A formula for mixed volumes

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Abstract

An identity for mixed volumes, discovered by Gusev and Esterov, which involves together with some convex bodies also the convex hull of their union, is given a new proof, using only the classical approach to mixed volumes.

In the introduction to his paper [2], Esterov writes (with slightly different notation) the following: "Counting Euler characteristics of the discriminant of the quadratic equation in terms of Newton polytopes in two different ways, G. Gusev [3] found an unexpected relation for mixed volumes of two polytopes P_1 and P_2 in $\mathbb{R}^n$ and the convex hull P of their union. For instance, assuming $n = 2$ and denoting the mixed area of polygons A, B by $V(A, B)$, this relation specializes to

$$V(P, P) - V(P, P_1) - V(P, P_2) + V(P_1, P_2) = 0.$$

We call it unexpected because it is not a priori invariant under parallel translations of P_1."

Esterov [2] proved a multidimensional generalization of this equality. To formulate it, we denote the support function of a convex body $K \subset \mathbb{R}^n$ by h_K and write the mixed volume $V(K_1, \dots, K_n)$ of the convex bodies $K_1, \dots, K_n$ in the form $V(h_{K_1}, \dots, h_{K_n})$, that is, we consider it equivalently as a functional of support functions. By linearity in each argument, the mixed volume can then be extended to differences of support functions. This extension appears already in the work of Aleksandrov [1], §6; see also [4], Section 5.2. Esterov's extension of Gusev's formula (in a slightly more general form, which is needed for the inductive proof below) then reads as follows. Here $\mathcal{K}^n$ denotes the set of convex bodies (nonempty, compact, convex subsets) of $\mathbb{R}^n$.

Theorem. *Let $2 \le k \le n$, let $A_1, \dots, A_k, B_1, \dots, B_{n-k}, C_1, \dots, C_{n-k} \in \mathcal{K}^n$, write*

$$A := \mathrm{conv}(A_1 \cup \dots \cup A_k),$$

and abbreviate the $(n-k)$-tuple $(h_{B_1} - h_{C_1}, \dots, h_{B_{n-k}} - h_{C_{n-k}})$ by $\mathcal{H}_{n-k}$. Then

$$V(h_A - h_{A_1}, \dots, h_A - h_{A_k}, \mathcal{H}_{n-k}) = 0. \tag{1}$$

Note that

$$h_A = \max\{h_{A_1}, \dots, h_{A_k}\},$$

hence, at each point of the unit sphere, at least one the functions $h_A - h_{A_1}, \dots, h_A - h_{A_k}$ appearing in (1) vanishes. This observation may serve as an intuitive hint to the validity of (1), but it is, of course, not a proof.

Using multilinearity, equation (1) can be written as an identity involving mixed volumes of the bodies $A_1, \dots, A_k$ and the convex hull of their union, and of the bodies B_i, C_i. By the symmetry of the mixed volume, the special role played by the first k arguments in (1) can be taken over by any k arguments.

In addition to Esterov's dictum of being 'unexpected', this identity may well be called a surprise, since it deals with the well-established topic of mixed volumes, but within this classical theory it had not been noticed before. In the following, we give a proof of the identity (1) that, other than the proofs of Gusev and Esterov, stays entirely within the classical theory.

This proof makes use of the representation of the mixed volume of $K_1, \dots, K_n \in \mathcal{K}^n$ by

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{K_1} \, \mathrm{d}S(K_2, \dots, K_n, \cdot), \tag{2}$$

where $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$ and $S(K_2, \dots, K_n, \cdot)$ denotes the mixed area measure of the bodies $K_2, \dots, K_n$ (see, e.g. [4], Section 5.1). By linearity in each argument, also the mixed area measures and formula (2) extend to differences of support functions (also this extension appears already in [1]).

The following lemma expresses in a precise way how the mixed area measures are 'determined locally'.

Lemma. *Let $K, L, K_2, \dots, K_{n-1} \in \mathcal{K}^n$, let $\omega \subset \mathbb{S}^{n-1}$ be an open set. If*

$$h_K(u) = h_L(u) \quad \text{for all } u \in \omega,$$

then

$$S(K, K_2, \dots, K_{n-1}, \omega') = S(L, K_2, \dots, K_{n-1}, \omega')$$

for all Borel sets $\omega' \subset \omega$.

Proof. Let $u \in \omega$. Since the functions h_K and h_L coincide in a neighbourhood of u, their directional derivatives at u also coincide. Therefore, Theorem 1.7.2 and the second displayed formula on p. 88 of [4] show that $\tau(K, \omega) = \tau(L, \omega)$, where $\tau(K, \omega)$ denotes the reverse spherical image of K at ω. This implies the equality $S_{n-1}(K, \cdot) \llcorner \omega = S_{n-1}(L, \cdot) \llcorner \omega$ for the restrictions of the surface area measures S_{n-1} to the set ω; see [4], p. 215. In this equation, K and L may be replaced (with the same proof) by $K + \lambda_2 K_2 + \dots + \lambda_{n-1} K_{n-1}$ and $L + \lambda_2 K_2 + \dots + \lambda_{n-1} K_{n-1}$, respectively, with any fixed $\lambda_2, \dots, \lambda_{n-1} \geq 0$. Now the assertion of the lemma follows from [4], formula (5.21). $\square$

In the lemma, we can again use multilinearity to replace $K_2, \dots, K_{n-1}$ by differences of support functions. Then the lemma immediately yields the case $k = 2$ of (1). For this, we put

$$\begin{aligned} \omega_1 &:= \{u \in \mathbb{S}^{n-1} : h_{A_1}(u) \geq h_{A_2}(u)\}, \\ \omega_2 &:= \{u \in \mathbb{S}^{n-1} : h_{A_1}(u) < h_{A_2}(u)\}. \end{aligned}$$

Then $h_A(u) = h_{A_1}(u)$ for $u \in \omega_1$ and $h_A(u) = h_{A_2}(u)$ for $u \in \omega_2$. Since ω_2 is open, the lemma gives

$$S(h_A - h_{A_2}, \mathcal{H}_{n-2}, \cdot) \llcorner \omega_2 = 0.$$

It follows that

$$\begin{aligned}
& nV(h_A - h_{A_1}, h_A - h_{A_2}, \mathcal{H}_{n-2}) \\
& = \int_{\mathbb{S}^{n-1}} (h_A - h_{A_1})\, \mathrm{d}S(h_A - h_{A_2}, \mathcal{H}_{n-2}, \cdot) \\
& = \int_{\omega_1} (h_A - h_{A_1})\, \mathrm{d}S(h_A - h_{A_2}, \mathcal{H}_{n-2}, \cdot) + \int_{\omega_2} (h_A - h_{A_1})\, \mathrm{d}S(h_A - h_{A_2}, \mathcal{H}_{n-2}, \cdot) \\
& = 0.
\end{aligned}$$

This is formula (1) for $k = 2$.

The general case of (1) is now proved by induction on k. Let $3 \leq k \leq n$ and suppose the assertion has been proved for the convex hull of less than k convex bodies. We abbreviate

$$h^{(m)} := h_{\mathrm{conv}(A_1 \cup \cdots \cup A_m)}.$$

Then the identity

$$\begin{aligned}
& V(h^{(k)} - h_1, \dots, h^{(k)} - h_k, \mathcal{H}_{n-k}) \\
& = \sum_{j=0}^{k-2} V(h^{(k-1)} - h_1, \dots, h^{(k-1)} - h_j, h^{(k)} - h^{(k-1)}, h^{(k)} - h_{j+2}, \dots, h^{(k)} - h_k, \mathcal{H}_{n-k}) \\
& \quad + V(h^{(k-1)} - h_1, \dots, h^{(k-1)} - h_{k-1}, h^{(k)} - h_k, \mathcal{H}_{n-k}) \qquad (3)
\end{aligned}$$

holds. For the proof, we note that the summand with $j = k - 2$, which is

$$V(h^{(k-1)} - h_1, \dots, h^{(k-1)} - h_{k-2}, h^{(k)} - h^{(k-1)}, h^{(k)} - h_k, \mathcal{H}_{n-k}),$$

and the last term (3) add up to

$$V(h^{(k-1)} - h_1, \dots, h^{(k-1)} - h_{k-2}, h^{(k)} - h_{k-1}, h^{(k)} - h_k, \mathcal{H}_{n-k}).$$

Continuing in this way, we obtain the identity.

Each term in the sum vanishes, by the induction hypothesis for two convex bodies, using the differences $h^{(k)} - h^{(k-1)}$ and $h^{(k)} - h_k$ and observing that

$$\mathrm{conv}(\mathrm{conv}(A_1 \cup \cdots \cup A_{k-1}) \cup A_k) = \mathrm{conv}(A_1 \cup \cdots \cup A_k).$$

The last term (3) vanishes by the induction hypothesis for $k - 1$ convex bodies.

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References

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