let $K \vee K'$ denote the convex hull of $K \cup K'$. A typical result of Glasauer concerns the mixed volumes (see, e.g., Schneider (1993)) with fixed convex bodies $K_{j+1}, \dots, K_n$ and states that

$$\begin{aligned}&\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n :\, gK'\subset rB^n\}} V(K \vee gK'[j], K_{j+1}, \dots, K_n)\mu(dg)\\ &= \frac{\kappa_{n-1}}{(n+1)\kappa_n} \sum_{k=0}^{j-1} V(K[k], B^n[j-k], K_{j+1}, \dots, K_n)V(K'[j-k-1], B^n[n-j+k+1]).\end{aligned}$$

This is a special case of Theorem 3 of Glasauer (1998). He has considerably more general results, for not necessarily invariant measures, and with mixed area measures instead of mixed volumes. For $K_{j+1} = \ldots = K_n = B^n$, the formula reduces to one for intrinsic volumes. For this result, there is also a local version, which is 'dual' to formula (8). It involves a law of composition for subsets of Σ which is adapted to the convex hull operation for pairs of convex bodies. For $\eta, \eta' \subset \Sigma$, let

$$\begin{aligned} \eta \vee \eta' := \quad & \{(x,u) \in \Sigma : \text{ there are } x_1, x_2 \in \mathbb{E}^n \text{ with} \\ & \langle x_1 - x_2, u \rangle = 0, (x_1, u) \in \eta, (x_2, u) \in \eta', x \in \operatorname{conv}\{x_1, x_2\}\}. \end{aligned}$$

Now suppose that $K, K' \in \mathcal{K}^n$, $\eta \subset \operatorname{Nor} K$ and $\eta' \subset \operatorname{Nor} K'$ are Borel sets, and $j \in \{0, \ldots, n-1\}$. Then Glasauer (1999) proved (with different notation) that

$$\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n :\, gK' \subset rB^n\}} \Lambda_j(K \vee gK', \eta \vee g\eta')\mu(dg) = \sum_{k=0}^{j-1} \beta_{njk} \Lambda_k(K, \eta)\Lambda_{j-k-1}(K', \eta'),$$

with explicit constants β_{njk}. The proof requires the following regularity result. A common support plane H of the convex bodies K, K' (leaving K and K' on the same side) is said to be exceptional if the affine hulls of the sets $H \cap K$ and $H \cap K'$ have a nonempty intersection or contain parallel lines. Then the set of all rigid motions g for which K and gK' have some exceptional common support plane is of Haar measure zero. This was conjectured by Glasauer and proved by Schneider (1999).

2.4 Analogues of kinematic formulae

Kinematic formulae are not restricted to Euclidean spaces. In fact, investigations for spaces of constant curvature or more general spaces date back to the early development of integral geometry. Santaló's book gives a rich picture. During the period covered by the present report (roughly the last decade) there appeared an important study by Howard (1993) on kinematic formulae in Riemannian homogeneous spaces. Such differential-geometric investigations are outside the scope of this survey.

Closer to our subject is the work of Glasauer (1995) (with a summary in Glasauer (1996)) on curvature measures of spherically convex sets. A perfect duality for support measures of convex bodies in spherical spaces makes these results particularly elegant. A spherically convex body in the sphere S^{n-1} is a set of the form $K = C \cap S^{n-1}$, where C is a closed convex cone in the vector space $\mathbb{E}^n$. The set of all spherically convex sets in S^{n-1} is denoted by $\mathcal{K}^n_s$. For $K \in \mathcal{K}^n_s$, the set $K^* := \{x \in S^{n-1} : \langle x, u \rangle \le 0 \text{ for } u \in K\}$ is the polar body of K, and $\operatorname{Nor} K := \{(x,u) \in K \times K^* : \langle x, u \rangle = 0\}$ is the generalized normal bundle of K. There are unique Borel measures $\Theta^s_j(K, \cdot)$ on $S^{n-1} \times S^{n-1}$ $(j \in \{0, \ldots, n-2\})$, concentrated on $\operatorname{Nor} K$, so that the Steiner-type formula

$$\begin{aligned} \int\limits_{S^{n-1}\setminus(K\cup K^*)} f\, d\sigma_{n-1} \quad = \quad & \sum_{j=0}^{n-2} \omega_{j+1}\omega_{n-j-1} \int\limits_0^{\pi/2} \cos^j t \sin^{n-j-2} t \times \\ & \int\limits_{S^{n-1}\times S^{n-1}} f(x\cos t + u \sin t)\Theta^s_j(K, d(x,u))\, dt \end{aligned}$$

holds for all σ_{n-1}-integrable functions $f : S^{n-1} \to \mathbb{R}$. These are the spherical counterparts of the Euclidean support measures, with a convenient normalization. They behave nicely under duality. For $\eta \subset S^{n-1} \times S^{n-1}$, let $\eta^{-1} := \{(u,x) : (x,u) \in \eta\}$. Then $\Theta_j^s(K^*, \eta) = \Theta_{n-2-j}^s(K, \eta^{-1})$ for Borel sets η. For bodies $K, K' \in \mathcal{K}_s^n$, their spherically convex hull can be defined by $K \vee K' = S^{n-1} \cap \mathrm{pos}\,(K \cup K')$; then $(K \vee K')^* = K^* \cap K'^*$. For sets $\eta, \eta' \subset S^{n-1} \times S^{n-1}$, the sets $\eta \wedge \eta'$ and $\eta \vee \eta'$ are defined similarly as in the Euclidean space. Now let $K, K' \in \mathcal{K}_s^n$ and let $\eta \subset \mathrm{Nor}\, K$ and $\eta' \subset \mathrm{Nor}\, K'$ be Borel sets. Then

$$\int_{SO_n} \Theta_j^s(K \cap \rho K', \eta \wedge \rho \eta') \nu(d\rho) = \sum_{i=j+1}^{n-2} \Theta_i^s(K, \eta) \Theta_{n-1+j-i}^s(K', \eta'),$$

$$\int_{SO_n} \Theta_j^s(K \vee \rho K', \eta \vee \rho \eta') \nu(d\rho) = \sum_{i=0}^{j-1} \Theta_i^s(K, \eta) \Theta_{j-i-1}^s(K', \eta'),$$

for $j = 0, \dots, n-2$. These results are due to Glasauer (1995). The equations are equivalent to each other, by duality. Glasauer's search for Euclidean analogues resulted in the papers Glasauer (1997), (1998), (1999), mentioned above.

The classical kinematic formula (1) for convex bodies involves the intrinsic volumes, which belong to the Brunn-Minkowski theory. There is also a dual Brunn-Minkowski theory, and within this theory there exists a striking analogue of the kinematic formula. This analogy becomes clearer in terms of the quermassintegrals $W_0, \dots, W_n$, which are the intrinsic volumes with a different normalization, given by $\binom{n}{i} W_{n-i} = \kappa_{n-i} V_i$. Equivalent to (2) is the formula

$$W_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \lambda_i(K|L) \nu_i(dL)$$

for $i = 0, \dots, n$, where $K|L$ is the image of K under orthogonal projection on to the subspace L. The kinematic formula (1) now reads

$$\int_{G_n} \chi(K \cap gK') \mu(dg) = \frac{1}{\kappa_n} \sum_{i=0}^{n} \binom{n}{i} W_i(K) W_{n-i}(K'). \tag{15}$$

Let $K \subset \mathbb{E}^n$ be a star body (a compact set containing 0, with continuous radial function). The *dual quermassintegrals* $\tilde{W}_0, \dots, \tilde{W}_n$ are defined by

$$\tilde{W}_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\mathcal{L}_i^n} \lambda_i(K \cap L) \nu_i(dL).$$

For $g \in G_n$, let N_g denote the segment joining 0 and $g0$. Zhang (1999) has proved the kinematic formula

$$\int_{G_n} \chi(K \cap gK' \cap N_g) \mu(dg) = \frac{1}{\kappa_n} \sum_{i=0}^{n} \binom{n}{i} \tilde{W}_i(K) \tilde{W}_{n-i}(K')$$

for star bodies $K, K' \subset \mathbb{E}^n$, which is very similar to (15).

For striking combinatorial analogues of the kinematic formula in the context of finite lattices, see Klain (1997), Klain and Rota (1997), p. 25.

2.5 A Crofton formula for functions

There is a fruitful interplay between convex sets and convex functions, which partly extends to integral geometry. Let $\Omega \subset \mathbb{E}^n$ be open and convex, let $u : \Omega \to \mathbb{R}$ be a convex function, and choose $\epsilon > 0$ and a Borel set $\beta \subset \Omega$. Defining the expansion of β by means of u as

$$P_\epsilon(u, \beta) := \{x + \epsilon v : x \in \beta,\ v \in \partial u(x)\},$$

where $\partial u(x)$ denotes the subdifferential of u at x, we obtain again a Steiner type formula

$$\mathcal{H}^n(P_\epsilon(u,\beta)) = \sum_{k=0}^{n} \epsilon^{n-k} \binom{n}{k} F_k(u,\beta).$$

The measures $F_0(u,\cdot),\dots,F_n(u,\cdot)$ are the Hessian measures of u, which have been studied by Colesanti (1997) and Colesanti and Hug (2000a, 2000b); see also Colesanti and Salani (1997, 1999), Colesanti, Salani and Francini (2000) and Trudinger and Wang (1997, 1999, 2000) for related work. It can be shown that $F_n(u,\beta) = \mathcal{H}^n(\beta)$ and $F_0(u,\beta) = \mathcal{H}^n(\{v \in \mathbb{R}^n : v \in \partial u(x), x \in \beta\})$. For u of class C^2 one obtains

$$\binom{n}{k} F_k(u,\beta) = \int_\beta S_{n-k}(D^2(u(x))\mathcal{H}^n(dx).$$

Here $S_{n-k}(D^2(u(x))$ is the $(n-k)$-th elementary symmetric function of the eigenvalues of the Hessian matrix of u at x.

The Crofton type result for functions involves the Hessian measures $F_j^{(k)}(u|_E,\cdot)$ of the restriction of u to a k-flat $E \subset \mathbb{E}^n$. For $k \in \{1,\dots,n\}$ and $j \in \{0,\dots,k\}$ one gets the formula

$$F_{n+j-k}(u,\beta) = \int_{\mathcal{E}_k^n} F_j^{(k)}(u|_E, \beta \cap E)\mu_k(dE).$$

An application of this Crofton type formula is given in Colesanti and Hug (2000b). Since the curvature and surface area measures of a convex body K can be obtained as Hessian measures of the distance function and the support function of K, respectively, it is conceivable that integral geometric Crofton and projection formulae for the former measures can be deduced from the previous result. This subject will be studied further in Colesanti and Hug (2002).

3 Translative integral geometry

In this section, we mainly consider integral means with respect to the translation group of $\mathbb{E}^n$ (or a linear subspace of $\mathbb{E}^n$) and a corresponding Haar measure. Such translative results are more fundamental than the analogous kinematic results; the latter very often can be deduced from the former by subsequent integrations over the rotation group. Surveys on translative integral geometry are available in Weil (1989, 1990a), it is also treated in the book by Schneider and Weil (1992) and applied in Schneider and Weil (2000). Since the translation group is smaller than the rigid motion group, results of translative integral geometry are more difficult to state, they usually involve mixed functionals of the sets considered. Translative mean value formulae are required for the study, in stochastic geometry, of stationary or even more general random sets and point processes, without isotropy assumptions.

3.1 Convex bodies and curvature measures

Results of translative integral geometry concerning various special situations date back to the late Thirties of the 20th century. A systematic investigation of the subject was initiated by Schneider and Weil (1986), where it was observed that the separation of the translative and the rotational group leads to simpler proofs and additional insights. This line of research was continued by Goodey and Weil (1987) and Weil (1990b).

The two translative formulae which correspond to (1) and (2) are

$$\int_{\mathbb{E}^n} \chi(K \cap (K'+t))\, dt = \sum_{i=0}^{n} \binom{n}{i} V_i(K,-K') \tag{16}$$

and

$$\int_{U^\perp} \chi(K \cap (U + t))\, dt = V_{n-q}(K|U^\perp), \tag{17}$$

where $V_i(K, -K')$ denotes the mixed volume of i copies of K and $n-i$ copies of $-K'$, $U \in \mathcal{L}^n_q$, $U^\perp \in \mathcal{L}^n_{n-q}$ is the subspace orthogonal to U and $K|U^\perp$ is the orthogonal projection of K onto $U^\perp$. In translative formulae, we often write dt for $\lambda_n(dt)$, etc. Relation (16) is essentially the defining relation for mixed volumes. By additivity, both results extend to polyconvex sets, and there are also results for the other intrinsic volumes.

A general translative integral formula, which includes the previous results as very special cases, for support measures of sets with positive reach, was obtained in Rataj and Zähle (1995) and later extended (and corrected) in Rataj and Zähle (2001) to finite unions of sets with positive reach. The technical assumptions which are made in the general case cannot be avoided (cf. Rataj (1997b) and Rataj and Zähle (2000)); they are satisfied for convex sets. On this case, we concentrate in the following. Let $K, K' \subset \mathbb{E}^n$ be closed convex sets, and let $h : (\mathbb{E}^n)^3 \to [0, \infty)$ be measurable, then

$$\begin{aligned} &\int_{\mathbb{E}^n} \int h(x, x - z, u) \Lambda_j(K \cap (K' + z), d(x, u)) dz \\ &= \sum_{\substack{0 \le m_1, m_2 \le n \\ m_1 + m_2 = n + j}} \int h(x, y, u) \Lambda^{(j)}_{m_1, m_2}(K, K'; d(x, y, u)), \end{aligned} \tag{18}$$

where $\Lambda^{(j)}_{m_1,m_2}(K, K'; \cdot)$ are certain mixed measures of K and K' which are determined by (18). For convex sets, this result was proved in Schneider and Weil (1986) for functions which are independent of u, and also the mixed measures $\Phi^{(j)}_{m_1,m_2}(K, K'; \cdot) := \Lambda^{(j)}_{m_1,m_2}(K, K'; \cdot \times \mathbb{E}^n)$ on $(\mathbb{E}^n)^2$ have been introduced there. In this case, one can first establish the result for polytopes and then apply a continuity argument. In the general case of equation (18), one considers the normal cycles of the sets and uses repeatedly the area/coarea formula as well as slicing of currents. An auxiliary result implicitly required in Rataj and Zähle (1995) was later provided in Zähle (1999). The measure geometric approach is required for the study of sets with positive reach, and it leads to explicit expressions for the mixed measures in terms of generalized curvature functions which are defined on the normal bundles of the sets. As an application of this approach one obtains, e.g., an alternative proof of relation (8).

More recently, for convex sets a more elementary approach to an extension of (18) in a different direction was discovered in Kiderlen (1999) and Hug (1999). First, for convex bodies $K, B \in \mathcal{K}^n$ in general relative position and with $0 \in B$, one introduces relative support measures $\Lambda_j(K; B; \cdot)$ as measures on $(\mathbb{E}^n)^2$ (see Schneider (1994), Kiderlen and Weil (1999), Hug and Last (2000)). If K, B are polytopes, these measures can be described in an explicit way, and from this a version of (18) for relative support measures can be deduced in the case of polytopes. By an approximation argument the general case follows. The special choice $B = B^n$ then yields (18), since $\Lambda_j(K; B^n; \cdot) = \Lambda_j(K, \cdot)$.

Motivated by applications in stochastic geometry, an iterated version of (18) for curvature measures has first been obtained in Weil (1990b). More generally, if $h : (\mathbb{E}^n)^{k+1} \to [0, \infty)$ is measurable, $K_1, \ldots, K_k \subset \mathbb{E}^n$ are closed and convex, $k \ge 2$ and $j \in \{0, \ldots, n-1\}$, then

$$\int_{\mathbb{E}^n} \ldots \int_{\mathbb{E}^n} \int h(x, x - z_2, \ldots, x - z_k, u) \Lambda_j(K_1 \cap (K_2 + z_2) \cap \ldots \cap (K_k + z_k), d(x, u))$$
$$dz_2 \ldots dz_k$$

$$= \sum_{\substack{0 \le m_1, \ldots, m_k \le n \\ m_1 + \ldots + m_k = (k-1)n + j}} \int h(x_1, \ldots, x_k, u) \Lambda^{(j)}_{m_1, \ldots, m_k}(K_1, \ldots, K_k; d(x_1, \ldots, x_k, u)), \tag{19}$$

where each of the measures $\Lambda^{(j)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k; \cdot)$ is a mixed (support) measure of $K_1, \ldots, K_k$ on $(\mathbb{E}^n)^{k+1}$. The corresponding total measures with respect to the last component are defined by $\Phi^{(j)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k; \cdot) := \Lambda^{(j)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k; \cdot \times \mathbb{E}^n)$ as measures on $(\mathbb{E}^n)^k$ and are called mixed curvature measures of $K_1, \ldots, K_k$. Relation (19) indicates that in contrast to the case of kinematic formulae, an increasing number of mixed measures is required if the number of intersections increases. A more general version of the iterated formula (19) for relative support measures is proved in Hug (1999), which implies the Euclidean case. A corresponding result for sets with positive reach and (Euclidean) support measures has been obtained in Rataj (1997a), where the mixed measures are expressed as currents evaluated for specially chosen differential forms. For convex sets, more explicit expressions for these mixed measures have been found in Hug (1999). From these expressions, all formulae which were obtained in Goodey and Weil (2002b) for special mixed measures can be derived immediately. This subject will be studied further in Hug and Rataj (2002). For polytopes $P_1, \ldots, P_k$, explicit summation formulae for the total mixed measures $V^{(0)}_{m_1,\ldots,m_k}(P_1, \ldots, P_k) := \Phi^{(0)}_{m_1,\ldots,m_k}(P_1, \ldots, P_k; (\mathbb{E}^n)^k)$ have been found by Schneider (2000b).

Crofton formulae for curvature measures are implied by corresponding kinematic formulae. The same is true for translative results (compare Schneider and Weil (1986)). However, as soon as support measures are considered, the situation becomes much more difficult. In Rataj (1999), a translative Crofton formula for support measures is proved. But even a kinematic formula for support measures cannot be expressed in a simple way, in the general case. Despite these technical difficulties such a result has several interesting consequences. For instance, one can retrieve a translative Crofton formula for surface area measures due to Goodey, Kiderlen and Weil (1998), or kinematic Crofton formulae for curvature measures of sets with positive reach.

For convex sets the support measures are nonnegative, for sets with positive reach these measures may be signed. This has led to the definition of nonnegative absolute curvature measures, which can be interpreted as measures of sets of flats (of a particular dimension) touching a given set in a prescribed set of support elements. We refer to Rother and Zähle (1992) and to recent work of Rataj (2000, 2001), where translative and kinematic Crofton formulae are established for (certain unions of) sets with positive reach.

A systematic study of translative integral and Crofton formulae for various mixed measures and functionals of translative integral geometry is carried out in Weil (2001). We describe some of these results.

The starting point in Weil (2001) is the formula

$$\int_{\mathbb{E}^n} \ldots \int_{\mathbb{E}^n} \Phi_j(K_1 \cap (K_2 + z_2) \cap \ldots \cap (K_k + z_k), A_1 \cap (A_2 + z_2) \cap \ldots \cap (A_k + z_k))$$
$$dz_2 \ldots dz_k$$
$$= \sum_{\substack{0 \le m_1, \ldots, m_k \le n \\ m_1 + \ldots + m_k = (k-1)n + j}} \Phi^{(j)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k; A_1 \times \ldots \times A_k), \tag{20}$$

which is a special case of (19); here $K_1, \ldots, K_k \subset \mathbb{E}^n$ are closed convex sets, $A_1, \ldots, A_k \subset \mathbb{E}^n$ are bounded Borel sets, $k \ge 2$, $j \in \{0, \ldots, n-1\}$ and $m_1, \ldots, m_k \in \{j, \ldots, n\}$.

From this result it is easy to deduce that the mixed curvature measures also satisfy translative integral formulae. In order to deduce kinematic formulae for mixed curvature measures, one can apply the rotation formula

$$\int_{SO_n} \Phi^{(j)}_{m_1,\dots,m_{k-1},m_k}(K_1,\dots,K_{k-1},\rho K_k;A_1\times\dots\times A_{k-1}\times\rho A_k)\nu(d\rho)$$
$$= \alpha_{njm_k}\Phi^{(n+j-m_k)}_{m_1,\dots,m_{k-1}}(K_1,\dots,K_{k-1};A_1,\dots,A_{k-1})\Phi_m(K_k,A_k). \tag{21}$$

To deduce this, one replaces K_k in (20) by ρK_k, applies Fubini's theorem and then the kinematic formula (4) to $K_1\cap(K_2+z_2)\cap\dots\cap(K_{k-1}+z_{k-1})$ and K_k, and finally one applies again (20) with $k-1$ convex bodies.

Next we describe a translative Crofton formula for mixed curvature measures. Let $k\geq 1$, $L\in\mathcal{L}^n_q$, $q\in\{m_k,\dots,n-1\}$ and let $B_L\subset L$ denote a measurable set with $\mathcal{H}^q(B_L)=1$, then

$$\int_{L^\perp} \Phi^{(j)}_{m_1,\dots,m_{k-1},m_k}(K_1,\dots,K_{k-1},K_k\cap(L+x);A_1\times\dots\times A_{k-1}\times(A_k\cap(L+x)))$$
$$\mathcal{H}^{n-q}(dx)$$
$$= \Phi^{(j)}_{m_1,\dots,m_{k-1},n+m_k-q,q}(K_1,\dots,K_{k-1},K_k,L;A_1\times\dots\times A_{k-1}\times A_k\times B_L). \tag{22}$$

A concise proof proceeds as follows. First, as an easy consequence of a translative integral formula for mixed curvature measures, we have

$$\int_{\mathbb{E}^n} \Phi^{(j)}_{m_1,\dots,m_{k-1},m_k}(K_1,\dots,K_{k-1},K_k\cap(L+x);A_1\times\dots\times A_{k-1}\times(A_k\cap(B_L+x)))dx$$
$$= \Phi^{(j)}_{m_1,\dots,m_{k-1},n+m-q,q}(K_1,\dots,K_k,L;A_1\times\dots\times A_k\times B_L).$$

The left-hand side can be simplified by identifying $\mathbb{E}^n$ with $L^\perp\times L$ and by applying Fubini's theorem several times.

From (21) one can derive a kinematic Crofton formula. Weil (2001) provides a variety of other useful results including formulae for intersections with halfspaces, formulae for special mixed volumes and special representations for mixed measures and functionals of generalized zonoids. Formulas for projection functions turn out to be special cases of formulae for particular mixed volumes involving two convex bodies, and therefore they can be deduced from integral formulae for total mixed measures of two convex bodies. Connections to Radon transforms on Grassmannians are investigated in Goodey, Schneider and Weil (1995).

We mention that translative integral formulae for nonintersecting sets involving relative support measures and generalized distance functions have recently been found, which extend various previous formulae and which can be used to derive results concerning contact distributions in stochastic geometry; see Hug, Last and Weil (2002a, 2002b).

In Section 2.2, we have quoted Hadwiger's general integral geometric theorem, which provides a kinematic formula for arbitrary additive continuous functions on the space of convex bodies. For integrations over the translation group, an analogous result can be proved for simply additive functions. Let φ be a continuous real function on the space $\mathcal{K}^n$ of convex bodies in $\mathbb{E}^n$ which is a simple valuation, that is, additive and satisfying $\varphi(K)=0$ for convex bodies of dimension less than n. Then

$$\int_{\mathbb{E}^n}\varphi(K\cap(M+x))\,dx = \varphi(K)V_n(M)+\int_{S^{n-1}} f_{K,\varphi}(u)S_{n-1}(M,du)$$

for convex bodies $K,M\in\mathcal{K}^n$, where the function $f_{K,\varphi}:S^{n-1}\to\mathbb{R}$ is given by

$$f_{K,\varphi}(u)=\int_{-h(K,-u)}^{h(K,u)}\varphi(K\cap H^-(u,\tau))\,d\tau-\varphi(K)h(K,u);$$

here $H^-(u,\tau)$ is the closed halfspace $\{x \in \mathbb{E}^n : \langle x, u\rangle \leq \tau\}$. The measure $S_{n-1}(M,\cdot)$ is the image measure of $\Lambda_{n-1}(M,\cdot)$ under the projection $(x,u) \mapsto u$ and is called the surface area measure of M. This formula was recently proved by the second author. The proof uses the characterization theorem obtained in Schneider (1996).

3.2 Support functions

An important and basic tool in convexity is the support function. For a given convex body $K \in \mathcal{K}^n$, it is defined by $h(K,x) := \max\{\langle x,y\rangle : y \in K\}$, $x \in \mathbb{E}^n$. The support function is related to mixed volumes and mixed functionals of translative integral geometry via

$$V^{(0)}_{1,n-1}(K,-L) = nV_1(K,L) = \int_{S^{n-1}} h(K,u) S_{n-1}(L,du),$$

where $K, L \in \mathcal{K}^n$ are convex bodies. In the following, it is convenient to consider the centred support function $h^*(K,\cdot) = h(K - s(K),\cdot)$, where $s(K)$ is the Steiner point of K (a particular vector-valued valuation). In Weil (1995), these definitions and relationships are used to derive, for convex bodies $K, K' \in \mathcal{K}^n$, the translative integral formula

$$\int_{\mathbb{E}^n} h^*(K \cap (K'+x),\cdot)\, dx = h^*(K,\cdot)V_n(K') + \sum_{j=2}^{n-1} h^*_j(K,K',\cdot) + V_n(K)h^*(K',\cdot) \tag{23}$$

with certain functions $h^*_j(K,K',\cdot)$, of which a number of properties are established, including an explicit representation in the case of polytopes. A simpler approach to this result was recently discovered in Schneider (2002a). There a quick proof is given for Weil's formula

$$h^*(K,u) = \Phi^{(0)}_{1,n-1}(K,u^+;\mathbb{E}^n \times \beta(u)), \tag{24}$$

where $u \in S^{n-1}$, $\beta(u) \subset u^\perp$ is a Borel set with $\mathcal{H}^{n-1}(\beta(u)) = 1$, and $u^+ := \{x \in \mathbb{E}^n : \langle x,u\rangle \geq 0\}$. Using (24) and known translative integral formulae for mixed curvature measures, (23) can easily be deduced, and thus one also obtains

$$h^*_j(K,K',u) = \Phi^{(0)}_{j,n+1-j,n-1}(K,K',u^+;\mathbb{E}^n \times \mathbb{E}^n \times \beta(u)).$$

Hadwiger's general integral geometric theorem, applied with an arbitrary but fixed argument of the support functions, gives

$$\int_{G_n} h^*(K \cap gK',\cdot)\mu(dg) = \sum_{j=1}^{n} h^*_j(K,\cdot)V_{n+1-j}(K'),$$

where

$$h^*_j(K,\cdot) = \int_{\mathcal{E}^n_{n+1-j}} h^*(K \cap E,\cdot)\mu_{n+1-j}(dE);$$

see Weil (1995) for a different approach to this kinematic formula for support functions. By definition, the right-hand side of the preceding equation is the support function of the mean section body $M_{n+1-j}(K)$ of K, which (up to a translation) was previously studied in Goodey and Weil (1992) and Goodey (1998). Up to a constant factor, $h(M_{n+1-j}(K),u)$ is equal to $\Phi^{(j-1)}_{j,n-1}(K,u^+;\mathbb{E}^n \times \beta(u))$, which follows from (24) and from a very special case of a kinematic formula for mixed curvature measures (Corollary 4.4 in Weil (2001)).

The left-hand side of equation (23) defines a support function, and the summands on the right-hand side have different degrees of homogeneity. This suggests that the mixed functions

$h_j^*(K, K', \cdot)$ are support functions as well. A first, rather indirect proof of this fact was given in Goodey and Weil (2002a). Subsequently, Schneider (2002a) found a simpler argument which is based on a new representation of these mixed functionals in the polytopal case. There also iterated formulae and corresponding mixed support functions are studied.

For applications of some of these results to stochastic geometry see Weil (1994, 1999).

3.3 Convex surfaces and Euler characteristic

A class of sets which has not been considered so far are convex surfaces, that is boundaries of n-dimensional convex bodies. Let $K, L \subset \mathbb{E}^n$ be convex bodies with nonempty interiors, and let $\partial K, \partial L$ denote their boundaries. According to a conjecture by Firey (see Problem 18 in the collection of Gruber and Schneider (1979)), two kinematic formulae, involving intersections of two convex surfaces,

$$\int_{G_n} \chi(\partial K \cap g\partial L)\,\mu(dg) = \frac{1+(-1)^n}{\kappa_n} \sum_{k=0}^{n-1} \binom{n}{k} (1-(-1)^k) W_{n-k}(K) W_k(L), \tag{25}$$

and intersections of convex surfaces and convex bodies

$$\int_{G_n} \chi(\partial K \cap gL)\,\mu(dg) = \frac{1}{\kappa_n} \sum_{k=0}^{n-1} \binom{n}{k} (1-(-1)^{n-k}) W_{n-k}(K) W_k(L), \tag{26}$$

should be satisfied. In fact, for polytopes these formulae can easily be verified. However, it is not clear at all that the left-hand sides of these equations depend continuously on the bodies involved. In Hug and Schätzle (2001), Firey's conjecture was confirmed by the following more general translative versions of the integral formulae (25) and (26), that is by proving

$$\int_{\mathbb{E}^n} \chi(\partial K \cap (\partial L + t))\,dt = (1+(-1)^n) \sum_{i=0}^{n} \binom{n}{i} \left\{ V_i(K, -L) + (-1)^{i-1} V_i(K, L) \right\}$$

and

$$\int_{\mathbb{E}^n} \chi(\partial K \cap (L + t))\,dt = \sum_{i=0}^{n} \binom{n}{i} \left\{ V_i(K, -L) + (-1)^{n-i-1} V_i(K, L) \right\}.$$

In fact, Firey's original question was answered implicitly by a result in Fu (1994), which does not cover the translative case. A main idea for the proof of these integral formulae, which was already suggested by Firey, is to establish the desired result first for the boundaries of parallel sets K_r, L_r at distance $r > 0$ of K, L. In this case, ∂K_r and the intersections $\partial K_r \cap (\partial L_r + t)$ are sets with positive reach, at least for $\mathcal{H}^n$ almost all translations $t \in \mathbb{E}^n$. Since $\chi(X) = \Phi_0(X, \mathbb{E}^n)$ is satisfied for any compact set X with positive reach, the translative integral formula for support measures, due to Rataj and Zähle (1995), can be applied to deduce the desired result for the parallel bodies K_r, L_r and $r > 0$. To complete the argument one then shows that the Euler characteristic of the intersection $\chi(\partial K_r \cap \partial L_r)$ is independent of $r \geq 0$, if K and L intersect almost transversally. The latter result is related to another conjecture by Firey. Intersections of convex surfaces and bodies are treated similarly.

In Hug, Mani-Levitska and Schätzle (2002), these integral geometric results have been extended further, to lower-dimensional sets by Fubini-type arguments. Furthermore, iterated formulae are established concerning intersections of several convex bodies, which then are applied to obtain formulae of stochastic geometry. Defining intrinsic volumes for intersections of convex surfaces in a suitable way by a Crofton type expression, one can also derive integral formulae for such functionals.

4 Minkowski spaces and projective Finsler spaces

The kinematic formulae (6) and Crofton formulae (7) do not involve curvature measures, but only Hausdorff measures and thus, for sufficiently regular sets, Euclidean areas of different dimensions. Areas are easier to generalize than curvatures, and one may ask for corresponding generalizations of Crofton formulae. A vague question, for the moment in unspecified terms, can be posed as follows. Let vol_k be some notion of k-dimensional area, defined for a class of subsets of $\mathbb{R}^n$ ($k = 0, \dots, n$; with vol_0 = number of elements, possibly ∞). Are there (possibly translation invariant) measures (or signed measures) η_j on the affine Grassmannians $\mathcal{E}_j^n$ such that

$$\int_{\mathcal{E}_{n-k}^n} \mathrm{card}\,(M \cap E)\eta_{n-k}(dE) = a_{nk}\mathrm{vol}_k(M) \tag{27}$$

or, more generally,

$$\int_{\mathcal{E}_j^n} \mathrm{vol}_{k+j-n}(M \cap E)\eta_j(dE) = a_{nkj}\mathrm{vol}_k(M) \tag{28}$$

($j \in \{n-k, \dots, n-1\}$), with constants a_{nk}, a_{nkj}, holds for all sets M of the given class? Such questions were first posed, for axiomatically defined affine areas, by Busemann (1960); see Schneider and Wieacker (1997), Section 3, for a slight extension of a criterion derived by Busemann. In the following, we report briefly on some work that has been done in this direction.

A first natural case to consider is that of a Minkowski space $(\mathbb{R}^n, \|\cdot\|)$; here $\|\cdot\|$ is a norm on $\mathbb{R}^n$. This norm induces a curve length, vol_1, for rectifiable curves. Hence, it makes sense to ask whether (27) can be satisfied for $k = 1$ and rectifiable curves M. Since the isometry group of $(\mathbb{R}^n, \|\cdot\|)$ contains the translation group, it is natural to require that (27) should hold with a translation invariant measure η_{n-1}. It is known (see Schneider and Wieacker (1997) and the references given there) that such a measure exists if and only if the Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ is hypermetric, that is, its induced metric is a hypermetric. A metric d on a set S is called a hypermetric if $\sum_{i,j=1}^k d(p_i, p_j)N_iN_j \le 0$ whenever $k \ge 2$, $p_1, \dots, p_k \in S$ and $N_1, \dots, N_k$ are integers satisfying $\sum_{i=1}^k N_i = 1$. The Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ is hypermetric if and only if its dual unit ball is a zonoid. Similarly one obtains that (27) can be satisfied with a translation invariant signed measure η_{n-1} if and only if the dual unit ball is a generalized zonoid. In the space of centrally symmetric convex bodies in $\mathbb{R}^n$, the zonoids are nowhere dense, whereas the generalized zonoids are dense. The ℓ_∞^n-norm is an example of a norm for which the dual unit ball is not a generalized zonoid.

Minkowski spaces are special cases of projective Finsler spaces. A Finsler metric on $\mathbb{R}^n$ is a continuous function $F : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ with the property that $F(x, \cdot) =: \|\cdot\|_x$ is a norm on $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. The pair $(\mathbb{R}^n, F)$ is then called a Finsler space, and it is called smooth if F is of class C^∞ on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. (We can omit here the additional assumption, made in the differential geometry of Finsler spaces, that the local unit spheres of F are quadratically convex.) Let $(\mathbb{R}^n, F)$ be a Finsler space. The Finsler length of a parameterized piecewise C^1 curve $\gamma : [a, b] \to \mathbb{R}^n$ is defined by $\int_a^b F(\gamma(t), \gamma'(t))\, dt$, and the induced metric d_F assigns to $p, q \in \mathbb{R}^n$ the infimum, $d_F(p, q)$, of the lengths of the piecewise C^1 curves connecting p and q. The Finsler space $(\mathbb{R}^n, F)$ is *projective* if its Finsler metric and the affine structure of $\mathbb{R}^n$ are compatible in the sense that line segments are shortest curves connecting their endpoints.

In a Finsler space, there is a canonical notion of curve length, but no canonical notion of higher dimensional area. Two notions of area have found particular interest, the Busemann area and the Holmes-Thompson area. For the case of Minkowski spaces (where $F(x, \cdot)$ is independent of x), they are thoroughly discussed in the book of Thompson (1996). In the Finsler space $(\mathbb{R}^n, F)$, let $\mathcal{H}_F^k$ denote the k-dimensional Hausdorff measure induced by the metric d_F. For a k-rectifiable Borel set $M \subset \mathbb{R}^n$, the value $\mathcal{H}_F^k(M)$ gives a natural notion of k-area. It is called the *Busemann area*. The *Holmes-Thompson area* was first introduced for Minkowski spaces, and occurred later as a symplectic volume in Finsler spaces. For the latter definition see, e.g., Álvarez and Fernandes (1998). We give here two convenient representations of these areas in terms of an auxiliary Euclidean structure on $\mathbb{R}^n$, given by a scalar product $\langle\cdot,\cdot\rangle$. Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure with respect to the induced Euclidean metric, and let $(\cdot)|L$ denote the orthogonal projection on a linear subspace L. One defines the *indicatrix* of the Finsler metric F at $x \in \mathbb{R}^n$ by $B_x := \{u \in \mathbb{R}^n : \|u\|_x \le 1\}$, and the *figuratrix* by $B_x^o := \{v \in \mathbb{R}^n : \langle u, v\rangle \le 1 \text{ for all } u \in B_x\}$. Then the Busemann area of a $(\mathcal{H}^k, k)$-rectifiable Borel subset M of $\mathbb{R}^n$ is given by

$$\mathcal{H}_F^k(M) = \kappa_k \int_M \frac{1}{\mathcal{H}^k(B_x \cap T_x M)} \mathcal{H}^k(dx).$$

Here $T_x M$ is the approximate tangent space of M at x, which exists as a k-dimensional linear subspace of $\mathbb{R}^n$, for $\mathcal{H}^k$-almost all $x \in M$. The Holmes-Thompson area of M, which we denote by $\mathrm{vol}_k(M)$, is given by

$$\mathrm{vol}_k(M) = \frac{1}{\kappa_k} \int_M \mathcal{H}^k(B_x^o | T_x M) \mathcal{H}^k(dx).$$

Proofs can be found in Schneider (2002b).

Now the stage is set for the consideration of Crofton formulae. It turns out that for integral geometry the Holmes-Thompson area vol_k is the right area notion to use. First we turn to Minkowski spaces again. It was observed, with different degrees of generality, by Busemann (1960), El-Ekhtiar (1992), Schneider and Wieacker (1997) that for vol_{n-1} in a Minkowski space there always exists a translation invariant (positive) measure η_1 on $\mathcal{E}_1^n$ so that (27) holds. On the other hand, for (27) to hold for $k = 1$ with a translation invariant measure η_{n-1}, it is necessary that the Minkowski space be hypermetric. If this assumption is satisfied, then there are translation invariant measures η_j on $\mathcal{E}_j^n$ so that (28) holds for all $k \in \{1, \dots, n\}$ and for k-rectifiable Borel sets M. This was proved by Schneider and Wieacker (1997), Theorem 7.3. In hypermetric Minkowski spaces, there also exist satisfactory analogues of the Euclidean quermassintegrals of convex bodies, see Schneider (1997b). For general (not necessarily smooth) hypermetric projective Finsler spaces, the existence of measures η_j so that (27) holds at least for k-dimensional compact convex sets M was established in Schneider (2001b). For $k = 1$, the assumption 'hypermetric' can be deleted. The proofs make use of techniques developed by Pogorelov (1974) in his treatment of Hilbert's fourth problem.

The special role played here by the Holmes-Thompson area can be made more explicit. One can define a general notion of Minkowskian $(n-1)$-area by a few natural axioms. It was shown by Schneider (1997a) that there exist Minkowski spaces for which, among all Minkowskian $(n-1)$-areas, only the Holmes-Thompson area and its multiples allow a Crofton formula (27) for $k = 1$ with a translation invariant measure η_1. For the Busemann area, the picture is diffuse. Let us say that for a Minkowski space $X = (\mathbb{R}^n, \|\cdot\|)$ the Busemann area is *integral-geometric* if (27) holds for X and for the Busemann $(n-1)$-area with a translation invariant measure η_1, and at least for all $(n-1)$-dimensional compact convex sets M. The following was shown by Schneider (2001a), for $n \ge 3$. Every neighbourhood (in the sense of the Banach-Mazur distance) of the Euclidean space ℓ_2^n contains Minkowski spaces for which the Busemann area is not integral-geometric, as well as spaces (different from ℓ_2^n) for which the Busemann area is integral-geometric. If n is sufficiently large, then a full neighbourhood of the Minkowski space ℓ_∞^n consists of Minkowski spaces for which the Busemann area is not integral-geometric. We conjecture that it is generically true (that is, for a dense open subset of the space of all n-dimensional Minkowski spaces) that the Busemann area is not integral-geometric.

In the smooth category, general investigations on Crofton densities have been undertaken by Gelfand and Smirnov (1994), Álvarez, Gelfand and Smirnov (1997), in part related to Hilbert's fourth problem and to symplectic geometry. Subsequent work by Álvarez and Fernandes (1998, 1999, 2000) and the thesis of Fernandes (2002) use double fibrations and the Gelfand transform as a unifying approach to integral-geometric intersection formulae and obtain, in particular, Crofton type formulae in smooth projective Finsler spaces. In such spaces, for smooth submanifolds and the Holmes-Thompson area, formula (27) and formula (28) for $k = n-1$ are obtained in Álvarez and Fernandes (1998) and in Fernandes (2002), with signed measures η_j (defined by smooth densities). The first of these papers makes use of the symplectic structure on the space of geodesics of a projective Finsler space.

It has turned out that the methods applied by Schneider and Wieacker (1997) for the case of hypermetric Minkowski spaces (where they yield measures η_j) can be adapted to the case of smooth projective Finsler spaces (where they yield signed measures). In this way, the up to now most general version of the Crofton formula (28) was obtained, namely for $k = 1, \ldots, n$, $j = n-k, \ldots, n-1$ and for Holmes-Thompson areas of $(\mathcal{H}^k, k)$-rectifiable Borel sets M in a smooth projective Finsler space (where the local unit spheres need not be quadratically convex).

We close with a remark on the extendability of the kinematic formula (6) to a Minkowski space $X = (\mathbb{R}^n, \|\cdot\|)$. Here the motion group G_n has to be replaced by the translation group. Such a formula, however, would not even hold in Euclidean space. It does hold, in a Euclidean space, in the special case where K' is a sphere. In analogy, it makes sense to ask whether in the Minkowski space X there exist a convex body K and a constant $c_{nk}(K)$ so that

$$\int_{\mathbb{R}^n} \mathrm{vol}_{k-1}(M \cap (\partial K + t))\, dt = c_{nk}(K)\mathrm{vol}_k(M) \tag{29}$$

holds for all k-dimensional C^1 submanifolds M of $\mathbb{R}^n$, $k \in \{1, \ldots, n-1\}$. The following was proved in Schneider (1997a). A convex body K satisfying (29) for $k = 1$ exists only if X is hypermetric. If X is hypermetric, then there exists a convex body K so that (29) holds for all $k \in \{1, \ldots, n-1\}$.

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