Reverse inequalities for zoonoids and their application

Daniel Hug[a,1,*], Rolf Schneider[b],

[a] *Department of Mathematics, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany*
[b] *Mathematisches Institut, Albert-Ludwigs-Universität, Eckerstr. 1, D-79104 Freiburg, Germany*

Abstract

We prove inequalities for mixed volumes of zonoids with isotropic generating measures. A special case is an inequality for zonoids that is reverse to the generalized Urysohn inequality, between mean width and another intrinsic volume; here the equality case characterizes parallelepipeds. We apply this to a question from stochastic geometry. It is known that among the stationary Poisson hyperplane processes of given positive intensity in n-dimensional Euclidean space, the ones with rotation invariant distribution are characterized by the fact that they yield, for $k \in \{2, \ldots, n\}$, the maximal intensity of the intersection process of order k. We show that, if the kth intersection density is measured in an affine-invariant way, the processes of hyperplanes with only n fixed directions are characterized by a corresponding minimum property.

Keywords: Zonoid, Isotropic measure, Mixed volume, Reverse Urysohn inequality, Characterization of parallelepipeds, Hyperplane process, Intersection density, Associated zonoid
2000 MSC: 52A40, 52A21, 60D05

1. Introduction

In the geometry of convex bodies, the term 'reverse inequality' is conventionally used when some geometric functional satisfies a sharp inequality, which is either well-known or trivial, and a counterpart in the opposite direction is sought. For Euclidean functionals, which often are unbounded in one direction, such counterparts may become possible if the functional under investigation is replaced by its infimum, or supremum, over all affine transforms of the argument. In the plane, this principle was first extensively employed by Behrend [3]. An example in higher dimensions is the investigation of the functional R/r, the ratio of circumradius to inradius, by Leichtweiss [8], who obtained an inequality reverse to the trivial one, $R/r \geq 1$, and showed that the extremal case characterizes simplices. The most famous example is undoubtedly Keith Ball's [2] reverse isoperimetric inequality in $\mathbb{R}^n$, stating that the maximum of the isoperimetric quotient

*Corresponding author
Email addresses: `daniel.hug@kit.edu` (Daniel Hug), `rolf.schneider@math.uni-freiburg.de` (Rolf Schneider)
URL: `http://www.math.kit.edu/iag4/~hug/de` (Daniel Hug), `http://home.mathematik.uni-freiburg.de/rschnei/` (Rolf Schneider)
[1] The author is grateful to the Fields Institute for hospitality and financial support.

V_n^{n-1}/V_{n-1}^n, taken over the affine transforms of an n-dimensional convex body, becomes minimal on simplices, respectively on parallelepipeds if only centrally symmetric bodies are taken into consideration.

In the present paper, we prove inequalities for zonoids which are reverse, in the explained sense, to some classical inequalities from the Brunn–Minkowski theory, namely to the inequalities

$$\frac{V_j(K)}{V_1(K)^j} \le \frac{V_j(B^n)}{V_1(B^n)^j} \tag{1}$$

for the intrinsic volumes V_i of convex bodies $K \subset \mathbb{R}^n$ with $\dim K \ge 1$, where $j \in \{2, \dots, n\}$. Here B^n is the unit ball in $\mathbb{R}^n$. Equality in (1) holds if and only if K is a ball. Note that V_n is the volume and V_1 is a constant multiple of the mean width, so that the case $j = n$ of (1) is Urysohn's inequality.

As a reverse counterpart of (1) for zonoids Z we prove that

$$\sup_{\Lambda \in \mathrm{GL}(n)} \frac{V_j(\Lambda Z)}{V_1(\Lambda Z)^j} \ge \frac{1}{n^j}\binom{n}{j}, \tag{2}$$

if $\dim Z = n$ and $j \ge 2$, with equality if and only if Z is a parallelepiped. Inequality (2) is derived from a more general inequality for zonoids (Theorem 1), a particular case of which says that a zonoid Z with isotropic generating measure satisfies

$$V_j(Z) \ge 2^j \binom{n}{j}, \tag{3}$$

with equality for $j \ge 2$ if and only if Z is a cube of side length 2. For $j = n$ (the case of the volume) and discrete generating measures, inequality (3), in a more general form, is due to Ball [1], Lemma 4, though without mention of the equality case. An extension to general isotropic generating measures, to L_p zonoids and to a dual situation, with full information on the equality cases, was achieved by Lutwak, Yang and Zhang [10]. It seems that their method is restricted to the case of the volume and does not admit an extension to the other intrinsic volumes. Therefore, we rather extend the method of Ball, using induction with respect to the dimension.

The restriction to zonoids is a regrettable drawback, but no approach for general convex bodies is in sight. There are some other instances of inequalities for convex bodies where a complete solution is available for zonoids, but not for general convex bodies. We mention Reisner's reverse Blaschke–Santaló inequality (see, e.g., Gordon, Meyer and Reisner [7]) and the discussion of equality in the Aleksandrov–Fenchel inequalities by Schneider [14].

On the other hand, inequalities for zonoids are precisely what is needed for our envisaged application to stochastic geometry. Such applications were, in fact, the motivation for the present investigation. We explain the starting point and present our result in Section 3.

2. Inequalities for zonoids

We work in n-dimensional Euclidean vector space $\mathbb{R}^n$ ($n \ge 2$), with scalar product $\langle \cdot, \cdot \rangle$, induced norm $\|\cdot\|$, unit ball B^n and unit sphere $\mathbb{S}^{n-1}$. For $u \in \mathbb{S}^{n-1}$, the linear subspace $u^\perp$ is the orthogonal complement of the linear hull of u. By $\mathcal{K}^n$ we denote the space of convex bodies (nonempty, compact, convex subsets) of $\mathbb{R}^n$, endowed with the Hausdorff metric.

Let V_n be the volume in $\mathbb{R}^n$ and $V(K_1, \dots, K_n)$ the mixed volume of the convex bodies $K_1, \dots, K_n$. Recall that the mixed volume is the symmetric function $V : (\mathcal{K}^n)^n \to \mathbb{R}$ with

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \dots, K_{i_n})$$

for $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$; it is n-linear with respect to Minkowski addition and nonnegative. The special case of the Steiner formula,

$$V_n(K + \epsilon B^n) = \sum_{j=0}^{n} \epsilon^{n-j} \kappa_{n-j} V_j(K),$$

defines the *intrinsic volumes* $V_0, \dots, V_n$, thus

$$\kappa_{n-j} V_j(K) = \binom{n}{j} V(K[j], B^n[n-j]) \tag{4}$$

(where the symbol $[j]$ on the right side indicates that the argument before it appears j times). The constant κ_n denotes the volume of the unit ball in $\mathbb{R}^n$. For more information on these functionals, and on convex bodies in general, we refer to [15].

The support function of a convex body $K \in \mathcal{K}^n$ is defined by $h(K, u) = \max\{\langle x, u \rangle : x \in K\}$, for $u \in \mathbb{R}^n$. A convex body $Z \subset \mathbb{R}^n$ is a *zonoid* if its support function has a representation

$$h(Z, u) = \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \, \mu(\mathrm{d}v), \qquad u \in \mathbb{R}^n, \tag{5}$$

with an even, finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$ (a measure μ on $\mathbb{S}^{n-1}$ is even if $\mu(A) = \mu(-A)$ for all Borel sets $A \subset \mathbb{S}^{n-1}$). Thus, in this paper, we consider only zonoids which have their centre of symmetry at the origin o; this does not restrict the generality of the results. The even measure μ in (5) is uniquely determined by Z. It is called the *generating measure* of the zonoid Z. Properties and applications of zonoids are described in the survey articles [16] and [6].

A finite Borel measure μ on $\mathbb{S}^{n-1}$ is called *isotropic* if

$$\int_{\mathbb{S}^{n-1}} \langle v, a \rangle \langle v, b \rangle \, \mu(\mathrm{d}v) = \langle a, b \rangle \qquad \text{for all } a, b \in \mathbb{R}^n$$

or, equivalently,

$$\int_{\mathbb{S}^{n-1}} \langle v, a \rangle^2 \, \mu(\mathrm{d}v) = \|a\|^2 \qquad \text{for all } a \in \mathbb{R}^n.$$

If μ is isotropic, then $\mu(\mathbb{S}^{n-1}) = n$, as follows by evaluating $\sum_i \int \langle v, u_i \rangle^2 \mu(\mathrm{d}v)$ for an orthonormal basis $(u_1, \dots, u_n)$.

Theorem 1. *If $j \in \{1, \dots, n\}$ and if $Z_1, \dots, Z_j \subset \mathbb{R}^n$ are zonoids with isotropic generating measures, then*

$$V(Z_1, \dots, Z_j, B^n[n-j]) \geq 2^j \kappa_{n-j}. \tag{6}$$

For $j = 1$, inequality (6) *holds with equality. For $j \geq 2$, equality in* (6) *holds if and only if $Z_1 = \cdots = Z_j$ is a cube of side length* 2.

By (4), we have the following corollary.

Corollary 1. *Let $Z \subset \mathbb{R}^n$ be a zonoid with isotropic generating measure μ. If $j \in \{1, \dots, n\}$, then*

$$V_j(Z) \geq 2^j \binom{n}{j}. \tag{7}$$

For $j = 1$, inequality (7) *holds with equality. For $j \geq 2$, equality in* (7) *holds if and only if Z is a cube of side length* 2.

Proof of Theorem 1. We need some formulas for mixed volumes involving zonoids.

For $u \in \mathbb{S}^{n-1}$, we denote by π_u the orthogonal projection to $u^\perp$. If Z is a zonoid with generating measure μ and if $K_1, \dots, K_{n-1}$ are convex bodies, then

$$V(K_1, \dots, K_{n-1}, Z) = \frac{2}{n} \int_{\mathbb{S}^{n-1}} v(\pi_u K_1, \dots, \pi_u K_{n-1})\, \mu(\mathrm{d}u), \tag{8}$$

where v denotes the mixed volume in the $(n-1)$-dimensional space $u^\perp$. This goes back, in special cases, to Matheron [12], p. 103, and also follows from the work of Weil [19]. For the reader's convenience, we indicate a short proof, using some well-known properties of mixed volumes (for which we refer to [15]). According to [15], (5.3.23), for any convex body $K \subset \mathbb{R}^n$ and for $u \in \mathbb{S}^{n-1}$,

$$V(K, \dots, K, [-u, u]) = \frac{2}{n} V_{n-1}(\pi_u K),$$

where $[-u, u]$ is the segment with endpoints $-u$ and u (recall that $V_{n-1}(C)$, for a convex body of dimension less than n, is the $(n-1)$-dimensional volume of C). Assuming first that μ is discrete, assigning mass $m_i/2$ to each of $\pm u_i$, we can write the resulting equation

$$V\left(K, \dots, K, \sum_i m_i [-u_i, u_i]\right) = \frac{2}{n} \sum_i m_i V_{n-1}(\pi_{u_i} K)$$

in the form

$$V(K, \dots, K, Z) = \frac{2}{n} \int_{\mathbb{S}^{n-1}} V_{n-1}(\pi_u K)\, \mu(\mathrm{d}u).$$

By approximation, this holds for general generating measures μ. By 'mixing' (that is, replacing K by a Minkowski combination, expanding, and comparing coefficients), we obtain (8).

Let $j \in \{1, \dots, n\}$, and for $i \in \{1, \dots, j\}$ let Z_i be a zonoid with generating measure μ_i. Then formula (14.34) in [17] states that

$$\begin{aligned} &V(Z_1, \dots, Z_j, B^n[n-j]) \\ &\quad = \frac{2^j (n-j)! \kappa_{n-j}}{n!} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \nabla_j(u_1, \dots, u_j)\, \mu_1(\mathrm{d}u_1) \cdots \mu_j(\mathrm{d}u_j), \end{aligned} \tag{9}$$

where $\nabla_j(u_1, \dots, u_j)$ denotes the j-dimensional volume of the parallelepiped spanned by the vectors $u_1, \dots, u_j$. The case $Z_1 = \cdots = Z_j$ of (9) is proved in [15], Theorem 5.3.3 (with different notation); from this, the general case is obtained by 'mixing'.

Next, we need some facts about generating measures. Let $u \in \mathbb{S}^{n-1}$ and write $\mathbb{S}_u := \mathbb{S}^{n-1} \cap u^\perp$ for the great subsphere with pole u. The spherical projection $\Pi_u : \mathbb{S}^{n-1} \setminus \{\pm u\} \to \mathbb{S}_u$ is defined by

$$\Pi_u v := \frac{\pi_u v}{\|\pi_u v\|} \qquad \text{for } v \in \mathbb{S}^{n-1} \setminus \{\pm u\}.$$

The projection $\pi_u Z$ of a zonoid Z is a zonoid in the $(n-1)$-dimensional space $u^\perp$ and hence has a generating measure concentrated on $\mathbb{S}_u$. If Z has generating measure μ, then $\pi_u Z$ has generating measure given by

$$\mu_u(A) = \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \mathbf{1}_A(\Pi_u v)\|\pi_u v\|\,\mu(\mathrm{d}v) \tag{10}$$

for Borel sets $A \subset \mathbb{S}^{n-1}$. Relation (10) is easily seen if μ is discrete, and the general case follows by approximation. See Weil [20] for the application of more general results of this kind.

We define a second measure $\overline{\mu}_u$ on $\mathbb{S}^{n-1}$, also concentrated on $\mathbb{S}_u$, by

$$\overline{\mu}_u(A) = \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \mathbf{1}_A(\Pi_u v)\|\pi_u v\|^2\,\mu(\mathrm{d}v) \tag{11}$$

for Borel sets $A \subset \mathbb{S}^{n-1}$.

For any nonnegative, measurable function f on $\mathbb{S}_u$ we deduce from (10) and (11) that

$$\int_{\mathbb{S}_u} f\,\mathrm{d}\mu_u = \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} f(\Pi_u v)\|\pi_u v\|\,\mu(\mathrm{d}v), \tag{12}$$

$$\int_{\mathbb{S}_u} f\,\mathrm{d}\overline{\mu}_u = \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} f(\Pi_u v)\|\pi_u v\|^2\,\mu(\mathrm{d}v). \tag{13}$$

If μ is isotropic, then for $a \in \mathbb{S}_u$ we obtain

$$\int_{\mathbb{S}_u} \langle w, a\rangle^2\,\overline{\mu}_u(\mathrm{d}w) = \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \left\langle \frac{\pi_u v}{\|\pi_u v\|}, a\right\rangle^2 \|\pi_u v\|^2\,\mu(\mathrm{d}v) = \int_{\mathbb{S}^{n-1}} \langle v, a\rangle^2\,\mu(\mathrm{d}v) = \|a\|^2.$$

Hence, $\overline{\mu}_u$ is isotropic in $u^\perp$.

Now we prove (6). Accordingly, we assume that the generating measure μ_i of Z_i is isotropic, for $i \in \{1, \dots, j\}$. For $u \in \mathbb{S}^{n-1}$, let $\mu_{i,u}$ be the generating measure of $\pi_u Z_i$, and let $\overline{\mu}_{i,u} := (\overline{\mu_i})_u$ be the measure defined by (11), with μ replaced by μ_i. Let $\overline{Z}_{i,u}$ be the zonoid with generating measure $\overline{\mu}_{i,u}$. Since $\overline{\mu}_{i,u}$ is concentrated on $\mathbb{S}_u$, the zonoid $\overline{Z}_{i,u}$ lies in $u^\perp$. From $\overline{\mu}_{i,u} \le (\mu_i)_u$ we have $\overline{Z}_{i,u} \subset \pi_u Z_i$. Together with the monotonicity of the mixed volume in each argument, this could be used directly in the following induction argument, but in view of the later equality discussion we need a slightly more elaborate argument.

For $n = 1$, the assertion of the theorem is easily verified. Now let $n \ge 2$ and assume that the assertion has been proved in smaller dimensions.

For $j = 1$, formula (8) yields

$$V(Z_1, B^n[n-1]) = \frac{2}{n}\int_{\mathbb{S}^{n-1}} V_{n-1}(\pi_u B^n)\,\mu_1(\mathrm{d}u) = \frac{2}{n}\kappa_{n-1}\cdot n.$$

Let $j \in \{2, \dots, n\}$. From (8) we have

$$V(Z_1, \dots, Z_j, B^n[n-j]) = \frac{2}{n}\int_{\mathbb{S}^{n-1}} v(\pi_u Z_1, \dots, \pi_u Z_{j-1}, \pi_u B^n[n-j])\,\mu_j(\mathrm{d}u).$$

Using (9) in dimension $n-1$, together with (12) and (13), and putting

$$c_{n,j} := \frac{2^{j-1}(n-j)!\kappa_{n-j}}{(n-1)!},$$

we obtain, for arbitrary $u \in \mathbb{S}^{n-1}$,

$$\begin{aligned}
& v(\pi_u Z_1, \dots, \pi_u Z_{j-1}, \pi_u B^n[n-j]) \\
&= c_{n,j} \int_{\mathbb{S}_u} \cdots \int_{\mathbb{S}_u} \nabla_{j-1}(v_1, \dots, v_{j-1})\, \mu_{1,u}(\mathrm{d}v_1) \cdots \mu_{j-1,u}(\mathrm{d}v_{j-1}) \\
&= c_{n,j} \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \cdots \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \nabla_{j-1}(\Pi_u v_1, \dots, \Pi_u v_{j-1}) \|\pi_u v_1\| \cdots \|\pi_u v_{j-1}\| \\
&\quad \times \mu_1(\mathrm{d}v_1) \cdots \mu_{j-1}(\mathrm{d}v_{j-1}) \\
&\geq c_{n,j} \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \cdots \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \nabla_{j-1}(\Pi_u v_1, \dots, \Pi_u v_{j-1}) \|\pi_u v_1\|^2 \cdots \|\pi_u v_{j-1}\|^2 \\
&\quad \times \mu_1(\mathrm{d}v_1) \cdots \mu_{j-1}(\mathrm{d}v_{j-1}) \\
&= c_{n,j} \int_{\mathbb{S}_u} \cdots \int_{\mathbb{S}_u} \nabla_{j-1}(v_1, \dots, v_{j-1})\, \overline{\mu}_{1,u}(\mathrm{d}v_1) \cdots \overline{\mu}_{j-1,u}(\mathrm{d}v_{j-1}) \\
&= v\left(\overline{Z}_{1,u}, \dots, \overline{Z}_{j-1,u}, \pi_u B^n[n-j]\right).
\end{aligned}$$

Since the zonoids $\overline{Z}_{1,u}, \dots, \overline{Z}_{j-1,u}$ have isotropic generating measures in $u^\perp$, we can apply the induction hypothesis, which states that

$$v\left(\overline{Z}_{1,u}, \dots, \overline{Z}_{j-1,u}, \pi_u B^n[n-j]\right) \geq 2^{j-1}\kappa_{(n-1)-(j-1)} = 2^{j-1}\kappa_{n-j}.$$

We conclude that

$$V(Z_1, \dots, Z_j, B^n[n-j]) \geq \frac{2}{n} 2^{j-1}\kappa_{n-j} \cdot n = 2^j \kappa_{n-j},$$

which is the asserted inequality (6).

Suppose that equality holds here. Then the equality

$$v(\pi_u Z_1, \dots, \pi_u Z_{j-1}, \pi_u B^n[n-j]) = v(\overline{Z}_{1,u}, \dots, \overline{Z}_{j-1,u}, \pi_u B^n[n-j]) \tag{14}$$

holds for μ_j-almost all $u \in \mathbb{S}^{n-1}$. By continuity, (14) holds for all $u \in \operatorname{supp}\mu_j$, the support of the measure μ_j.

Let $u \in \operatorname{supp}\mu_j$ be given. Since (14) holds, we have

$$\nabla_{j-1}(\Pi_u v_1, \dots, \Pi_u v_{j-1}) \|\pi_u v_1\| \cdots \|\pi_u v_{j-1}\| (1 - \|\pi_u v_1\| \cdots \|\pi_u v_{j-1}\|) = 0 \tag{15}$$

for $\mu_1 \otimes \cdots \otimes \mu_{j-1}$-almost all $(v_1, \dots, v_{j-1}) \in \left(\mathbb{S}^{n-1} \setminus \{\pm u\}\right)^{j-1}$. By continuity, (15) holds whenever

$$v_i \in (\operatorname{supp}\mu_i) \setminus \{\pm u\} \qquad \text{for } i = 1, \dots, j-1.$$

Let $v_1 \in (\operatorname{supp}\mu_1) \setminus \{\pm u\}$ be given. None of the isotropic measures $\mu_2, \dots, \mu_{j-1}$ is concentrated on a great subsphere. If $j \geq 3$, we can find a vector $v_2 \in (\operatorname{supp}\mu_2) \setminus \{\pm u\}$ such that $\Pi_u v_1, \Pi_u v_2$ are linearly independent. Continuing in this way, we see that there exist vectors $v_i \in (\operatorname{supp}\mu_i) \setminus \{\pm u\}$, $i = 1, \dots, j-1$, such that $\Pi_u v_1, \dots, \Pi_u v_{j-1}$ are linearly independent. From (15) we now conclude that $\|\pi_u v_1\| \cdots \|\pi_u v_{j-1}\| = 1$. In particular, this shows that $v_1 \in u^\perp$. It follows that $\operatorname{supp}\mu_1 \subset \{\pm u\} \cup u^\perp$. Since μ_1 is not concentrated on a great subsphere, we must have $u \in \operatorname{supp}\mu_1$. Since u was an arbitrary element of $\operatorname{supp}\mu_j$, we see that $\operatorname{supp}\mu_j \subset \operatorname{supp}\mu_1$. The roles of μ_j and μ_1 can be interchanged, hence we have $\operatorname{supp}\mu_1 = \operatorname{supp}\mu_j$. Since the assumptions are symmetric in $Z_1, \ldots, Z_j$, we conclude that there is a set $S \subset \mathbb{S}^{n-1}$ with

$$\operatorname{supp}\mu_i = S \quad \text{for } i = 1, \ldots, j.$$

The o-symmetric set S is not concentrated on a great subsphere and has the property that $u \in S$ implies $S \subset \{\pm u\} \cup u^\perp$. Choose $u_1 \in S$. Then there is a vector $u_2 \in u_1^\perp \cap S$, and $S \subset [u_2] \cup u_2^\perp$, where $[u]$ denotes the linear span of the vector u. Hence, $S \subset [u_1] \cup [u_2] \cup \operatorname{lin}\{u_1, u_2\}^\perp$, and if $n \geq 3$, we can choose $u_3 \in \operatorname{lin}\{u_1, u_2\}^\perp \cap S$. Continuing in this way, we see that S is the set $\{\pm u_1, \ldots, \pm u_n\}$ for some orthonormal basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$. Since each μ_i is isotropic, it must assign the value $1/2$ to each of these points. Therefore, $Z_1, \ldots, Z_j$ are all equal to the same cube of side length 2. That, conversely, equality in (6) does hold in this case, is clear from the proof, but follows also directly from (9). This completes the proof of Theorem 1. □

Remark 1. Inspection of the inductive proof of Theorem 1 shows that it only needs the measure properties $\mu(\mathbb{S}^{n-1}) = n$ and $\overline{\mu}_u(\mathbb{S}_u) = n - 1$ for all u. But these properties already imply that μ is isotropic. Indeed, let μ be a finite Borel measure on $\mathbb{S}^{n-1}$ with $\mu(\mathbb{S}^{n-1}) = n$, and suppose that the measures $\overline{\mu}_u$, defined as above, satisfy $\overline{\mu}_u(\mathbb{S}_u) = n - 1$ for all $u \in \mathbb{S}^{n-1}$. Then, for arbitrary $u \in \mathbb{S}^{n-1}$,

$$\begin{aligned} n - 1 &= \overline{\mu}_u(\mathbb{S}_u) = \int_{\mathbb{S}^{n-1}\setminus\{\pm u\}} \|\pi_u v\|^2 \, \mu(\mathrm{d}v) = \int_{\mathbb{S}^{n-1}} (1 - \langle u, v\rangle^2) \, \mu(\mathrm{d}v) \\ &= \mu(\mathbb{S}^{n-1}) - \int_{\mathbb{S}^{n-1}} \langle u, v\rangle^2 \, \mu(\mathrm{d}v) = n - \int_{\mathbb{S}^{n-1}} \langle u, v\rangle^2 \, \mu(\mathrm{d}v) \end{aligned}$$

and therefore

$$\int_{\mathbb{S}^{n-1}} \langle u, v\rangle^2 \, \mu(\mathrm{d}v) = 1.$$

This shows that μ is isotropic.

Our application of Theorem 1 is based on the following lemma. It follows from a more general result of Lewis [9], which has a functional analytic proof. We give here a short derivation from known geometric results.

Lemma 1. *Let $Z \subset \mathbb{R}^n$ be a zonoid with interior points. There exists a linear transformation $\Lambda \in \mathrm{GL}(n)$ such that the generating measure of ΛZ is isotropic.*

Proof. We use the well-known fact that every full-dimensional zonoid is a projection body. Let μ_Z be the generating measure of Z. This measure is even and not concentrated on a great subsphere, since $\dim Z = n$. By Minkowski's existence theorem (see, e.g., [15], Section 7.1) there exists a unique o-symmetric convex body B_Z with surface area measure $S_{n-1}(B_Z, \cdot) = 2\mu_Z$, and we have $\Pi B_Z = Z$, that is, Z is the projection body of B_Z. Petty [13] has shown that among all volume preserving linear transformations of $\mathbb{R}^n$ there exists one, say Λ_0, for which the surface area of $\Lambda_0 B_Z$ is minimal, and that the surface area measure of $\Lambda_0 B_Z$ is proportional to an isotropic measure. We have $\Pi\Lambda_0 B_Z = \Lambda_0^{-t}\Pi B_Z = \Lambda_0^{-t}Z$, where Λ_0^t denotes the transpose of Λ_0 (see [15], p. 414, for references). Therefore, the generating measure of the zonoid $\Lambda_0^{-t}Z$ is equal to $\frac{1}{2}S_{n-1}(\Lambda_0 B_Z, \cdot)$ and thus is proportional to an isotropic measure. Multiplying Λ_0 by an appropriate factor, we obtain the assertion. □

We can now state reverse inequalities, in the sense explained in the introduction. Let $Z \in \mathcal{K}^n$ be a zonoid with interior points. By Lemma 1, there exists a linear transformation $\Lambda_0 \in \mathrm{GL}(n)$ such that $\Lambda_0 Z$ has isotropic generating measure. Then Corollary 1 states that $V_1(\Lambda_0 Z) = 2n$ and $V_j(\Lambda_0 Z) \geq 2^j\binom{n}{j}$, thus, for $j \in \{2, \dots, n\}$,

$$\sup_{\Lambda\in\mathrm{GL}(n)} \frac{V_j(\Lambda Z)}{V_1(\Lambda Z)^j} \geq \frac{V_j(\Lambda_0 Z)}{V_1(\Lambda_0 Z)^j} \geq \frac{1}{n^j}\binom{n}{j}. \tag{16}$$

If here the left-hand side is equal to the right-hand side, then $\Lambda_0 Z$ is a cube and hence Z is a parallelepiped. Conversely, if Z is a parallelepiped, we can choose Λ_0 such that $\Lambda_0 Z$ is a cube, and then the second inequality in (16) holds with equality. We show that also the first inequality holds with equality. Generally, for a polytope P, formula (4.2.17) in [15] states that

$$V_j(P) = \sum_{F\in\mathcal{F}_j(P)} \gamma(F, P)V_j(F),$$

where $\mathcal{F}_j(P)$ is the set of j-faces of P and $\gamma(F, P)$ denotes the outer angle of P at its face F. If P is a parallelepiped and $F \in \mathcal{F}_j(P)$, then all j-faces of P parallel to F have the same j-volume, and the exterior angles of P at these faces add up to one, as seen by projecting P orthogonally to the orthogonal complement of F. Therefore,

$$V_j(P) = 2^{j-n} \sum_{F\in\mathcal{F}_j(P)} V_j(F) \tag{17}$$

for a parallelepiped P. Let Q be a rectangular parallelepiped with the same edge lengths as P. Then (17) shows that $V_1(Q) = V_1(P)$. Also, clearly, $V_n(Q) \geq V_n(P)$, and by the corresponding inequality for the j-faces and by (17), also $V_j(Q) \geq V_j(P)$ for each $j \in \{2, \dots, n\}$, with equality if and only if P is rectangular. For rectangular parallelepipeds, the intrinsic volumes are expressed by elementary symmetric functions of their edge lengths, hence the equality condition for Newton's inequalities shows that V_j/V_1^j attains its maximum on rectangular parallelepipeds precisely at the cubes. Thus, we have

$$\sup_{\Lambda\in\mathrm{GL}(n)} \frac{V_j(\Lambda P)}{V_1(\Lambda P)^j} = \frac{V_j(C)}{V_1(C)^j} \tag{18}$$

for a parallelepiped P and a cube C. Now we can state the following corollary.

Corollary 2. *If $j \in \{2, \dots, n\}$ and $Z \in \mathcal{K}^n$ is a zonoid with interior points, then*

$$\sup_{\Lambda\in\mathrm{GL}(n)} \frac{V_j(\Lambda Z)}{V_1(\Lambda Z)^j} \geq \frac{1}{n^j}\binom{n}{j}.$$

Equality holds if and only if Z is a parallelepiped.

In the case of Theorem 1, the 'classical' inequality to which (6) could be considered as a reverse one for zonoids with isotropic generating measures, has not yet been proved. However, for zonoids $Z_1, \dots, Z_j$ ($j \in \{1, \dots, n\}$), we immediately obtain from (9) and $\nabla_j(u_1, \dots, u_j) \leq 1$ that

$$V(Z_1, \dots, Z_j, B^n[n-j]) \leq \frac{(n-j)!\kappa_{n-j}}{n!} V_1(Z_1) \cdots V_1(Z_j). \tag{19}$$

Suppose that $\dim Z_i \geq 1$ ($i = 1, \dots, j$) and that equality holds in (19); then $\mu_1 \otimes \cdots \otimes \mu_j$-almost all j-tuples $(u_1, \dots, u_j) \in (\mathbb{S}^{n-1})^j$ are orthonormal. It follows that the j-tuple $(u_1, \dots, u_j)$ is orthonormal whenever $u_i \in \operatorname{supp} \mu_i$ for $i = 1, \dots, j$. Let L_i be the smallest linear subspace of $\mathbb{R}^n$ containing $\operatorname{supp} \mu_i$, or equivalently, the linear hull of Z_i (recall that $o \in Z_i$). Then the subspaces $L_1, \dots, L_j$ are pairwise totally orthogonal. Conversely, if this is satisfied, then equality holds in (19). We state this as a theorem.

Theorem 2. *If $j \in \{1, \dots, n\}$ and $Z_1, \dots, Z_j \in \mathcal{K}^n$ are zonoids, then* (19) *holds. If* $\dim Z_i \geq 1$ ($i = 1, \dots, j$)*, then equality holds if and only if $Z_1, \dots, Z_j$ are pairwise totally orthogonal.*

It is conjectured that the assertion of Theorem 2 holds for arbitrary convex bodies. In the plane, this was proved by Betke and Weil [4], who also conjectured a more general version of the case $j = n$ of (19) (inequality (16) in [4]).

3. Intersection densities in stochastic geometry

The inequalities of the previous section will now be applied to a special topic of stochastic geometry, namely intersection densities of stationary hyperplane processes. We briefly sketch the background.

In a short note of 1974, Davidson [5] suggested stationary Poisson line processes in the plane as a model for the fibres in a sheet of paper and then continued: "the strength of paper surely lies in the density of matting of its fibres; it is thus of interest to find what line-processes have the greatest specific density of intersections per unit area." Davidson gave an answer, using Fourier series. From a geometric point of view, various higher-dimensional generalizations of this question are of considerable intrinsic interest. We describe such versions in general terms, referring the reader to the book [17], in particular Sections 4.4 and 4.6, for the foundations.

For $k \in \{0, \dots, n\}$, a *k-flat process X* in $\mathbb{R}^n$ is a simple point process (with locally finite intensity measure) in the space $A(n, k)$ of k-flats (k-dimensional affine subspaces) of $\mathbb{R}^n$, with its usual topology. Thus, X is a measurable mapping from some probability space into the space of locally finite subsets of $A(n, k)$, with a suitable measurability structure. A realization of X is a system of k-flats with the property that every compact subset of $\mathbb{R}^n$ meets almost surely only finitely many of the flats. It is always assumed in the following that the k-flat process X is stationary, which means that its distribution is invariant under translations. Under this assumption, there are a number $\gamma \geq 0$ and a Borel probability measure ϕ on the Grassmannian $G(n, k)$ of k-dimensional linear subspaces of $\mathbb{R}^n$ such that, for each Borel subset B of $A(n, k)$, the expected number of k-flats of X falling in B is given by

$$\gamma \int_{G(n,k)} \int_{L^\perp} \mathbf{1}_B(L + x)\, \lambda_{L^\perp}(\mathrm{d}x)\, \phi(\mathrm{d}L).$$

Here $\lambda_{L^\perp}$ denotes the $(n-k)$-dimensional Lebesgue measure on $L^\perp$, the orthogonal complement of $L \in G(n,k)$. We assume that $\gamma > 0$. The number γ is called the *intensity* and ϕ is the *directional distribution* of X. The following special case gives an intuitive interpretation. If A is a Borel subset of $G(n,k)$, then

$$\gamma\phi(A) = \frac{1}{\kappa_{n-k}} \mathbb{E}\, \mathrm{card}\{E \in X : E \cap B^n \neq \emptyset,\ E_0 \in A\},$$

where $\mathbb{E}$ denotes mathematical expectation, and E_0 is the translate of the flat E passing through the origin. The k-flat process X is a *Poisson process* if, for each Borel subset B of $A(n,k)$, the random variable $\mathrm{card}\{E \in X : E \in B\}$ has a Poisson distribution. We refer to [17] for more information and mention here only that to given $\gamma > 0$ and Borel probability measure ϕ on $G(n,k)$ there exists a stationary Poisson k-flat process with intensity γ and directional distribution ϕ, and that it is uniquely determined up to stochastic equivalence.

Now we specialize this to the case $k = n-1$. Let X be a stationary hyperplane process in $\mathbb{R}^n$. We assume that it is *nondegenerate*, in the sense that its directional distribution is not concentrated on the set of hyperplanes parallel to some fixed line. Associated with such a hyperplane process X are the intersection processes $X_{(1)}, \dots, X_{(n)}$. Here $X_{(k)}$, the *intersection process of order* k, is obtained by intersecting any k hyperplanes of X which are in general position. The result is a stationary process of $(n-k)$-flats (with $X_{(1)} = X$). Its intensity, denoted by $\gamma_k = \gamma_k(X)$, defines the *kth intersection density* of X, with $\gamma_1 = \gamma$ being the intensity of X.

These intersection densities are particularly accessible in the case of Poisson hyperplane processes, due to the strong intrinsic independence properties of Poisson processes. It is a natural question to ask which Poisson hyperplane processes X of given intensity $\gamma > 0$ have the largest kth intersection density, for $k \in \{2, \dots, n\}$. An answer became possible after Matheron [11] (see also [12], Section 6.1) had introduced his 'Steiner compact set' of X, later called 'associated zonoid', as an auxiliary convex body. Its kth intrinsic volume is precisely the kth intersection density of X. Applying the classical Aleksandrov–Fenchel inequalities from the theory of convex bodies, Thomas [18] then showed that the maximum of the quotient γ_k/γ^k (which is invariant under dilatations) is attained precisely if the process X is stochastically invariant under rotations. This result is reproduced as Theorem 4.6.5 in [17].

The analogous question for minimal intersection densities obviously does not make sense. For example, consider a stationary Poisson line process in the plane where all lines are either 'horizontal' or 'vertical'. By choosing the density of the horizontal lines sufficiently high and the density of the vertical lines sufficiently small, one can achieve that the intensity γ of X is a prescribed positive number, whereas the intersection density γ_2 is below a given positive bound. However, one might argue that this drawback is caused by the fact that intersections of hyperplanes, an affine notion, are measured, unsuitably, by means of Euclidean tools. Instead, we propose here to measure the strength of intersections from an affine-invariant point of view, namely by the quantities

$$\Gamma_k(X) := \sup_{\Lambda \in \mathrm{GL}(n)} \frac{\gamma_k(\Lambda X)}{\gamma(\Lambda X)^k}, \qquad k = 2, \dots, n.$$

We call $\Gamma_k(X)$ the kth *affine intersection density* of X.

Theorem 3. *If X is a nondegenerate stationary Poisson hyperplane process in $\mathbb{R}^n$ and if $k \in$*

$\{2, \ldots, n\}$, *then the kth affine intersection density of X satisfies*

$$\Gamma_k(X) \geq \frac{1}{n^k}\binom{n}{k}. \tag{20}$$

Equality holds if and only if the hyperplanes of X almost surely attain only n fixed directions.

Proof. Let X be a stationary hyperplane process in $\mathbb{R}^n$. By our general assumptions, it has intensity $\gamma > 0$ and is nondegenerate. Its directional distribution induces, by passing from hyperplanes through the origin to their unit normal vectors, an even Borel probability measure φ on the unit sphere $\mathbb{S}^{n-1}$, which is not concentrated on any great subsphere. The measure φ is called the spherical directional distribution of X. For this and for more details about the following, we refer to [17], in particular Sections 4.4 and 4.6.

The associated zonoid Π_X of X is defined by its support function

$$h(\Pi_X, u) = \frac{\gamma}{2}\int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\, \varphi(\mathrm{d}v), \qquad u \in \mathbb{R}^n.$$

If X is a Poisson process, then the kth intersection density of X is given by Matheron's formula (see [17], (4.63)),

$$\gamma_k(X) = V_k(\Pi_X), \tag{21}$$

for $k = 1, \ldots, n$.

The group $\mathrm{GL}(n)$ of linear transformations of $\mathbb{R}^n$ operates measurably on the space of hyperplanes in $\mathbb{R}^n$. Hence, if Λ is a linear transformation of $\mathbb{R}^n$, then ΛX is again a hyperplane process (see [17], p. 53), and obviously stationary and Poisson if X has these properties.

Let $\Lambda \in \mathrm{GL}(n)$. By formula (4.60) of [17] (and using $X(B)$ synonymously with $\mathrm{card}\{E \in X : E \in B\}$),

$$2h(\Pi_X, u) = \mathbb{E}X(\mathcal{H}_{[o,u]}) \qquad \text{for } u \in \mathbb{R}^n,$$

where $\mathcal{H}_{[o,u]}$ is the set of hyperplanes having nonempty intersection with the segment $[o, u]$ with endpoints o and u. This gives

$$h(\Pi_{\Lambda X}, u) = \frac{1}{2}\mathbb{E}(\Lambda X)(\mathcal{H}_{[o,u]}) = \frac{1}{2}\mathbb{E}X(\mathcal{H}_{[o,\Lambda^{-1}u]}) = h(\Pi_X, \Lambda^{-1}u) = h(\Lambda^{-t}\Pi_X, u).$$

We deduce that

$$\Pi_{\Lambda X} = \Lambda^{-t}\Pi_X. \tag{22}$$

From Lemma 1 of the previous section, we now conclude that there exists a linear transformation $\Lambda \in \mathrm{GL}(n)$ such that $\Pi_{\Lambda X}$ has isotropic generating measure. From the definition of $\Gamma_k(X)$ and from (21) we get, with $Z := \Pi_{\Lambda X}$,

$$\Gamma_k(X) \geq \frac{\gamma_k(\Lambda X)}{\gamma(\Lambda X)^k} = \frac{V_k(Z)}{V_1(Z)^k}.$$

Corollary 1 gives $V_1(Z) = 2n$ and $V_k(Z) \geq 2^k\binom{n}{k}$. This yields the inequality (20). If equality holds for some $k \geq 2$, then Z is a cube, hence Π_X is a parallelepiped. Therefore, its generating measure, which is a multiple of the spherical directional distribution of X, is concentrated in $\pm e_1, \ldots, \pm e_n$, for some basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$. This completes the proof of Theorem 3. $\square$

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