The polytopes in a Poisson hyperplane tessellation

Rolf Schneider

Abstract

For a stationary Poisson hyperplane tessellation X in $\mathbb{R}^d$, whose directional distribution satisfies some mild conditions (which hold in the isotropic case, for example), it was recently shown that with probability one every combinatorial type of a simple d-polytope is realized infinitely often by the polytopes of X. This result is strengthened here: with probability one, every such combinatorial type appears among the polytopes of X not only infinitely often, but with positive density.

2010 Mathematics Subject Classification. Primary 60D05, Secondary 51M20, 52C22

Key words and phrases. Poisson hyperplane tessellation; ergodicity; simple polytope; combinatorial type; density in a tessellation

1 Introduction

Imagine a system H of hyperplanes in Euclidean space $\mathbb{R}^d$ ($d \geq 2$) that induces a tessellation T_H of $\mathbb{R}^d$. This means that any bounded subset of $\mathbb{R}^d$ meets only finitely many hyperplanes of H and that the components of $\mathbb{R}^d \setminus \bigcup_{H\in\mathsf{H}} H$ are bounded. The closures of these components are then convex polytopes which cover $\mathbb{R}^d$ and have pairwise no common interior points. The set of these polytopes is denoted by T_H. We impose the additional assumption that the hyperplanes of H are in general position; then each polytope of T_H is simple, that is, each of its vertices is contained in precisely d facets. The polytopes appearing in T_H may be rather boring; they could, for example, all be parallelepipeds. However, if the hyperplanes of H have sufficiently many different directions, one can imagine that quite different shapes of polytopes appear in T_H. Is it possible that every combinatorial type of a simple d-polytope is realized in T_H? This can be achieved in a much stronger sense.

In fact, suppose that $\widehat{X}$ is a stationary and isotropic Poisson hyperplane process in $\mathbb{R}^d$ (explanations are found in [7], for example). Its hyperplanes are almost surely in general position and induce a random tessellation of $\mathbb{R}^d$, denoted by X. The general character of the polytopes in X was recently investigated in [4]. For example, it was shown there that almost surely (a.s.) the translates of the polytopes in X are dense in the space of convex bodies in $\mathbb{R}^d$ (with the Hausdorff metric). Another result was that a.s. the polytopes of X realize every combinatorial type of a simple d-polytope infinitely often. In the following, we improve the latter result considerably, replacing 'infinitely often' by 'with positive density'. In the subsequent definition, B_n is the ball in $\mathbb{R}^d$ with center at the origin and radius $n \in \mathbb{N}$, and λ_d denotes Lebesgue measure in $\mathbb{R}^d$. Further, $\mathbb{1}_A$ is the indicator function of A.

Definition 1. *Let T be a tessellation of $\mathbb{R}^d$, and let A be a translation invariant set of polytopes in $\mathbb{R}^d$. We say that A* **appears in T with density** δ *if*

$$\liminf_{n\to\infty} \frac{1}{\lambda_d(B_n)} \sum_{P\in T,\, P\subset B_n} \mathbb{1}_A(P) = \delta.$$

With this definition, we prove below that in a Poisson hyperplane tessellation in $\mathbb{R}^d$ which is stationary and isotropic (that is, has a motion invariant distribution), almost surely every combinatorial type of a simple d-polytope appears with positive density. The actual result will, in fact, be more general: it is sufficient that the Poisson hyperplane tessellation is stationary and that its directional distribution, a measure on the unit sphere, is not zero on any nonempty open set and is zero on any great subsphere. The precise theorem is formulated in the next section.

2 Explanations

We work in the d-dimensional Euclidean space $\mathbb{R}^d$ ($d \geq 2$) with its usual scalar product $\langle \cdot\,, \cdot \rangle$. By λ_d we denote its Lebesgue measure, by o its origin, by B^d its unit ball (with $nB^d =: B_n$), and by $\mathbb{S}^{d-1}$ its unit sphere. The space of hyperplanes in $\mathbb{R}^d$, with its usual topology, is denoted by $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})$ is the σ-algebra of Borel sets in $\mathcal{H}$. Hyperplanes in $\mathbb{R}^d$ are often written in the form

$$H(u, \tau) = \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau\}$$

with $u \in \mathbb{S}^{d-1}$ and $\tau \in \mathbb{R}$.

We assume that $\widehat{X}$ is a stationary Poisson hyperplane process in $\mathbb{R}^d$, thus, a Poisson process in the space $\mathcal{H}$ of hyperplanes, with the property that its distribution is invariant under translations (we refer, e.g., to [7] for more details). The *intensity measure* $\widehat{\Theta}$ of $\widehat{X}$ is defined by

$$\widehat{\Theta}(A) = \mathbb{E}\,\widehat{X}(A) \quad \text{for } A \in \mathcal{B}(\mathcal{H}).$$

Here $\mathbb{E}$ denotes expectation, and we write $(\Omega, \mathcal{A}, \mathbb{P})$ for the underlying probability space. It is assumed that $\widehat{\Theta}$ is locally finite and not identically zero. That $\widehat{X}$ is a Poisson process includes that

$$\mathbb{P}(\widehat{X}(A) = k) = e^{-\widehat{\Theta}(A)} \frac{\widehat{\Theta}(A)^k}{k!} \quad \text{for } k \in \mathbb{N}_0,$$

for any $A \in \mathcal{B}(\mathcal{H})$ with $\widehat{\Theta}(A) < \infty$.

Since $\widehat{X}$ is stationary, the measure $\widehat{\Theta}$ has a decomposition

$$\widehat{\Theta}(A) = \widehat{\gamma} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} \mathbb{1}_A(H(u, \tau))\, \mathrm{d}\tau\, \varphi(\mathrm{d}u)$$

for $A \in \mathcal{B}(\mathcal{H})$ (see [7], Theorem 4.4.2 and (4.33)). The number $\widehat{\gamma} > 0$ is the *intensity* of $\widehat{X}$, and φ is a finite, even Borel measure on the unit sphere. It is called the *spherical directional distribution* of $\widehat{X}$. For any such measure φ and any number $\widehat{\gamma} > 0$, there exists a stationary Poisson hyperplane process in $\mathbb{R}^d$ with these data, and it is unique up to stochastic equivalence.

The hyperplane process $\widehat{X}$ induces a random tessellation of $\mathbb{R}^d$, which we denote by X. As usual, a random tessellation is formalized as a particle process; we refer again to [7].

Since we are considering only simple processes, it is convenient to identify such a process, which by definition is a counting measure, with its support, which is a locally finite set. In particular, a realization $\widehat{X}(\omega)$ of $\widehat{X}$ is also considered as a set of hyperplanes, and a realization of X is considered as a set of polytopes. The notations $\widehat{X}(\omega)(\{H\}) = 1$ and $H \in \widehat{X}(\omega)$ for a hyperplane H, for example, are therefore used synonymously.

The combinatorial type of a polytope P in $\mathbb{R}^d$ is the set of all polytopes in $\mathbb{R}^d$ that are combinatorially isomorphic to P. Now we can formulate our result.

Theorem 1. *Let X be a tessellation of $\mathbb{R}^d$ that is induced by a stationary Poisson hyperplane process $\widehat{X}$ with spherical directional distribution φ. Suppose that the support of φ is the whole unit sphere $\mathbb{S}^{d-1}$ and that φ assigns measure zero to each great subsphere of $\mathbb{S}^{d-1}$. Then, with probability one, each combinatorial type of a simple d-polytope appears with positive density in X.*

Theorem 1 implies, trivially, that under its assumptions almost surely each combinatorial type of a simple d-polytope appears infinitely often in X. When the latter fact was proved, among other results, in [4], a tool was a strengthened version of the Borel–Cantelli lemma, due to Erdös and Rényi [3] (see also [5, p. 327]). When the note [4] was submitted, an anonymous referee wrote “that the use of ergodicity of the mosaic could lead to a possibly shorter alternative proof”, and he/she briefly indicated a possible approach. After thorough consideration, we preferred the more elementary Borel–Cantelli lemma. However, reconsideration revealed that ergodicity, applied in a different way, would lead to a stronger result, as far as the occurrence of combinatorial types is concerned. This is carried out in the following.

3 Proof

Let X satisfy the assumptions of Theorem 1. Under the only assumption that the spherical directional distribution of the stationary Poisson hyperplane tessellation X is zero on every great subsphere, it was shown in [7, Thm. 10.5.3] that X is mixing and hence ergodic. This requires a few explanations. To model X as a point process, we consider the space $\mathcal{K}$ of convex bodies (nonempty, compact, convex subsets) in $\mathbb{R}^d$ with the Hausdorff metric. By $\mathcal{B}(\mathcal{K})$ we denote the σ-algebra of Borel sets in $\mathcal{K}$. Let $\mathsf{N}_s(\mathcal{K})$ be the set of simple, locally finite counting measures on $\mathcal{B}(\mathcal{K})$ and $\mathcal{N}_s(\mathcal{K})$ its usual σ-algebra (for details see, e.g., [7, Sect. 3.1]). As underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$, on which X is defined, we can use $(\mathsf{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X)$, where $\mathbb{P}_X$ is the distribution of X. For $t \in \mathbb{R}^d$, a bijective map $\mathsf{T}_t : \eta \mapsto \mathsf{T}_t\eta$ of $\mathsf{N}_s(\mathcal{K})$ onto itself is defined by

$$(\mathsf{T}_t\eta)(B) := \eta(B - t), \quad B \in \mathcal{B}(\mathcal{K}),\ \eta \in \mathsf{N}_s(\mathcal{K}).$$

Since X is stationary, we have

$$\mathbb{P}_X(\mathsf{T}_t A) = \mathbb{P}_X(A) \quad \text{for } A \in \mathcal{N}_s(\mathcal{K}),$$

thus T_t induces a measure preserving map of $\mathcal{N}_s(\mathcal{K})$ into itself. Let $\mathcal{T} := \{\mathsf{T}_t : t \in \mathbb{R}^d\}$. As shown in [7, Thm. 10.5.3], the dynamical system $(\mathsf{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X, \mathcal{T})$ is mixing, that is,

$$\lim_{\|t\| \to \infty} \mathbb{P}_X(A \cap \mathsf{T}_t B) = \mathbb{P}_X(A)\mathbb{P}_X(B)$$

holds for all $A, B \in \mathcal{N}_s(\mathcal{K})$. It follows that the system is ergodic, which means that $\mathbb{P}_X(A) \in \{0, 1\}$ for all $A \in \mathbf{T} := \{A \in \mathcal{N}_s(\mathcal{K}) : \mathsf{T}_t A = A \text{ for all } t \in \mathbb{R}^d\}$. Therefore, the ‘Individual Ergodic Theorem for d-dimensional Shifts’ yields the following.

Proposition 1. *Let f be an integrable random variable on $(\mathsf{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X)$. Then*

$$\lim_{n \to \infty} \frac{1}{\lambda_d(B_n)} \int_{B_n} f(\mathsf{T}_t\,\omega)\,\lambda(\mathrm{d}t) = \mathbb{E}\, f$$

holds for $\mathbb{P}_X$-almost all $\omega \in \mathsf{N}_s(\mathcal{K})$.

We refer to Daley and Vere–Jones [2, Proposition 12.2.II] for a more general formulation (with hints to proofs of more general results in Tempel'man [6]). However, we have already incorpated into our Proposition 1 the information that in our case $(\mathsf{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X, \mathcal{T})$ is ergodic, which yields that the limit is equal to the expectation of f.

We apply this Proposition in the following way. First we choose a center function c on $\mathcal{K}$; for example, let $c(K)$ denote the circumcenter of $K \in \mathcal{K}$, which is the center of the smallest ball containing K. Let $A \in \mathcal{B}(\mathcal{K})$ be a translation invariant Borel set of convex bodies. Given any bounded Borel set $B \in \mathcal{B}(\mathbb{R}^d)$, we define

$$f(B, \omega) := \sum_{K \in X(\omega),\, c(K) \in B} \mathbb{1}_A(K)$$

for $\omega \in \Omega$, where we use $(\Omega, \mathcal{A}, \mathbb{P}) = (\mathsf{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X)$ as the underlying probability space. Then $f(B, \cdot)$ is measurable, and $f(B + t, \omega) = f(B, \mathsf{T}_{-t}\, \omega)$ for $t \in \mathbb{R}^d$. The following generalizes an approach of Cowan [1] in the plane ("Tricks with small disks"). Assuming that $n > 1$, we have

$$\begin{aligned}
&\int_{B_{n-1}} f(B_1 + t, \omega)\, \lambda_d(\mathrm{d}t) \\
&= \sum_{K \in X(\omega)} \int_{\mathbb{R}^d} \mathbb{1}\{t \in B_{n-1}\} \mathbb{1}\{K \in A\} \mathbb{1}\{c(K) \in B_1 + t\}\, \lambda_d(\mathrm{d}t).
\end{aligned}$$

Since

$$\mathbb{1}\{t \in B_{n-1}\} \mathbb{1}\{c(K) \in B_1 + t\} \leq \mathbb{1}\{t \in -B_1 + c(K)\} \mathbb{1}\{c(K) \in B_n\},$$

we get

$$\begin{aligned}
&\int_{B_{n-1}} f(B_1 + t, \omega)\, \lambda_d(\mathrm{d}t) \\
&\leq \sum_{K \in X(\omega)} \int_{\mathbb{R}^d} \mathbb{1}\{t \in -B_1 + c(K)\} \mathbb{1}\{K \in A\} \mathbb{1}\{c(K) \in B_n\}\, \lambda_d(\mathrm{d}t) \\
&= \lambda_d(B_1) f(B_n, \omega).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{B_{n+1}} f(B_1 + t, \omega)\, \lambda_d(\mathrm{d}t) \\
&\geq \sum_{K \in X(\omega)} \int_{\mathbb{R}^d} \mathbb{1}\{t \in -B_1 + c(K)\} \mathbb{1}\{K \in A\} \mathbb{1}\{c(K) \in B_n\}\, \lambda_d(\mathrm{d}t) \\
&= \lambda_d(B_1) f(B_n, \omega).
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\frac{\lambda_d(B_{n-1})}{\lambda_d(B_n)} \frac{1}{\lambda_d(B_{n-1})} \int_{B_{n-1}} f(B_1, \mathsf{T}_{-t}\, \omega)\, \lambda_d(\mathrm{d}t) \\
&\leq \frac{\lambda_d(B_1)}{\lambda_d(B_n)} f(B_n, \omega) \\
&\leq \frac{\lambda_d(B_{n+1})}{\lambda_d(B_n)} \frac{1}{\lambda_d(B_{n+1})} \int_{B_{n+1}} f(B_1, \mathsf{T}_{-t}\, \omega)\, \lambda_d(\mathrm{d}t).
\end{aligned}$$

By Proposition 1, the lower and the upper bound converge, for $n \to \infty$, almost surely to $\mathbb{E}\, f(B_1, \cdot)$, hence a.s.

$$\lim_{n\to\infty} \frac{1}{\lambda_d(B_n)} f(B_n, \cdot) = \frac{\mathbb{E}\, f(B_1, \cdot)}{\lambda_d(B_1)}. \tag{1}$$

Now we assume in addition that there is a constant $D > 0$ such that all convex bodies $K \in A$ satisfy $\operatorname{diam} K \le D$, where diam denotes the diameter. The center function c satisfies $c(K) \in K$, hence if $c(K) \in B_{n-D}$ (with $n > D$) and $\operatorname{diam} K \le D$, then $K \subset B_n$. It follows that, for $n > D$,

$$\begin{aligned}
&\frac{\lambda_d(B_{n-D})}{\lambda_d(B_n)} \frac{1}{\lambda_d(B_{n-D})} \sum_{K\in X} \mathbb{1}_A(K)\mathbb{1}\{c(K) \in B_{n-D}) \\
&\le \frac{1}{\lambda_d(B_n)} \sum_{K\in X,\, K\subset B_n} \mathbb{1}_A(K) \\
&\le \frac{1}{\lambda_d(B_n)} \sum_{K\in X} \mathbb{1}_A(K \in A)\mathbb{1}\{c(K) \in B_n\}.
\end{aligned}$$

As $n \to \infty$, the lower and the upper bound converge a.s. to the right side of (1), hence a.s. we have

$$\delta(X, A) := \lim_{n\to\infty} \frac{1}{\lambda_d(B_n)} \sum_{K\in X,\, K\subset B_n} \mathbb{1}_A(K) = \frac{1}{\lambda_d(B^d)}\, \mathbb{E} \sum_{K\in X,\, c(K)\in B^d} \mathbb{1}_A(K). \tag{2}$$

Now we consider the special case where A_D is the set of polytopes that are combinatorially isomorphic to a given simple d-polytope P and have diameter at most D, for some fixed number $D > 0$. We remark that (2) shows that

$$\delta(X, A_D) = \frac{1}{\lambda_d(B^d)}\, \mathbb{E} \sum_{K\in X,\, c(K)\in B^d} \mathbb{1}\{K \in A_D\}, \tag{3}$$

It remains to show that

$$\mathbb{E} \sum_{K\in X,\, c(K)\in B^d} \mathbb{1}\{K \in A_D\} > 0. \tag{4}$$

For this, we use an argument from [4], which we recall for completeness.

Without loss of generality, we can assume that $c(P) = o$. Let $F_1, \dots, F_m$ be the facets of P. We denote by $B(x, \varepsilon)$ the ball with center x and radius $\varepsilon > 0$, set $[B(x, \varepsilon)]_{\mathcal{H}} := \{H \in \mathcal{H} : H \cap B(x, \varepsilon) \neq \emptyset\}$, and define

$$A_j(P, \varepsilon) := \bigcap_{v\in \mathrm{vert} F_j} [B(v, \varepsilon)]_{\mathcal{H}}, \quad j = 1, \dots, m,$$

where vert denotes the set of vertices. Each hyperplane from $A_j(P, \varepsilon)$ is said to be *ε-close* to F_j. A polytope Q is said to be *ε-close* to P if it has m facets $G_1, \dots, G_m$ and, after suitable renumbering, the affine hull of G_j is ε-close to F_j, for $j = 1, \dots, m$. Since P is simple and $c(P) = o$, we can choose numbers $D, \varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, the following is true:

- the sets $A_1(P, \varepsilon), \dots, A_m(P, \varepsilon)$ are pairwise disjoint, and any hyperplanes $H_j \in A_j(P, \varepsilon)$, $j = 1, \dots, m$, are the facet hyperplanes of a polytope Q that is ε-close to P.

• Any polytope Q that is ε-close to P satisfies the following:
• Q is combinatorially isomorphic to P,
• $Q \subset P + B^d$,
• $\operatorname{diam} Q \leq D$,
• $c(Q) \in B^d$.
That this can be achieved by suitable choices of D and ε_0, follows from easy continuity considerations and the fact that P is simple.

Now we define

$$C(P, \varepsilon) := \{H \in \mathcal{H} : H \cap (P + B^d) \neq \emptyset,\ H \notin A_j(P, \varepsilon) \text{ for } j = 1, \dots, m\}$$

and consider the event $E(P, \varepsilon)$ defined by

$$\widehat{X}(A_j(P, \varepsilon)) = 1 \text{ for } j = 1, \dots, m \text{ and } \widehat{X}(C(P, \varepsilon)) = 0.$$

Let $0 < \varepsilon \leq \varepsilon_0$. The following was proved in [4]:
• If the event $E(P, \varepsilon)$ occurs, then some polytope Q of the tessellation X is ε-close to P and hence satisfies $Q \in A_D$ and $c(Q) \in B^d$,
• The event $\mathbb{P}(E(P, \varepsilon))$ has positive probability.

Now it follows that

$$\mathbb{E} \sum_{K \in X,\, c(K) \in B^d} \mathbb{1}\{K \in A_D\} \geq \mathbb{P}(E(P, \varepsilon)) > 0,$$

which proves (4).

The result is that $\delta(X, A_D) > 0$ a.s. This implies, in particular, that with probability one the polytopes of the combinatorial type of P appear in X with positive density. Since there are only countably many combinatorial types, it also holds with probability one that each combinatorial type of a simple d-polytope appears in X with positive density.

References

[1] Cowan, R., Properties of ergodic random mosaic processes. *Math. Nachr.* **97** (1980), 89–102.

[2] Daley, D., Vere–Jones, D., *An Introduction to the Theory of Point Processes, Volume II: General Theory and Structure.* 2nd edn., Springer, New York, 2008.

[3] Erdös, P., Rényi, A., On Cantor's series with convergent $\sum 1/q_n$. *Ann. Univ. Sci. Budapest. Eötvös. Sect. Math.* **2** (1959), 93–109.

[4] Reitzner, M., Schneider, R., On the cells in a stationary Poisson hyperplane mosaic. *Adv. Geom.* (accepted), arXiv:1609.04230 (2016).

[5] Rényi, A., *Wahrscheinlichkeitsrechnung.* 3rd edn., VEB Deutsch. Verl. d. Wiss., Berlin, 1971.

[6] Tempel'man, A.A., Ergodic theorems for general dynamical systems (in Russian). *Trudy Moskov. Mat. Obsc.* **26** (1972). Engl. transl.: *Trans. Moscow Math. Soc.* **26** (1972), 94–132.

[7] Schneider, R., Weil, W., *Stochastic and Integral Geometry.* Springer, Berlin, 2008.

Author's address:

Rolf Schneider
Mathematisches Institut, Albert-Ludwigs-Universität
D-79104 Freiburg i. Br., Germany
E-mail: rolf.schneider@math.uni-freiburg.de