Convexity in Stochastic Geometry

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Stochastic Geometry studies randomly generated geometric objects. In recent decades, this field has developed considerably, mainly due to its applications in various sciences, where two and three dimensions are dominant, but also due to its inherent mathematical interest. In these lectures, I want to introduce and study two basic and frequently employed models of stochastic geometry in d-dimensional space, which deserve particular interest also from a purely mathematical point of view. The first of these is the so-called Boolean model, providing a class of random closed sets which are well accessible to mathematical investigation. The second one are special random mosaics – tessellations of space generated by random hyperplanes or by the Dirichlet-Voronoi cells of a point process. The underlying probabilistic object in each case is a Poisson point process; the 'points' are either convex bodies, or hyperplanes, or ordinary points of $\mathbb{R}^d$. The emphasis is on the combination of probabilistic and convex-geometric arguments. We will, in fact, meet several instances where results from convex geometry are crucial for obtaining explicit answers to questions from stochastic geometry.

The first topic, Boolean models, is well established; the second topic comprises also some recent research.

1 Poisson point processes

We start with a brief introduction to Poisson point processes. Since the 'points' will later be subsets of Euclidean space, we choose a general and abstract viewpoint.

For a topological space E, we denote by $\mathcal{B}(E)$ the σ-algebra of Borel sets of E; this is the smallest σ-algebra containing all open sets. A measure on E in the following is always understood as a measure on $\mathcal{B}(E)$.

Let E be a locally compact space with a countable base. A subset of E is **locally finite** if its intersection with every compact set is finite. Let N denote the system of locally finite subsets of E. If $N \in \mathsf{N}$ and $A \in \mathcal{B}(E)$ is a Borel set, card $(N \cap A)$ is the number (possibly ∞) of elements in $N \cap A$. By $\mathcal{N}$ we denote the smallest σ-algebra in N for which all functions $N \mapsto \operatorname{card}(N \cap A)$ $(N \in \mathsf{N})$, with $A \in \mathcal{B}(E)$, are measurable.

Definition. A simple **point process** in E is a measurable map X from some probability space $(\Omega, \mathbf{A}, \mathbb{P})$ into $(\mathsf{N}, \mathcal{N})$. The **intensity measure** of X is the measure Θ on E defined by

$$\Theta(A) := \mathbb{E} \operatorname{card}(X \cap A) \qquad \text{for } A \in \mathcal{B}(E).$$

Here $\mathbb{E}$ denotes mathematical expectation. As usual, the image measure $\mathbb{P}_X$ of $\mathbb{P}$ under the map X is called the **distribution** of X, thus

$$\mathbb{P}_X(\mathcal{A}) = \mathbb{P}(X^{-1}(\mathcal{A})) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in \mathcal{A}\}) =: \mathbb{P}(X \in \mathcal{A})$$

for $\mathcal{A} \in \mathcal{N}$.

Since in the following we consider only simple point processes, we will omit the word 'simple'.

Definition. The point process X is a **Poisson process** if its intensity measure Θ is locally finite (i.e., finite on compact sets) and if

$$\mathbb{P}(\operatorname{card}(X \cap A) = j) = \frac{\Theta(A)^j}{j!} e^{-\Theta(A)}$$

holds for all $A \in \mathcal{B}(E)$ with $\Theta(A) < \infty$ and all $j \in \mathbb{N}_0$.

As known from elementary probability, a Poisson distribution can be obtained from binomial distributions by a limit procedure. This fact is reflected in the important independence properties that a Poisson process has, similar to those of the models leading to binomial distributions.

Let X be a Poisson process in E with intensity measure Θ.

1.1 Proposition. *If $A_1, A_2, \ldots \in \mathcal{B}(E)$ are pairwise disjoint and $\Theta(A_i) < \infty$ for $i = 1, 2, \ldots$, then the point processes $X \cap A_1$, $X \cap A_2, \ldots$ are stochastically independent.*

1.2 Proposition. *Let $A \in \mathcal{B}(E)$ be a Borel set with $0 < \Theta(A) < \infty$, let $k \in \mathbb{N}$. Under the condition that A contains precisely k points of X, the process $X \cap A$ is stochastically equivalent to the point process defined by the set of k independent, identically distributed random points in E with distribution $(\Theta \llcorner A)/\Theta(A)$.*

Here $\llcorner$ denotes the restriction of a measure, thus $(\Theta \llcorner A)(B) := \Theta(A \cap B)$ for $B \in \mathcal{B}(E)$.

More formally, the assertion of Proposition 1.2 says that

$$\mathbb{P}(X \cap A \in \cdot \mid \operatorname{card}(X \cap A) = k) = \mathbb{P}_{\{\xi_1, \ldots, \xi_k\}},$$

where $\xi_1, \ldots, \xi_k$ are independent, identically distributed random points in E with distribution

$$\mathbb{P}_{\xi_i} := \frac{\Theta \llcorner A}{\Theta(A)}, \qquad i = 1, \dots, k.$$

The following more technical property, to be used later, combines the so-called Campbell formula with the independence properties of the Poisson process. Here $A^m_{\neq}$ denotes the set of all m-tuples $(x_1, \dots, x_m)$ in the cartesian product A^m for which $x_1, \dots, x_m$ are pairwise distinct.

1.3 Proposition. *For $m \in \mathbb{N}$ and any nonnegative measurable function f on E^m,*

$$\mathbb{E} \sum_{(x_1,\dots,x_m) \in X^m_{\neq}} f(x_1, \dots, x_m) = \int_E \cdots \int_E f(x_1, \dots, x_m)\, \Theta(\mathrm{d}x_1) \cdots \Theta(\mathrm{d}x_m).$$

The existence of many Poisson processes is guaranteed by the following proposition.

1.4 Proposition. *Let Θ be a locally finite measure on E satisfying $\Theta(\{x\}) = 0$ for all $x \in E$. Then there exists a Poisson process on E with intensity measure Θ. Two Poisson processes on E with the same intensity measure are stochastically equivalent, that is, they have the same distribution.*

2 Particle processes

In the preceding section, we have introduced point processes in a general locally compact, second countable space E. This will be applied to the following concrete spaces:

- $E = \mathbb{R}^d$, the d-dimensional real Euclidean vector space,
- $E = \mathcal{K}^d$, the space of convex bodies (nonempty, compact, convex sets) in $\mathbb{R}^d$, equipped with the Hausdorff metric,
- $E = \mathcal{H}^d$, the space of hyperplanes of $\mathbb{R}^d$, with its standard topology.

Let E be one of these spaces. Then E is locally compact and has a countable base. The group of translations of $\mathbb{R}^d$, the group SO_d of rotations of $\mathbb{R}^d$, and the group G_d of (proper) rigid motions of $\mathbb{R}^d$ operate also on E, in the canonical way. Each of these operations is continuous. A point process X in E is called **stationary** (or **homogeneous**) if X and $X + t$ have the same distribution, for every $t \in \mathbb{R}^d$, and it is called **isotropic** if X and ϑX have the same distribution, for every rotation $\vartheta \in SO_d$. Stationary point processes are easier to handle than general ones, they are more aesthetic from a geometric point of view, and even in applications they are preferred as long as possible. We will, therefore, restrict ourselves to stationary point processes.

Let X be a stationary point process in $\mathbb{R}^d$ with a locally finite intensity measure Θ. For $t \in \mathbb{R}^d$, the point processes X and $X + t$ have the same distribution, hence $\Theta(A+t) = \Theta(A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$. Since the Lebesgue measure λ_d on $\mathbb{R}^d$ is, up to a constant factor, the only translation invariant, locally finite measure on $\mathbb{R}^d$, we have $\Theta = \gamma\lambda_d$ with a constant $\gamma \geq 0$. This number is called the **intensity** of the point process X.

Let X be a stationary point process in $\mathcal{K}^d$. In this case, we impose a stronger finiteness condition on the intensity measure. We put

$$\mathcal{K}_C := \{K \in \mathcal{K}^d : K \cap C \neq \emptyset\} \qquad \text{for } C \subset \mathbb{R}^d$$

and assume that

$$\Theta(\mathcal{K}_C) < \infty \qquad \text{for every compact set } C \subset \mathbb{R}^d. \tag{1}$$

If this is satisfied, X is called a **particle process** in $\mathbb{R}^d$ (with convex grains, but we omit this specification, since no other particle processes will be considered here).

Let X be a stationary particle process in $\mathbb{R}^d$. The stationarity implies a decomposition property of its intensity measure Θ. For this, let $c(K)$ denote the circumcentre of the convex body K (the centre of the smallest ball containing K), and put

$$\mathcal{K}_0 := \{K \in \mathcal{K}^d : c(K) = 0\}.$$

We define a homeomorphism $\Phi : \mathcal{K}_0 \times \mathbb{R}^d \to \mathcal{K}^d$ by $\Phi(K,t) := K + t$.

2.1 Lemma. *Suppose that $\Theta \not\equiv 0$. There exist a number $\gamma \in (0,\infty)$ and a probability measure $\mathbb{Q}$ on $\mathcal{K}_0$ such that*

$$\Theta = \gamma\, \Phi(\mathbb{Q} \otimes \lambda_d),$$

hence, for every Θ-integrable function f on $\mathcal{K}^d$,

$$\int_{\mathcal{K}^d} f \,\mathrm{d}\Theta = \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} f(K+x)\, \lambda_d(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K). \tag{2}$$

Proof. Fix $A \in \mathcal{B}(\mathcal{K}_0)$ and define $\mu_A(B) := \Theta(\Phi(A \times B))$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Then μ_A is a translation invariant measure on $\mathbb{R}^d$. If $C \subset \mathbb{R}^d$ is compact, then $\mu_A(C) = \Theta(\Phi(A \times C)) \leq \Theta(\mathcal{K}_C) < \infty$, thus μ_A is locally finite. It follows that $\mu_A = \varphi(A)\lambda_d$ with $0 \leq \varphi(A) < \infty$. If C^d denotes a unit cube, then $\varphi(A) = \mu_A(C^d) = \Theta(\Phi(A \times C^d))$ for $A \in \mathcal{B}(\mathcal{K}_0)$, hence φ is a measure on $\mathcal{K}_0$, and $\gamma := \varphi(\mathcal{K}_0)$ satisfies $0 < \gamma < \infty$. For the probability measure $\mathbb{Q} := \gamma^{-1}\varphi$ we have

$$\Theta(\Phi(A \times B)) = \gamma(\mathbb{Q} \otimes \lambda_d)(A \times B)$$

for $A \in \mathcal{B}(\mathcal{K}_0)$, $B \in \mathcal{B}(\mathbb{R}^d)$. From this, the assertion follows. □

We call γ the **intensity** and $\mathbb{Q}$ the **grain distribution** of the particle process X.

A similar decomposition property for the intensity measure of a stationary point process in the space $\mathcal{H}^d$ of hyperplanes will be stated in Section 4.

Let us now consider a particle process X in $\mathbb{R}^d$ and a fixed convex body $L \in \mathcal{K}^d$. It is a natural question to ask for the distribution of the random variable

$$\operatorname{card}\{K \in X : K \cap L \neq \emptyset\},$$

the number of particles hitting the 'test body' L. Thus, we are asking for the probabilities

$$p_j := \mathbb{P}(\operatorname{card}(X \cap \mathcal{K}_L) = j), \qquad j \in \mathbb{N}_0.$$

In general, this question seems hopeless, but we will see that an explicit answer is possible by combining

- geometric and probabilistic assumptions on the particle process,
- results from convex geometry.

One such geometric assumption on the particle process X has already been made, namely that the particles are convex. Another geometric assumption will be the stationarity. From the probabilistic side, it seems unavoidable to have strong independence properties; we will, in fact, assume that X is a Poisson process.

So we assume now that X is a stationary Poisson particle process, with intensity measure $\Theta \not\equiv 0$. Then we immediately have

$$p_j = \frac{\Theta(\mathcal{K}_L)^j}{j!} e^{-\Theta(\mathcal{K}_L)} \tag{3}$$

(observe that $\Theta(\mathcal{K}_L) < \infty$ by assumption (18)). It remains to determine the parameter $\Theta(\mathcal{K}_L)$. Applying the decomposition formula (19) with $f = \mathbf{1}_{\mathcal{K}_L}$, the indicator function of $\mathcal{K}_L$ on $\mathcal{K}^d$, we get

$$\Theta(\mathcal{K}_L) = \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{K}_L}(K + x)\, \lambda_d(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K) \tag{4}$$

$$= \gamma \int_{\mathcal{K}_0} \lambda_d(K - L)\, \mathbb{Q}(\mathrm{d}K), \tag{5}$$

since $\mathbf{1}_{\mathcal{K}_L}(K + x) = 1 \Leftrightarrow (K + x) \cap L \neq \emptyset \Leftrightarrow x \in L - K$ $(:= \{l - k : l \in L, k \in K\})$ and $\lambda_d(L - K) = \lambda_d(K - L)$.

Now convex geometry enters the scene. Let us first consider the case where $L = rB^d$, the ball of radius r and centre 0 in $\mathbb{R}^d$. The classical **Steiner formula** tells us that $\lambda_d(K + rB^d)$ is a polynomial in r, of degree at most d. It is usually written in the form

$$\lambda_d(K + rB^d) = \sum_{m=0}^{d} r^{d-m} \kappa_{d-m} V_m(K), \tag{6}$$

with $\kappa_d := \lambda_d(B^d)$. This defines important functionals $V_m : \mathcal{K}^d \to \mathbb{R}$, the **intrinsic volumes**. For example, $V_d(K)$ is the volume of K, $2V_{d-1}(K)$ is the surface area of K (if K has interior points), and $V_0(K) = 1 = \chi(K)$, the **Euler characteristic** of K. We see that the intrinsic volumes are inevitable if we want an explicit answer to our question. In particular, inserting (23) in (21), we obtain

$$\Theta(\mathcal{K}_{rB^d}) = \sum_{m=0}^{d} r^{d-m}\kappa_{d-m}\overline{V}_m(X), \tag{7}$$

where we have put

$$\overline{V}_m(X) := \gamma \int_{\mathcal{K}_0} V_m(K)\,\mathbb{Q}(\mathrm{d}K). \tag{8}$$

The number $\overline{V}_m(X)$ is called the m**th intrinsic volume intensity** of the particle process X. It can be defined, by (25), for general stationary (not necessarily Poisson) particle processes X. The intensities are means in a twofold sense: they are obtained by spatial as well as by stochastic averaging. This averaging is made evident by some more intuitive representations of the intensity. If $B \in \mathcal{B}(\mathbb{R}^d)$ is any Borel set with $\lambda_d(B) > 0$, then

$$\overline{V}_m(X) = \frac{1}{\lambda_d(B)}\mathbb{E} \sum_{K\in X,\, c(K)\in B} V_m(K). \tag{9}$$

For the proof, we use Campbell's formula

$$\mathbb{E}\sum_{K\in X} f(K) = \int_{\mathcal{K}^d} f\,\mathrm{d}\Theta,$$

which holds for all nonnegative measurable functions f on $\mathcal{K}^d$ (for indicator functions of Borel sets, it holds by the definition of Θ, and the extension to nonnegative measurable functions is a standard argument). This gives

$$\begin{aligned}
&\mathbb{E} \sum_{K\in X,\, c(K)\in B} V_m(K)\\
&= \int_{\mathcal{K}^d} \mathbf{1}_B(c(K))V_m(K)\,\Theta(\mathrm{d}K)\\
&= \gamma \int_{\mathcal{K}_0}\int_{\mathbb{R}^d} \mathbf{1}_B(c(K+x))V_m(K+x)\,\lambda_d(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}K)\\
&= \gamma \int_{\mathcal{K}_0} V_m(K)\lambda_d(B-c(K))\,\mathbb{Q}(\mathrm{d}K)\\
&= \lambda_d(B)\overline{V}_m(X),
\end{aligned}$$

by the translation invariance of V_m and λ_d.

We mention without proof two other representations of $\overline{V}_m(X)$. They involve an arbitrary convex body $W \in \mathcal{K}^d$ with $V_d(W) > 0$ (an 'observation window') and assert that

$$\overline{V}_m(X) = \lim_{r\to\infty} \frac{1}{V_d(rW)} \mathbb{E} \sum_{K\in X,\, K\subset rW} V_m(K)$$

and

$$\overline{V}_m(X) = \lim_{r\to\infty} \frac{1}{V_d(rW)} \mathbb{E} \sum_{K\in X} V_m(K\cap rW).$$

We return to a stationary Poisson particle process X and recall that we have obtained an explicit formula for the probability (20) in the case where L is a ball. Through (24), the result involves the intensities of the intrinsic volumes of the particle process X. For a general convex body L, a similarly explicit result can be obtained if we introduce a further geometric assumption on the particle process X, namely that it be isotropic.

Let X be a stationary, isotropic Poisson particle process in $\mathbb{R}^d$. Its intensity measure Θ is now invariant under translations and rotations, hence the grain distribution $\mathbb{Q}$ is invariant under rotations. Therefore, the right-hand side of (4) remains unchanged if we replace K by ϑK, where $\vartheta \in SO_d$ is a rotation. We can then integrate the resulting expression over all $\vartheta \in SO_d$, with respect to the invariant probability measure ν on the rotation group SO_d. After an application of Fubini's theorem we obtain

$$\Theta(\mathcal{K}_L) = \gamma \int_{\mathcal{K}_0} \left[\int_{SO_d} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{K}_L}(\vartheta K + x)\, \lambda_d(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \right] \mathbb{Q}(\mathrm{d}K).$$

The double integral in brackets can be written as an integral over the motion group G_d with respect to its (suitably normalized) invariant measure μ, namely

$$[\cdots] = \int_{G_d} \mathbf{1}_{\mathcal{K}_L}(gK)\, \mu(\mathrm{d}g) = \int_{G_d} \chi(L\cap gK)\, \mu(\mathrm{d}g),$$

where $\chi(M) = 1$ for a (nonempty) convex body M and $\chi(\emptyset) = 0$. Another classical result from convex geometry, the **principal kinematic formula** for convex bodies, tells us that

$$\int_{G_d} \chi(L\cap gK)\, \mu(\mathrm{d}g) = \sum_{m=0}^{d} \alpha_{dm} V_m(L) V_{d-m}(K)$$

with

$$\alpha_{dm} := \frac{m!\kappa_m (d-m)!\kappa_{d-m}}{d!\kappa_d}.$$

This gives

$$\Theta(\mathcal{K}_L) = \sum_{m=0}^{d} \alpha_{dm} V_m(L) \overline{V}_{d-m}(X). \tag{10}$$

We have obtained the following explicit result: *The probability that the fixed convex body L is hit by precisely j bodies of the stationary isotropic Poisson particle process X is given by*

$$p_j = \frac{\Theta(\mathcal{K}_L)^j}{j!} e^{-\Theta(\mathcal{K}_L)},$$

where the parameter $\Theta(\mathcal{K}_L)$ *is given by* (10). *Thus, this probability depends only on the intrinsic volumes of* L *and the intrinsic volume intensities of* X.

3 Boolean models

Particle processes, as considered in the previous section, are often used to generate random closed sets. To explain the notion of a random closed set in $\mathbb{R}^d$, let $\mathcal{F}$ denote the system of all closed subsets of $\mathbb{R}^d$ (including the empty set). For $A \subset \mathbb{R}^d$, one sets

$$\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \emptyset\}, \qquad \mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \emptyset\}.$$

The system

$$\{\mathcal{F}_G : G \subset \mathbb{R}^d \text{ open}\} \cup \{\mathcal{F}^C : C \subset \mathbb{R}^d \text{ compact}\}$$

is the subbasis of a topology on $\mathcal{F}$, which is called the **topology of closed convergence**. By $\mathcal{B}(\mathcal{F})$ we denote the corresponding σ-algebra of Borel sets. It can be shown that $\mathcal{B}(\mathcal{F})$ is generated by $\{\mathcal{F}_G : G \subset \mathbb{R}^d \text{ open}\}$, for example. Now a **random closed set** in $\mathbb{R}^d$, briefly a RACS, is defined as a random variable with values in $\mathcal{F}$, more explicitly, as a measurable map Z from some probability space $(\Omega, \mathbf{A}, \mathbb{P})$ into the measurable space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. The image measure $\mathbb{P}_Z := Z(\mathbb{P})$ is called the **distribution** of Z. The RACS Z is called **stationary** if $Z + t$ and Z have the same distribution for all $t \in \mathbb{R}^d$, and **isotropic** if ϑZ and Z have the same distribution for all $\vartheta \in SO_d$.

General random closed sets, although the subject of some deep results, are not easy to handle. One seeks, therefore, for classes of random closed sets which are more accessible. Suitable such sets are obtained as union sets of particle processes. If X is a particle process in $\mathbb{R}^d$, then

$$Z_X := \bigcup_{K \in X} K$$

is its union set. One can deduce from condition (1) that Z_X is almost surely a closed set. Also the necessary measurability property can be verified, so that Z_X is a random closed set. If X is especially a Poisson particle process, then Z_X is called a **Boolean model**. If X is stationary (isotropic), then the Boolean model Z_X is stationary (isotropic).

Let Z_X be a stationary Boolean model, generated by the stationary Poisson particle process X. The investigation of such a RACS begins with a search for simple numerical parameters describing quantitative properties. A parameter immediately coming to mind is given by

$$p := \mathbb{P}(0 \in Z_X),$$

the probability that 0 is covered by the random set Z_X. For $y \in \mathbb{R}^d$, the random sets Z_X and Z_{X-y} have the same distribution, hence

$$p = \mathbb{P}(y \in Z_X) = \mathbb{E}\,\mathbf{1}_{Z_X}(y).$$

Let $W \subset \mathbb{R}^d$ be a Borel set with $0 < \lambda_d(W) < \infty$. Using Fubini's theorem, we get

$$p\lambda_d(W) = \int_W \mathbb{E}\,\mathbf{1}_{Z_X}(y)\,\lambda_d(\mathrm{d}y) = \mathbb{E}\int_W \mathbf{1}_{Z_X}(y)\,\lambda_d(\mathrm{d}y) = \mathbb{E}\,\lambda_d(Z_X \cap W),$$

thus

$$p = \frac{\mathbb{E}\,\lambda_d(Z_X \cap W)}{\lambda_d(W)} =: \overline{V}_d(Z_X)$$

is independent of the set W. This number is called the **volume intensity** of Z_X.

We can find a connection with the volume intensity $\overline{V}_d(X)$ of the underlying Poisson particle process. In fact,

$$\begin{aligned}\overline{V}_d(Z_X) &= \mathbb{P}(0 \in Z_X) = 1 - \mathbb{P}(0 \notin Z_X)\\ &= 1 - \mathbb{P}(\mathrm{card}\,(X \cap \mathcal{K}_{\{0\}}) = 0) = 1 - e^{-\Theta(\mathcal{K}_{\{0\}})}\end{aligned}$$

and

$$\begin{aligned}\Theta(\mathcal{K}_{\{0\}}) &= \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{K}_{\{0\}}}(K + x)\,\lambda_d(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}K)\\ &= \gamma \int_{\mathcal{K}_0} V_d(K)\,\mathbb{Q}(\mathrm{d}K) = \overline{V}_d(X).\end{aligned}$$

Thus we have found

$$\overline{V}_d(Z_X) = 1 - e^{-\overline{V}_d(X)}. \tag{11}$$

This equality should have come as a surprise: it says that the volume intensity $\overline{V}_d(X)$ of the particle process X can be determined from the volume intensity $\overline{V}_d(Z_X)$ of the union set. This is surprising since in a given realization of Z_X one cannot identify the generating particles, since they overlap, and some particles may even be covered totally by others. The reason for the existence of the exact relation above lies in the strong independence properties of Poisson processes.

The elegant connection between quantitative properties of a stationary Boolean model and its underlying particle process is not restricted to the volume. Let us consider, in heuristic terms, a question which has its origin in practice. Assume that we observe a realization of a random system of convex sets in the plane, for example a microscopic image of blood cells or, in material sciences, the polished surface of some material that contains particles of some other material. Assume we need to know some quantitative aspects, like the mean number of particles per unit area, or the mean perimeter, or the mean area. In general, we will not be able to observe individual particles, but only their union set. We assume that for the union set we can measure, for a given realization inside an observation window, the area, the perimeter, the Euler characteristic. Can we obtain estimators for the corresponding parameters of the underlying particle process? Such a correspondence can only be expected if the particle process satisfies strong independence assumptions. We shall see that stationary Poisson particle processes and their union sets provide a perfect model to permit such conclusions.

We replace the volume by a general continuous function $\varphi : \mathcal{K}^d \to \mathbb{R}$. Since we intend to investigate sets arising as unions of convex bodies, we must be able to control the behaviour of φ under unions, therefore φ is assumed to satisfy

$$\varphi(K \cup L) = \varphi(K) + \varphi(L) - \varphi(K \cap L) \tag{12}$$

whenever $K,\ L,\ K \cup L \in \mathcal{K}^d$; we also set $\varphi(\emptyset) = 0$. Such a function φ is called **additive** or a **valuation**. By a theorem of Groemer, a continuous additive function $\varphi : \mathcal{K}^d \to \mathbb{R}$ has an additive extension to the system $\mathcal{R}^d$ of **polyconvex sets**, which are defined as unions of finitely many convex bodies. The extension, also denoted by φ, satisfies (12) for $K, L \in \mathcal{R}^d$. If we start with $\varphi = V_d$ on $\mathcal{K}^d$, the extension will, of course, be the volume on $\mathcal{R}^d$. The extension of the surface area is the surface area, and the extension of the function V_0 (which is 1 on $\mathcal{K}^d$) gives the **Euler characteristic** of polyconvex sets.

It follows by induction that an additive function φ on $\mathcal{R}^d$ satisfies the **inclusion-exclusion principle**

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{i_1 < \cdots < i_r} \varphi(K_{i_1} \cap \cdots \cap K_{i_r}). \tag{13}$$

Now let X be a stationary Poisson particle process with intensity measure Θ, and let Z_X be the generated Boolean model. Motivated by practical applications (in small dimensions), we assume that a **sampling window**, a convex body W with $V_d(W) > 0$, is given in which $Z_X \cap W$ can be observed. Since $Z_X \cap W$ is a polyconvex set, $\varphi(Z_X \cap W)$ is defined and yields a random variable. We want to investigate how its expectation is related to the characteristics of the underlying particle process, that is, to the intensity γ and the grain distribution $\mathbb{Q}$ of X. In applications, such relations may be used to fit a Boolean model to given data, or to estimate functional densities of the particle process, in particular its intensity, from measurements at realizations of the union set.

To begin with the computation of $\mathbb{E}\,\varphi(Z_X \cap W)$, let ν be the random number of particles of X hitting W, and let $M_1, \ldots, M_\nu$ be these particles (with any numbering). Then (13) gives

$$\begin{aligned}
\varphi(Z_X \cap W) &= \varphi\left(\bigcup_{K\in X} K \cap W\right)\\
&= \sum_{k=1}^{\nu}(-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le \nu} \varphi(W \cap M_{i_1} \cap \cdots \cap M_{i_k})\\
&= \sum_{k=1}^{\nu} \frac{(-1)^{k-1}}{k!} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}} \varphi(W \cap K_1 \cap \cdots \cap K_k). \qquad (14)
\end{aligned}$$

Here $X^k_{\neq}$ is the set of pairwise distinct k-tuples from X. In (14) we may extend the first summation to ∞, since $\varphi(\emptyset) = 0$.

The function φ is continuous on $\mathcal{K}^d$ and hence bounded on the set of convex bodies contained in W. Thus, there exists a number c (depending on W) with $|\varphi(L)| \le c$ for all $L \in \mathcal{K}^d$ with $L \subset W$. This gives

$$\begin{aligned}
&\left|\sum_{k=1}^{\nu} \frac{(-1)^{k-1}}{k!} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}} \varphi(W \cap K_1 \cap \cdots \cap K_k)\right|\\
&\le \sum_{k=1}^{\nu} \binom{\nu}{k} c \le 2^{\nu} c = 2^{\operatorname{card}(X\cap\mathcal{K}_W)} c.
\end{aligned}$$

Since $\operatorname{card}(X \cap \mathcal{K}_W)$ has a Poisson distribution,

$$\begin{aligned}
\mathbb{E}\, 2^{\operatorname{card}(X\cap\mathcal{K}_W)} &= \sum_{k=0}^{\infty} 2^k\, \mathbb{P}(\operatorname{card}(X \cap \mathcal{K}_W) = k)\\
&= e^{-\Theta(\mathcal{K}_W)} \sum_{k=0}^{\infty} \frac{[2\Theta(\mathcal{K}_W)]^k}{k!}\\
&= e^{-\Theta(\mathcal{K}_W)} e^{2\Theta(\mathcal{K}_W)} = e^{\Theta(\mathcal{K}_W)} < \infty,
\end{aligned}$$

by (1). It follows that $\varphi(Z_X \cap W)$ is integrable. By the bounded convergence theorem, we can interchange expectation and summation. Using Proposition 1.3, we obtain

$$\begin{aligned}
&\mathbb{E}\, \varphi(Z_X \cap W)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \mathbb{E} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}} \varphi(W \cap K_1 \cap \cdots \cap K_k)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}^d} \cdots \int_{\mathcal{K}^d} \varphi(W \cap K_1 \cap \cdots \cap K_k)\, \Theta(\mathrm{d}K_1) \cdots \Theta(\mathrm{d}K_k).
\end{aligned}$$

So far, we have not used the stationarity. But if we now employ this assumption, we can use the decomposition (3) of the intensity measure, put

$$\begin{aligned}&\Phi(W,K_1,\dots,K_k)\\ &:=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\varphi(W\cap(K_1+x_1)\cap\cdots\cap(K_k+x_k))\,\lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_k)\end{aligned}$$

and end up with the formula

$$\begin{aligned}&\mathbb{E}\,\varphi(Z_X\cap W)\\ &=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\gamma^k\int_{\mathcal{K}_0}\cdots\int_{\mathcal{K}_0}\Phi(W,K_1,\dots,K_k)\,\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k).\end{aligned}$$

Further progress requires the computation of the integrals

$$I:=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\varphi(W\cap(K_1+x_1)\cap\cdots\cap(K_k+x_k))\,\lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_k).$$

This is possible either for special choices of φ or under isotropy assumptions on the Boolean model (alternatively, by randomizing the observation window by an isotropic rotation).

Let us first consider the volume, $\varphi=V_d$. In that case, it is not difficult to show that

$$\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}V_d(K_0\cap(K_1+x_1)\cap\cdots\cap(K_k+x_k))\,\lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_k)=\prod_{i=0}^{k}V_d(K_i).$$

Thus we obtain

$$\mathbb{E}V_d(Z_X\cap W)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}V_d(W)\overline{V}_d(X)^k=V_d(W)\left(1-e^{-\overline{V}_d(X)}\right).$$

This is nothing but relation (11) again.

More interesting is the case of the intrinsic volume V_{d-1}, which is half the surface area (for convex bodies with interior points). It is again not difficult to prove that

$$\begin{aligned}&\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}V_{d-1}(K_0\cap(K_1+x_1)\cap\cdots\cap(K_k+x_k))\,\lambda_d(\mathrm{d}x_1)\cdots\lambda_d(\mathrm{d}x_k)\\ &=\sum_{j=0}^{k}\frac{V_{d-1}(K_j)}{V_d(K_j)}\prod_{i=0}^{k}V_d(K_i).\end{aligned}$$

This leads to

$$\mathbb{E}V_{d-1}(Z_X\cap W)=V_d(W)\overline{V}_{d-1}(X)e^{-\overline{V}_d(X)}+V_{d-1}(W)\left(1-e^{-\overline{V}_d(X)}\right).$$

In contrast to the case of the volume, the quotient

$$\frac{\mathbb{E}V_{d-1}(Z_X \cap W)}{V_d(W)} = \overline{V}_{d-1}(X)e^{-\overline{V}_d(X)} + \frac{V_{d-1}(W)}{V_d(W)}\left(1 - e^{-\overline{V}_d(X)}\right)$$

still depends on the observation window W. This influence disappears for increasing W. More precisely, we see that

$$\lim_{r\to\infty} \frac{\mathbb{E}V_{d-1}(Z_X \cap rW)}{V_d(rW)} = \overline{V}_{d-1}(X)e^{-\overline{V}_d(X)}.$$

The limit on the left-hand side is denoted by $\overline{V}_{d-1}(Z_X)$ and is, up to a factor $1/2$, the **surface area intensity** of Z_X.

We repeat that we have obtained the two relations

$$\overline{V}_d(Z_X) = 1 - e^{-\overline{V}_d(X)},$$
$$\overline{V}_{d-1}(Z_X) = \overline{V}_{d-1}(X)e^{-\overline{V}_d(X)},$$

connecting intrinsic volume intensities of the Boolean model Z_X with those of the underlying particle process X.

Now we assume that the considered Boolean model Z_X is also isotropic. Then we can obtain an explicit formula for a general additive function φ (continuous on $\mathcal{K}^d$). Since the grain distribution $\mathbb{Q}$ of X is now rotation invariant, we can argue as in the case of isotropic particle processes. We insert rotations, integrate over the rotation group and apply Fubini's theorem, to obtain

$$\begin{aligned}
&\int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \Phi(W, K_1, \ldots, K_k)\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k) \\
&= \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{G_d} \cdots \int_{G_d} \varphi(W \cap g_1 K_1 \cap \cdots \cap g_k K_k)\, \mu(\mathrm{d}g_1) \cdots \mu(\mathrm{d}g_k) \\
&\quad \times \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k).
\end{aligned}$$

To compute the inner integrals over the motion group, again heavy use is made of convex geometry. To calculate, for example, the integral

$$\int_{G_d} \varphi(W \cap gK)\, \mu(\mathrm{d}g), \tag{15}$$

one uses Hadwiger's celebrated characterization theorem for the intrinsic volumes. In order to get simple formulas, it is now advisable to renormalize the intrinsic volumes by putting

$$v_j := \frac{j!\kappa_j}{d!\kappa_d} V_j,$$

with corresponding definitions of the intensities $\overline{v}_j(X), \overline{v}_j(Z_X)$.

As a function of K, the integral (15) turns out to be additive, continuous, and invariant under rigid motions. By Hadwiger's theorem, it must be a linear combination of the intrinsic volumes, thus

$$\int_{G_d} \varphi(W \cap gK)\, \mu(\mathrm{d}g) = \sum_{j=0}^{d} \varphi^{(d-j)}(W) v_j(K).$$

For the coefficients, one finds that

$$\varphi^{(d-j)}(W) = \frac{d!\kappa_d}{j!\kappa_j} \int_{\mathcal{E}_j^d} \varphi(W \cap E)\, \mu_j(\mathrm{d}E),$$

where $\mathcal{E}_j^d$ is the space of j-dimensional planes in $\mathbb{R}^d$ and μ_j is its motion invariant measure, suitably normalized. By induction, we then get the general formula

$$\int_{G_d} \cdots \int_{G_d} \varphi(W \cap g_1 K_1 \cap \cdots \cap g_k K_k)\, \mu(\mathrm{d}g_1) \cdots \mu(\mathrm{d}g_k)$$
$$= \sum_{\substack{r_0,\dots,r_k=0 \\ r_0+\cdots+r_k=kd}}^{d} \varphi^{(r_0)}(W) v_{r_1}(K_1) \cdots v_{r_k}(K_k).$$

Inserting this in the expression for $\mathbb{E}\, \varphi(Z_X \cap W)$ and rearranging, we finally obtain the following result.

3.1 Theorem. *Let Z_X be the Boolean model generated by the stationary, isotropic Poisson particle process X. If $\varphi : \mathcal{K}^d \to \mathbb{R}$ is an additive, continuous functional, additively extended to the polyconvex sets, then, for any $W \in \mathcal{K}^d$ with $V_d(W) > 0$,*

$$\mathbb{E}\, \varphi(Z_X \cap W) = \varphi(W) \left(1 - e^{-\overline{V}_d(X)}\right) -$$
$$- e^{-\overline{V}_d(X)} \sum_{m=1}^{d} \varphi^{(m)}(W) \sum_{s=1}^{m} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=0 \\ m_1+\cdots+m_s=sd-m}}^{d-1} \prod_{i=1}^{s} \overline{v}_{m_i}(X).$$

Now we choose $\varphi = v_j$, the renormalized jth intrinsic volume. By the Crofton formula from integral geometry, we have

$$(v_j)^{(m)} = v_{m+j},$$

with $v_{m+j} = 0$ if $m + j > d$. Inserting this, we obtain

$$\mathbb{E}\, v_j(Z_X \cap W) = v_j(W) \left(1 - e^{-\overline{V}_d(X)}\right) -$$
$$- e^{-\overline{V}_d(X)} \sum_{m=j+1}^{d} v_m(W) \sum_{s=1}^{m-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\cdots+m_s=sd+j-m}}^{d-1} \prod_{i=1}^{s} \overline{v}_{m_i}(X).$$

Here we can replace W by rW with $r > 0$ and then let r tend to infinity. We obtain the following result. The limit

$$\lim_{r\to\infty} \frac{\mathbb{E}\, v_j(Z_X \cap rW)}{V_d(rW)} =: \overline{v}_j(Z_X)$$

exists and is given by

$$\overline{v}_j(Z_X) = e^{-\overline{V}_d(X)} \left[\overline{v}_j(X) - \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1 \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \prod_{i=1}^{s} \overline{v}_{m_i}(X) \right]$$

for $j = 0, \dots, d-1$. The cases $j = d$ and $j = d-1$ have been obtained earlier, without the isotropy assumption.

We specialize the formulas to two and three dimensions, using classical notation:

$n = 2:$		$n = 3:$	
$A = V_2,$	area	$V = V_3,$	volume
$P = 2V_1,$	perimeter	$S = 2V_2,$	surface area
$\chi = V_0,$	Euler characteristic	$M = \pi V_1,$	integral of mean curvature
		$\chi = V_0,$	Euler characteristic.

We obtain the following relations: For $n = 2$,

$$\begin{aligned}
\overline{A}(Z_X) &= 1 - e^{-\overline{A}(X)}, \\
\overline{P}(Z_X) &= e^{-\overline{A}(X)}\overline{P}(X), \\
\overline{\chi}(Z_X) &= e^{-\overline{A}(X)} \left(\overline{\chi}(X) - \frac{1}{4\pi}\overline{P}(X)^2 \right).
\end{aligned}$$

For $n = 3$,

$$\begin{aligned}
\overline{V}(Z_X) &= 1 - e^{-\overline{V}(X)}, \\
\overline{S}(Z_X) &= e^{-\overline{V}(X)}\overline{S}(X), \\
\overline{M}(Z_X) &= e^{-\overline{V}(X)} \left(\overline{M}(X) - \frac{\pi^2}{32}\overline{S}(X)^2 \right), \\
\overline{\chi}(Z_X) &= e^{-\overline{V}(X)} \left(\overline{\chi}(X) - \frac{1}{4\pi}\overline{M}(X)\overline{S}(X) + \frac{\pi}{384}\overline{S}(X)^3 \right).
\end{aligned}$$

Observe that $\overline{\chi}(X) = \gamma$, the intensity of X. Thus, these formulas provide a possibility to determine the intensity of the underlying particle process from measurements at the union set. It was a priori clear that such a possibility cannot exist without strong independence properties. It was less easy to expect which functional intensities of the union set would be necessary to achieve this goal. As we have seen, the answer comes from convex geometry, in particular from integral geometric results for convex bodies.

Hints to the literature. The standard books on Stochastic Geometry are Matheron [7] and Stoyan-Kendall-Mecke [13]. Closer to the foregoing presentation are the Lecture Notes [12] on Stochastic Geometry and [11] on Integral Geometry. The monograph by Molchanov [8] is devoted particularly to the Boolean model. Further details from Convex Geometry are found in [10].

4 Poisson hyperplane and Poisson-Voronoi mosaics

The second part of these lectures is devoted to another prominent model of stochastic geometry, which has many applications in two and three dimensions and which will be studied here in $\mathbb{R}^d$ from a theoretical point of view. We will consider random mosaics which are generated either by Poisson processes of hyperplanes or by the Voronoi cells of a Poisson point process. Again there will be a close connection to results from convex geometry.

A **mosaic** in $\mathbb{R}^d$, or a **tessellation** of $\mathbb{R}^d$, is a locally finite system of d-dimensional polytopes in $\mathbb{R}^d$ which cover $\mathbb{R}^d$ and have pairwise no common interior points. If m is a mosaic in $\mathbb{R}^d$, its elements are called the **cells** of m. We restrict ourselves to two particular types of mosaics. First, let $\mathcal{H}$ be a locally finite system of hyperplanes in $\mathbb{R}^d$. The connected components of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{H}} H$ are open polyhedral sets. Their closures are the **cells induced by** $\mathcal{H}$. A mosaic is called a **hyperplane mosaic** if its cells are induced by some system of hyperplanes. Second, let A be a locally finite set of points in $\mathbb{R}^d$. For $x \in A$, the set

$$C(x, A) := \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - a\| \text{ for all } a \in A\}$$

consists of all points of $\mathbb{R}^d$ for which x is the nearest point in A. It is a closed polyhedral set, called the **Voronoi cell** (or **Dirichlet cell**) of x (with respect to A). A mosaic is called a **Voronoi mosaic** if its cells are the Voronoi cells of some point set.

A **random mosaic** in $\mathbb{R}^d$ is defined as a particle process in $\mathbb{R}^d$ which is almost surely a mosaic. We consider two types of random mosaics which are particularly accessible to mathematical investigation.

Let $\hat{X}$ be a stationary Poisson point process in the space $\mathcal{H}^d$ of hyperplanes in $\mathbb{R}^d$; it is called a stationary **Poisson hyperplane process**. Its intensity measure Θ is a measure on $\mathcal{H}^d$ which is finite on compact sets. If $C \subset \mathbb{R}^d$ is compact, the set $\{H \in \mathcal{H}^d : H \cap C \neq \emptyset\}$ is compact and hence has finite Θ measure. It follows that the realizations of $\hat{X}$ are almost surely locally finite systems of hyperplanes. We assume that $\hat{X}$ is **nondegenerate**, which means that not almost surely all hyperplanes of $\hat{X}$ are parallel to some fixed line. Under this assumption, the system of the cells induced by $\hat{X}$ forms a random mosaic; it is denoted by X and called a stationary **Poisson hyperplane mosaic**.

In analogy to Lemma 2.1, the intensity measure Θ of the stationary Poisson hyperplane process $\hat{X}$ has a useful decomposition. For $u \in S^{d-1}$ and $t \in \mathbb{R}$, we write

$$H(u,t) := \{x \in \mathbb{R}^d : \langle x,u\rangle = t\}, \qquad H^-(u,t) := \{x \in \mathbb{R}^d : \langle x,u\rangle \le t\}.$$

4.1 Lemma. *There exist a number $\gamma \in (0,\infty)$ and an even probability measure φ on the unit sphere S^{d-1} such that*

$$\Theta = \gamma \int_{S^{d-1}} \int_{-\infty}^{\infty} \mathbf{1}\{H(u,t) \in \cdot\}\, \mathrm{d}t\, \varphi(\mathrm{d}u). \tag{16}$$

We call γ the **intensity** and φ the **direction distribution** of $\hat{X}$. The assumption that $\hat{X}$ be nondegenerate is equivalent to the fact that φ is not concentrated on some great subsphere of S^{d-1}.

Let $\tilde{X}$ be a stationary Poisson point process in $\mathbb{R}^d$. Then

$$X := \{C(x,\tilde{X}) : x \in \tilde{X}\}$$

is a stationary mosaic, called the **Poisson-Voronoi mosaic** induced by $\tilde{X}$. The point x is called the **nucleus** of the cell $C(x,\tilde{X})$.

If one wants to investigate the shapes of the cells in a stationary random mosaic m, one needs a notion of 'average' cell. One possibility is to consider the cell containing a given fixed point in its interior. By the stationarity of the mosaic, the resulting random shape will not depend on the choice of the fixed point, hence we may take 0 as this point. With probability one there is a unique cell containing 0 in its interior; this random polytope is called the **zero cell** or the **Crofton cell** of the mosaic m.

Another natural way of selecting an average cell of a mosaic m makes use of a 'centre function', like the circumcentre or, in the case of Voronoi cells, the nucleus. Within a large region one picks out a cell at random, with equal chances for each cell to be picked, and translates it so that its centre becomes the origin. The random polytope obtained in this way is called the **typical cell** of the mosaic. We give the precise definition only for a stationary Poisson-Voronoi mosaic, using the nucleus. In this case, the distribution $\mathbb{Q}$ of the typical cell can be defined by

$$\mathbb{Q}(\mathcal{A}) = \gamma^{-1}\mathbb{E}\,\mathrm{card}\{x \in \tilde{X} \cap B : C(x,\tilde{X}) - x \in \mathcal{A}\}$$

for Borel sets $\mathcal{A} \subset \mathcal{K}^d$; here γ is the intensity of $\tilde{X}$ and $B \subset \mathbb{R}^d$ is an arbitrary Borel set with $\lambda_d(B) = 1$. It is also true (using ergodicity properties) that

$$\mathbb{Q}(\mathcal{A}) = \lim_{r\to\infty} \frac{\mathrm{card}\{x \in \tilde{X} \cap rB^d : C(x,\tilde{X}) - x \in \mathcal{A}\}}{\mathrm{card}(\tilde{X} \cap rB^d)} \quad \text{almost surely.}$$

A particular property of Poisson processes (Slivnyak's theorem) entails that the typical cell of the Poisson-Voronoi mosaic X is stochastically equivalent to the random polytope

$$Z = C(0, \tilde{X} \cup \{0\}).$$

The Voronoi cell $C(0, \tilde{X} \cup \{0\})$, by its definition, can be obtained as the intersection

$$\bigcap_{x \in \tilde{X}} H^-(x),$$

where $H(x)$ is the mid hyperplane of 0 and x and $H^-(x)$ is the closed halfspace bounded by it and containing 0. Therefore, the typical cell Z of the Poisson-Voronoi mosaic X is the zero cell of the hyperplane mosaic generated by the hyperplane process

$$Y := \{H(x) : x \in \tilde{X}\}.$$

This is a nonstationary Poisson hyperplane process. For its intensity measure Θ one finds an expression similar to (16), namely

$$\Theta = 2^d \gamma \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\}\, t^{d-1}\, \mathrm{d}t\, \sigma(\mathrm{d}u), \tag{17}$$

where σ denotes the spherical Lebesgue measure on the unit sphere S^{d-1}.

5 Asymptotic shapes of large cells

The starting point for the following was a conjecture of David G. Kendall from the early 1940s. It became wider known when it was reformulated in the preface to the first edition of the book by Stoyan-Kendall-Mecke [13]. Kendall considered stationary and isotropic Poisson line processes in the plane and the induced mosaics. He asked whether cells of large area must be approximately circular. The question makes sense for the zero cell Z_0 and for the typical cell. The following is a slight modification of Kendall's conjecture.

D.G. Kendall's conjecture (slightly modified). *The conditional law for the shape of Z_0, given a lower bound for the area $A(Z_0)$ of Z_0, converges weakly, as the lower bound tends to ∞, to the degenerate law concentrated at the circular shape.*

A proof was given by I.N. Kovalenko [4], who also obtained in [5] an analogous result for the typical cell of a stationary Poisson-Voronoi mosaic in the plane. In the following, we want to present joint work with Daniel Hug and Matthias Reitzner [1], [2], [3], which extends this work to higher dimensions and generalizes and strengthens it under various aspects. Generally speaking, we investigate asymptotic shapes of large zero cells of Poisson hyperplane mosaics. The generality of our approach involves the following features:

- more general Poisson hyperplane processes,
- more general functionals to measure how 'large' a cell is,
- identification of asymptotic shapes where limit shapes do not exist,

- explicit estimates for deviations from asymptotic shapes.

The class of Poisson hyperplane processes to be considered is chosen so that the cases of zero cells of stationary Poisson hyperplane mosaics and of typical cells of stationary Poisson-Voronoi mosaics are both covered, but the concept is considerably more general, as we now explain.

By $\mathcal{H}^d$ we denote the space of hyperplanes in $\mathbb{R}^d$ not containing 0, with its usual topology and Borel structure. Every hyperplane $H \in \mathcal{H}^d$ has a unique representation $H = H(u,t)$ with $u \in S^{d-1}$ and $t > 0$, thus $0 \in H^-(u,t)$; we call this the standard representation. For $H \in \mathcal{H}^d$, we denote by H^- the closed halfspace bounded by H that contains 0. For a set $\mathcal{A} \subset \mathcal{H}^d$, we define

$$P(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H^-.$$

Let X be a Poisson hyperplane process in $\mathbb{R}^d$. We assume that the intensity measure $\Theta = \mathbb{E}\,\mathrm{card}\,(X \cap \cdot)$ is of the form

$$\Theta = \gamma \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\} t^{r-1}\,\mathrm{d}t\,\varphi(\mathrm{d}u). \tag{18}$$

Here $\gamma > 0$, $r \geq 1$, and φ is a probability measure on S^{d-1} with the property that its support is not contained in some closed hemisphere. The measure φ is called the **direction distribution** of the hyperplane process X, and to the number r we refer as the **distance exponent**. Note that (18) includes the two cases (16) (but with different γ) and (17).

The random polytope

$$Z_0 := P(X) = \bigcap_{H \in X} H^-$$

is the **zero cell**, or **Crofton cell**, of the mosaic induced by X.

Let $\mathcal{K}_o^d$ denote the space of convex bodies in $\mathbb{R}^d$ containing the origin, but not only the origin. Our investigation of asymptotic shapes of large zero cells is governed by three continuous homogeneous functionals on the space $\mathcal{K}_o^d$: the parameter, size, and deviation functional, respectively. We introduce them now.

For $K \in \mathcal{K}_o^d$, we define

$$\mathcal{H}_K := \{H \in \mathcal{H}^d : H \cap K \neq \emptyset\}.$$

We have

$$\mathbb{E}\,\mathrm{card}\,(X \cap \mathcal{H}_K) = \gamma \Phi(K) \tag{19}$$

with

$$\Phi(K) := \gamma^{-1}\Theta(\mathcal{H}_K) = \frac{1}{r}\int_{S^{d-1}} h(K,u)^r \varphi(\mathrm{d}u), \tag{20}$$

as follows from (18). Here, $h(K,u) = \max\{\langle x,u\rangle : x \in K\}$ is the value of the **support function** of K at u. We call Φ the **parameter functional** of the process X, since multiplied by the intensity γ it gives the parameter of the Poisson distribution of the random variable $\mathrm{card}\,(X \cap \mathcal{H}_K)$, for $K \in \mathcal{K}_o^d$:

$$\mathbb{P}(\mathrm{card}\,(X \cap \mathcal{H}_K) = n) = \frac{[\Phi(K)\gamma]^n}{n!} \exp\{-\Phi(K)\gamma\}$$

for $n \in \mathbb{N}_0$. The function Φ is continuous on $\mathcal{K}_o^d$ and homogeneous of degree r, that is, it satisfies $\Phi(\alpha K) = \alpha^r \Phi(K)$ for $\alpha \geq 0$.

The size of the zero cell can be measured by any real function Σ on $K \in \mathcal{K}_o^d$ satisfying only the following natural axioms:

(a) Σ is continuous,
(b) not identically zero,
(c) homogeneous of some degree $k > 0$,
(d) increasing under set inclusion ($K \subset M \Rightarrow \Sigma(K) \leq \Sigma(M)$).

Let a function Σ with these properties be given. We call it the **size functional**. Typical examples are volume, surface area, diameter, inradius.

It is easy to see (using continuity and homogeneity properties) that Φ and Σ satisfy a sharp inequality of isoperimetric type, of the form

$$\Phi(K) \geq \tau\, \Sigma(K)^{r/k} \qquad \text{for } K \in \mathcal{K}_o^d, \tag{21}$$

with some number $\tau > 0$. That the inequality is sharp means that (after the correct choice of τ) there exist convex bodies $K \in \mathcal{K}_o^d$ for which equality holds; every such body is called an **extremal body** (for given Φ and Σ).

We remark that the extremal bodies have the following probabilistic characterization. *Among all convex bodies $K \in \mathcal{K}_o^d$ of size $\Sigma(K) = 1$, precisely the extremal bodies maximize the probability* $\mathbb{P}(K \subset Z_0)$. In fact, if K satisfies the assumptions, then

$$\begin{aligned}&\mathbb{P}(K \subset Z_0) = \mathbb{P}(\mathrm{card}\,(X \cap \mathcal{H}_K) = 0)\\ &= \exp\{-\Phi(K)\gamma\} \leq \exp\{-\tau\Sigma(K)^{r/k}\gamma\} = \mathrm{e}^{-\tau\gamma},\end{aligned}$$

with equality if and only if equality holds in (21).

The realizations and the asymptotic shapes of the zero cell belong to a special class of convex bodies, which we now introduce.

By $\mathrm{supp}\,\varphi$ we denote the support of the direction distribution φ. This is a closed set on S^{d-1}, not lying in a closed hemisphere. We say that a convex body $K \in \mathcal{K}_o^d$ is **φ-adapted** if

$$K = \bigcap_{u \in \mathrm{supp}\,\varphi} H^-(u, h(K,u)),$$

that is, if K is the intersection of its supporting halfspaces which have an outer unit normal vector in the support of φ. The class of all φ-adapted convex bodies in $\mathcal{K}_o^d$ is denoted by $\mathcal{K}_\varphi$. In the subspace of d-dimensional convex bodies, the subset of φ-adapted bodies is closed. It is not difficult to show that the isoperimetric inequality (21) always has extremal bodies which are φ-adapted.

Our third functional measures the deviation of a convex body from the class of extremal bodies in $\mathcal{K}_\varphi$. Again, we introduce it axiomatically. We assume that Φ and Σ are given. A real function ϑ on $\{K \in \mathcal{K}_\varphi : \Sigma(K) > 0\}$ is called a **deviation functional** if

(a) ϑ is continuous,
(b) nonnegative,
(c) homogeneous of degree zero,
(d) $\vartheta(K) = 0$ for some $K \in \mathcal{K}_\varphi$ holds if and only if K is an extremal body.

Such deviation functionals always exist. For example, one could take

$$\vartheta(K) := \frac{\Phi(K)}{\tau \Sigma(K)^{r/k}} - 1. \tag{22}$$

However, in concrete examples, the deviation functional should be chosen in such a way that the deviation has a simple intuitive geometric meaning, and an inequality $\vartheta(K) < \epsilon$ should allow an explicit estimate of some geometric distance of K from an extremal body in $\mathcal{K}_\varphi$.

It follows from the properties of the involved functionals that the inequality (21) admits a strengthening in the form of a stability estimate: there exists a continuous function f with $f(\epsilon) > 0$ for $\epsilon > 0$ and $f(0) = 0$ such that

$$\Phi(K) \geq (1 + f(\epsilon))\tau \Sigma(K)^{r/k} \qquad \text{if} \quad \vartheta(K) \geq \epsilon, \tag{23}$$

for $K \in \mathcal{K}_\varphi$. Any such function f will be called a **stability function** for Φ, Σ, ϑ.

After these preparations, we can formulate a general result.

5.1 Theorem. *Suppose that a Poisson hyperplane process X with direction distribution φ and distance exponent r (which determine the parameter functional Φ), a size functional Σ, a deviation functional ϑ, and a stability function f for Φ, Σ, ϑ as explained are given. With a suitable constant $c_0 > 0$, the following holds. If $\epsilon \in (0, \infty)$ and $I = [a, b)$ is an interval (possibly $b = \infty$) with $a^{r/k}\gamma \geq \sigma_0$, where $\sigma_0 > 0$ is a constant, then*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \in I) \leq c \exp\left\{-c_0 f(\epsilon) a^{r/k}\gamma\right\}, \tag{24}$$

where c is a constant depending only on $\varphi, r, \Sigma, f, \epsilon, \sigma_0$.

Before turning to (a sketch of) the proof of this theorem, we want to explain what it tells us about asymptotic shapes of large cells in more concrete cases. First, we see that (24) implies

$$\lim_{a \to \infty} \mathbb{P}(\vartheta(Z_0) < \epsilon \mid \Sigma(Z_0) \geq a) = 1$$

for every $\epsilon > 0$. Roughly, this shows that zero cells which are 'large' in the sense of Σ have a small deviation from an extremal body, with high probability.

In order to draw precise conclusions about the existence of limit shapes, we introduce a notion of shape. It is common to consider two convex bodies to be of the same shape if they are similar to each other. We need a more general notion. Let G be a subgroup of the group S of similarities of $\mathbb{R}^d$ which contains the group D of all positive dilatations. A typical example is the group H of homotheties. Two convex bodies $K, M \in \mathcal{K}^d$ have the same **G-shape**, also written as $K \sim_{\mathsf{G}} M$, if $K = gM$ with some $g \in \mathsf{G}$. The quotient space $\mathcal{S}_{\mathsf{G}} := \mathcal{K}^d/\!\sim_{\mathsf{G}}$ is called the **space of G-shapes**. Let $s_{\mathsf{G}} : \mathcal{K}^d \to \mathcal{S}_{\mathsf{G}}$ be the projection, thus $s_{\mathsf{G}}(K) = \{gK : g \in \mathsf{G}\}$ is the class of all convex bodies in $\mathcal{K}^d$ having the same G-shape as K.

Let the Poisson hyperplane process X, the zero cell Z_0 and the size functional Σ be as above.

Definition. The **conditional law of the G-shape** of Z_0, given the lower bound a for the size Σ, is the image measure μ_a of the probability measure $\mathbb{P}(Z_0 \in \cdot \mid \Sigma(Z_0) \geq a)$ under the map s_{G}. A shape $s_{\mathsf{G}}(B)$, where $B \in \mathcal{K}_o^d$, is the **limit shape** of Z_0 with respect to Σ if the measure μ_a converges weakly, as $a \to \infty$, to the Dirac measure $\delta_{s_{\mathsf{G}}(B)}$ concentrated at the fixed G-shape $s_{\mathsf{G}}(B)$.

Now we can formulate a general theorem on the existence of limit shapes.

5.2 Theorem. *Let X, Z_0, Σ be as above. Suppose there exists a subgroup G of the group of similarities such that $\mathcal{K}_\varphi$ and the function (22) are invariant under G and that the extremal bodies of the inequality (21) in $\mathcal{K}_\varphi$ have a unique G-shape $s_{\mathsf{G}}(B)$. Then $s_{\mathsf{G}}(B)$ is the limit shape of Z_0 with respect to Σ.*

Proof. We deduce this from Theorem 5.1, assuming that all data are as given in that theorem and ϑ is chosen according to (22). For proving the asserted weak convergence of the measure μ_a, we have to show that

$$\limsup_{a\to\infty} \mu_a(\mathcal{C}) \leq \delta_{s_{\mathsf{G}}(B)}(\mathcal{C}) \tag{25}$$

for every closed set $\mathcal{C} \subset \mathcal{S}_{\mathsf{G}}$. This holds if $s_{\mathsf{G}}(B) \in \mathcal{C}$, hence we assume that $s_{\mathsf{G}}(B) \notin \mathcal{C}$. Every zero cell Z_0 is φ-adapted with probability one, hence $\mu_a(\mathcal{C}) = \mathbb{P}(s_{\mathsf{G}}(Z_0) \in \mathcal{C} \mid \Sigma(Z_0) \geq a) = \mu_a(\mathcal{C} \cap s_{\mathsf{G}}(\mathcal{K}_\varphi))$. Suppose there exists a convex body $K \in \mathcal{K}_\varphi$ such that $s_{\mathsf{G}}(K) \in \mathcal{C}$ and $\vartheta(K) = 0$. Then K is an extremal body. Since it is in $\mathcal{K}_\varphi$, its G-shape is uniquely determined, hence $s_{\mathsf{G}}(K) = s_{\mathsf{G}}(B)$ and thus $s_{\mathsf{G}}(B) \in \mathcal{C}$, a contradiction. Thus, ϑ is positive on $\mathcal{K}_\varphi \cap s_{\mathsf{G}}^{-1}(\mathcal{C})$. Since this set is closed and invariant under positive dilatations, and since the function ϑ is continuous and positively homogeneous of degree zero, ϑ attains a positive minimum ϵ on $\mathcal{K}_\varphi \cap s_{\mathsf{G}}^{-1}(\mathcal{C})$, hence $\mathcal{K}_\varphi \cap s_{\mathsf{G}}^{-1}(\mathcal{C}) \subset \{K \in \mathcal{K}_o^d : \vartheta(K) \geq \epsilon\}$. This gives

$$\mu_a(\mathcal{C}) = \mathbb{P}(Z_0 \in \mathcal{K}_\varphi \cap s_{\mathsf{G}}^{-1}(\mathcal{C}) \mid \Sigma(Z_0 \geq a) \leq \mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \to 0$$

for $a \to \infty$, by Theorem 5.1. This proves Theorem 5.2. □

Some special cases

We consider some special cases of the preceding theorems.

(1) *The zero cell of a stationary Poisson hyperplane process; the size measured by the volume* (this was treated in [1])

This is the higher-dimensional version of Kendall's problem, extended to the non-isotropic case. In this case, $r = 1$, $\Sigma = V_d$, and the parameter functional can be expressed as a mixed volume:

$$\Phi(K) = 2dV_1(B, K) = 2dV(B, \dots, B, K),$$

where B is the convex body with centre 0 for which the direction distribution φ is the area measure; it exists by Minkowski's existence theorem from convex geometry. The isoperimetric inequality (21) is now Minkowski's classical inequality

$$V_1(B, K)^d \geq V_d(B)^{d-1} V_d(K).$$

Equality holds if and only if K is homothetic to B. Hence, $s_{\mathsf{H}}(B)$ is the limit shape of the zero cell with respect to the volume. If the hyperplane process is isotropic, then B is a ball, thus we get a higher dimensional version of Kendall's original problem.

A deviation functional with a simple intuitive meaning is given by

$$r_B(K) := \inf\{s/r - 1 : rB \subset K + z \subset sB,\ z \in \mathbb{R}^d,\ r, s > 0\}. \tag{26}$$

A stability version of Minkowski's inequality due to Groemer then leads to the following deviation estimate:

$$\mathbb{P}(r_B(Z_0) \geq \epsilon \mid V(Z_0) \in [a, b)) \leq c \exp\left\{-c_0 \epsilon^{d+1} a^{1/d} \gamma\right\}.$$

(2) *The typical cell of a stationary Poisson-Voronoi mosaic; the size measured by the kth intrinsic volume* V_k (this is treated in [2], among other results)

In this case, $r = d$, the direction distribution φ is rotation invariant, $\Sigma = V_k$, and the parameter functional is

$$\Phi(K) = \frac{1}{d} \int_{S^{d-1}} h(K, u)^d \, \sigma(\mathrm{d}u).$$

The isoperimetric inequality (21) now reads

$$\Phi(K) \geq \tau(d, k) V_k(K)^{d/k} \tag{27}$$

with an explicit constant $\tau(d, k)$, which is obtained by combining Hölder's inequality with the Aleksandrov-Fenchel inequalities. The extremal bodies are precisely the balls with centre 0, hence the set of centred balls is the limit shape of the typical cell Z with respect to V_k. A convenient deviation functional is given by

$$\vartheta(K) := \frac{R_o(K) - \rho_o(K)}{R_o(K) + \rho_o(K)}, \tag{28}$$

where $R_o(K)$ (respectively $\rho_o(K)$) is the smallest (largest) ball with centre 0 containing K (contained in K). Using this deviation function, a stability version of (27) can be proved; the obtained estimate corresponding to (24) is

$$\mathbb{P}(\vartheta(Z) \geq \epsilon \mid V_k(Z) \in [a, b)) \leq c \exp\left\{-c_0 \epsilon^{(d+3)/2} a^{d/k} \gamma\right\}.$$

(3) *The zero cell of a stationary, nonisotropic Poisson hyperplane process; the size measured by the inradius*

For a convex body $K \in \mathcal{K}^d$, the inradius $\rho(K)$ is the radius of a largest ball contained in K. For the zero cell Z_0 of a stationary and isotropic Poisson hyperplane process X it was proved in [2] that the limit shape with respect to the inradius is the class of balls. We are now in a position to treat the nonisotropic case. In this case, the consideration of φ-adapted convex bodies is essential.

Since X is stationary, the direction distribution φ is even, hence the parameter functional

$$\Phi(K) = \int_{S^{d-1}} h(K, u)\, \varphi(\mathrm{d}u), \qquad K \in \mathcal{K}^d,$$

is translation invariant. We may therefore assume that 0 is the centre of a largest ball contained in K. Then $h(K, u) \geq \rho(K)$, hence

$$\Phi(K) \geq \rho(K). \tag{29}$$

Equality holds if and only if $h(K, u) = \rho(K)$ for all u in the support of the measure φ. Thus, equality in (29) holds for the convex body

$$B_\varphi := \bigcap_{u \in \operatorname{supp} \varphi} H^-(B^d, 1),$$

and for $K \in \mathcal{K}_\varphi$ equality in (29) holds if and only if K is homothetic to B_φ (in general, there are many convex bodies not in $\mathcal{K}_\varphi$ which yield equality in (29)). Thus, $s_{\mathsf{H}}(B_\varphi)$ is the limit shape of Z_0 with respect to the inradius ρ.

A stability improvement of (29) involving a simple geometrically resasonable deviation functional, like (26) or (28), can apparently not be achieved without special assumptions on the direction distribution φ.

In the nonstationary case, the parameter functional

$$\Phi(K) = \frac{1}{r} \int_{S^{d-1}} h(K,u)^r \, \varphi(\mathrm{d}u)$$

is not translation invariant, therefore we replace the inradius $\rho(K)$ by the centred inradius $\rho_o(K)$. As above, we obtain

$$\Phi(K) \geq \frac{1}{r} \rho_o(K)^r \qquad \text{for } K \in \mathcal{K}_o^d,$$

with equality for $K \in \mathcal{K}_\varphi$ if and only if K is a dilate of B_φ. Hence, $s_{\mathsf{D}}(B_\varphi)$ is the limit shape of Z_0 with respect to the centred inradius ρ_o.

(4) *The zero cell of a stationary, isotropic Poisson hyperplane process; the size measured by the diameter*

Let D denote the diameter. If $K \in \mathcal{K}^d$, then K contains a segment of length $D(K)$, without loss of generality with centre at 0. We conclude that

$$\Phi(K) \geq \kappa_{d-1} D(K),$$

with equality if and only if K is a segment. Thus, the limit shape of Z_0 with respect to the diameter is the class of segments.

A suitable deviation functional $\eta(K)$ can be defined as the Hausdorff distance of K from the set of all segments, divided by the diameter D(K). With this choice, the deviation estimate

$$\mathbb{P}(\eta(Z_0) \geq \epsilon \mid D(Z_0) \in [a,b)) \leq c \, \exp\left\{-c_0 \epsilon^2 a \gamma\right\}$$

can be proved.

(5) *The typical cell of a stationary Poisson-Voronoi process; the size measured by the largest distance of a vertex from the nucleus*

Similarly as above, one obtains the inequality

$$\Phi(K) \geq \tau(d) R_o(K)^d$$

with an explicit constant $\tau(d)$. Equality holds if and only if K is a segment with one endpoint at 0. Thus, in this case the limit shape is the class of all segments with one endpoint at the origin.

6 Principle ideas of the proof

To explain the approach to Theorem 5.1, we first extend a heuristic argument from [1], trying to make plausible why an estimate as in Theorem 5.1 can be expected. In these heuristics, we restrict ourselves to an interval $I = [a, \infty)$, with $a > 0$. We have to estimate the conditional probability

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) = \frac{\mathbb{P}(\vartheta(Z_0) \geq \epsilon, \ \Sigma(Z_0) \geq a)}{\mathbb{P}(\Sigma(Z_0) \geq a)}.$$

Estimation of the denominator is easy. As mentioned above, there exists an extremal body $B \in \mathcal{K}_\varphi$. Let B_a be the dilate of B with $\Sigma(B_a) = a$. Then, by the monotonicity of Σ,

$$\mathbb{P}(\Sigma(Z_0) \geq a) \geq \mathbb{P}(\mathrm{card}\,(X \cap \mathcal{H}_{B_a}) = 0) = \exp\{-\Phi(B_a)\gamma\}.$$

Since B_a is an extremal body, we have

$$\Phi(B_a) = \tau \Sigma(B_a)^{r/k} = \tau a^{r/k}, \tag{30}$$

hence

$$\mathbb{P}(\Sigma(Z_0) \geq a) \geq \exp\{-\tau a^{r/k}\gamma\}. \tag{31}$$

For the estimation of the numerator, we try to compare the occurring zero cells with a deterministic convex body with similar properties, that is, not cut by hyperplanes of the process, with large size and large deviation from B. Suppose that $K \in \mathcal{K}_\varphi$ is a convex body satisfying

$$\vartheta(K) \geq \epsilon > 0 \qquad \text{and} \qquad \Sigma(K) \geq a.$$

Then, by (23),

$$\mathbb{P}(\mathrm{card}\,(X \cap \mathcal{H}_K) = 0) = \exp\{-\Phi(K)\gamma\} \leq \exp\{-(1+f(\epsilon))\tau a^{r/k}\gamma\}.$$

Heuristically, we hope that here we may replace the deterministic body K satisfying

$$\mathrm{card}\,(X \cap \mathcal{H}_K) = 0, \qquad \vartheta(K) \geq \epsilon, \qquad \Sigma(K) \geq a$$

by the random polytope Z_0 satisfying

$$\mathrm{card}\,(X \cap \mathcal{H}_{\mathrm{int}\, Z_0}) = 0, \qquad \vartheta(Z_0) \geq \epsilon, \qquad \Sigma(Z_0) \geq a,$$

at the cost of only a slight weakening of the inequality, say

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \geq a) \leq c' \exp\{-(1+c'' f(\epsilon))\tau a^{r/k}\gamma\} \tag{32}$$

with $c', c'' > 0$. If (32) can be proved, then together with (31) this implies

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \leq c' \exp\{-c'' f(\epsilon)\tau a^{r/k}\gamma\},$$

which is of the form asserted in Theorem 5.1. The bulk of the proof consists in replacing this heuristic argument by precise reasoning.

The actual proof is too technical to allow more than a short sketch of some ideas. Returning to the general case, we have to estimate the conditional probability

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \in a(1, 1+h)) = \frac{\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \in a(1, 1+h))}{\mathbb{P}(\Sigma(Z_0) \in a(1, 1+h))},$$

where $\Sigma(Z_0)$ now ranges in an interval $(a,b) = a(1,1+h)$. In a first stage, this is only possible for sufficiently small positive h. The case of a more general range can later be deduced from the local version by means of a covering argument. We concentrate on the more difficult part, the estimation of the numerator. Here it is convenient to first assume that $h = 1$; a transformation will later give the general case. Thus we are aiming at an upper estimate for the probability

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \in (a, 2a)).$$

The random polytope Z_0 can, in principle, have an arbitrarily large diameter and arbitrarily many facets. To deal with this, we introduce the 'relative diameter'

$$\delta(K) := \frac{D(K))}{c\Sigma(K)^{1/k}} \qquad \text{for } K \in \mathcal{K}_0^d,$$

where D is the diameter and the constant c is chosen so that $\delta(K) \geq 1$ and the value 1 is attained. Putting

$$\mathcal{K}_{a,\epsilon}(m) := \{K \in \mathcal{K}_\varphi : \vartheta(K) \geq \epsilon,\ \Sigma(K) \in (a, 2a),\ \delta(K) \in [m, m+1)\}$$

and

$$q_{a,\epsilon}(m) := \mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon}(m)),$$

we have

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \in (a, 2a)) = \sum_{m=1}^{\infty} q_{a,\epsilon}(m).$$

The reason for introducing the additional restriction $\delta(Z_0) \in [m, m+1)$ lies in the fact that it allows us to consider in a first step only zero cells lying in some fixed bounded set. More precisely:

If $Z_0 \in \mathcal{K}_{a,\epsilon}(m)$ then

$$Z_0 \subset c_1 m a^{1/k} B^d =: C, \tag{33}$$

and Z_0 has a vertex v with

$$\|v\| \geq c_2 m a^{1/k}. \tag{34}$$

By $c_1, c_2, \dots$ we denote constants depending on various data, but not on a.

Now comes the moment to exploit the fact that the hyperplane process X is Poisson. The essential property is Proposition 1.2. Therefore, we consider separately each case where the set C defined by (33) is hit by exactly N hyperplanes of the process. We have

$$q_{a,\epsilon}(m) = \sum_{N=d+1}^{\infty} \mathbb{P}(\mathrm{card}\,(X \cap \mathcal{H}_C) = N)\, \mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon}(m) \mid \mathrm{card}\,(X \cap \mathcal{H}_C) = N).$$

By the Poisson distribution,

$$\mathbb{P}(\mathrm{card}\,(X \cap \mathcal{H}_C) = N) = \frac{[\Phi(C)\gamma]^N}{N!} \exp\{-\Phi(C)\gamma\}.$$

We must estimate the conditional probability

$$p_N := \mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon}(m) \mid \mathrm{card}\,(X \cap \mathcal{H}_C) = N).$$

By Proposition 1.2,

$$p_N = \frac{1}{[\Phi(C)\gamma]^N} \int_{\mathcal{H}_C} \cdots \int_{\mathcal{H}_C} \mathbf{1}\{H_1^- \cap \cdots \cap H_N^- \in \mathcal{K}_{a,\epsilon}(m)\}\, \Theta(\mathrm{d}H_1) \cdots \Theta(\mathrm{d}H_N).$$

Suppose that the integrand is equal to 1, that is,

$$P(H_{(N)}) := H_1^- \cap \cdots \cap H_N^- \in \mathcal{K}_{a,\epsilon}(m),$$

in particular $\vartheta(P(H_{(N)})) \geq \epsilon$. Thus the polytope $P(H_{(N)})$ is not too close to an extremal body of the isoperimetric inequality (21). We choose an extremal body B_a with $\Sigma(B_a) = a$. By the stability version (23) of the isoperimetric inequality,

$$\Phi(P(H_{(N)})) \geq (1 + f(\epsilon))\Phi(B_a).$$

In principle, the polytope $P(H_{(N)})$ can have as many as N facets. For an effective estimation, we must restrict its number of vertices. Using an approximation theorem from convex geometry, we can show, for given $\alpha > 0$, the existence of a number ν independent of N such that the convex hull $Q(H_{(N)})$ of ν suitably chosen vertices of $P(H_{(N)})$ satisfies

$$\Phi(Q(H_{(N)})) \geq (1 - \alpha)\Phi(P(H_{(N)})).$$

With $g(\epsilon) := f(\epsilon)/(2 + f(\epsilon))$ we obtain

$$\Phi(Q(H_{(N)})) \geq (1 + g(\epsilon))\Phi(B_a). \tag{35}$$

For each N-tuple $(H_1, \ldots, H_N)$ such that $P(H_{(N)}) \in \mathcal{K}_{a,\epsilon}(m)$, we make a definite choice of $Q = Q(H_{(N)})$. This selection can be made so that $Q(H_{(N)})$ is a measurable function of $(H_1, \ldots, H_N)$.

Excluding a set of N-tuples $(H_1, \ldots, H_N)$ of Θ^N measure zero, we can assume that each of the vertices of Q lies in precisely d of the hyperplanes $H_1, \ldots, H_N$, and the remaining hyperplanes are disjoint from Q. Hence, at most $d\nu$ of the hyperplanes $H_1, \ldots, H_N$ meet Q; let $j \in \{d+1, \ldots, d\nu\}$ denote their precise number. This leads to

$$
\begin{aligned}
&p_N[\Phi(C)\gamma]^N \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \underbrace{\int_{\mathcal{H}_C} \cdots \int_{\mathcal{H}_C}}_{j} \Bigg[\underbrace{\int_{\mathcal{H}_C} \cdots \int_{\mathcal{H}_C}}_{N-j} \\
&\quad \mathbf{1}\left\{P(H_{(N)}) \in \mathcal{K}_{a,\epsilon}(m)\right\} \\
&\quad \mathbf{1}\{H_i \cap Q(H_{(N)}) \neq \emptyset \text{ for } i = 1, \ldots, j\} \\
&\quad \mathbf{1}\{H_i \cap Q(H_{(N)}) = \emptyset \text{ for } i = j+1, \ldots, N\}\, \Theta(\mathrm{d}H_{j+1}) \cdots \Theta(\mathrm{d}H_N) \Bigg] \\
&\hspace{20em} \Theta(\mathrm{d}H_1) \cdots \Theta(\mathrm{d}H_j).
\end{aligned}
$$

If the integrand is equal to 1, then (35) holds. Since, for any convex body $K \subset C$,

$$\int_{\mathcal{H}_C} \mathbf{1}\{H \cap K = \emptyset\}\, \Theta(\mathrm{d}H) = \Phi(C)\gamma - \Phi(K)\gamma,$$

the integral in brackets (where $Q(H_{(N)})$ is fixed and independent of $H_{j+1}, \ldots, H_N$) can be estimated by

$$[\ldots] \leq [\Phi(C)\gamma - \Phi(Q(H_{(N)}))\gamma]^{N-j} \leq [\Phi(C)\gamma - (1+g(\epsilon))\Phi(B_a)\gamma]^{N-j},$$

and we obtain

$$p_N[\Phi(C)\gamma]^N \leq \sum_{j=d+1}^{d\nu} \binom{N}{j} [\Phi(C)\gamma - (1+g(\epsilon))\Phi(B_a)\gamma]^{N-j} [\Phi(C)\gamma]^j.$$

Here

$$\Phi(C) = c_3 m^r a^{r/k},$$

by the definition (33) of C and the homogeneity of Φ. Summation over N finally leads to

$$q_{a,\epsilon}(m) \leq c_4 m^{rd\nu} \exp\{-(1+f(\epsilon)/3)\tau a^{r/k}\gamma\}.$$

This estimate can be applied for small numbers m. For large m, the estimate

$$q_{a,\epsilon}(m) \leq c_5 \exp\{-c_6 m^r a^{r/k}\gamma\}$$

is used, which is obtained in a similar though somewhat easier way, using (34).

Now we have to combine both estimates and extend the considered range of $\Sigma(Z_0)$ from intervals $a(1,2)$ to intervals $a(1,1+h)$. This extension is achieved by a kind of transformation. We end up with the following estimate for the numerator of our conditional probability:

6.1 Lemma. *Let $\epsilon \in (0,1)$, $h \in (0,1/2)$ and $a^{r/k}\gamma \geq \sigma_0$, where $\sigma_0 > 0$ is a constant. Then*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \in a(1, 1+h)) \leq c_7 h \exp\left\{-(1+f(\epsilon)/6)\tau a^{r/k}\gamma\right\}.$$

Since this upper bound for the numerator contains the number h as a factor, it is necessary to estimate the denominator from below by a suitable bound which is also linear in h, so that this factor cancels out. This is achieved by the following lemma.

6.2 Lemma. *For each $\beta > 0$, there are constants $h_0 > 0$, $N \in \mathbb{N}$ and $c_8 > 0$ such that, for $a > 0$ and $0 < h < h_0$,*

$$\mathbb{P}(\Sigma(Z_0) \in a(1, 1+h)) \geq c_8 h (a^{r/k}\lambda)^N \exp\{-(1+\beta)\tau a^{r/k}\lambda\}.$$

The proof of this lemma is essentially constructive, exhibiting sufficiently many situations in which the event $\Sigma(Z_0) \in a(1, 1+h)$ occurs. In both lemmas the number h must be sufficiently small. The final proof of Theorem 1 extends the estimates from the special intervals $a(1, 1+h)$, with small h, to general intervals (a, b) by a covering argument.

The complete proof requires many more details, but already this sketch should make clear how essential the Poisson assumption was. Without it, we could not have worked with finitely many independent hyperplanes, that is, with product integrals over spaces like $\mathcal{H}_C \times \cdots \times \mathcal{H}_C$, and would not have been able to deduce any general estimate.

Hints to the literature. General information about random mosaics can be found in [13] and [12]; random Voronoi tessellations are treated in [9]. First solutions of Kendall's problem in the plane are due to Kovalenko [4], [5], [6]. Higher-dimensional versions of Kendall's problem were investigated in [1] and [2]. The general theorem 5.1 is contained in [3], together with further results.

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