Convex Cones

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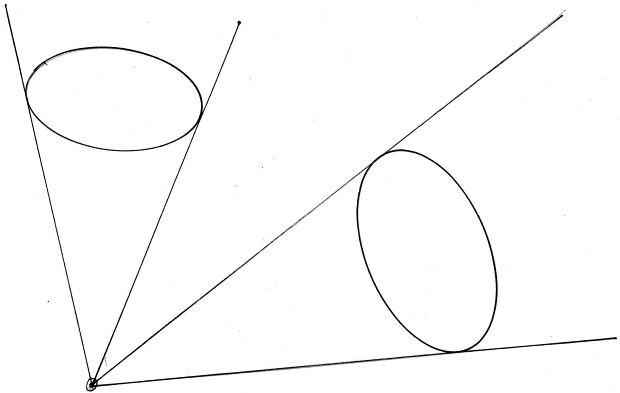
A closed convex cone in $\mathbb{R}^d$, briefly a cone, is a closed, convex set $\emptyset \neq C \subseteq \mathbb{R}^d$ with $\lambda x \in C$ for $x \in C$ and $\lambda \geq 0$.

We consider the following question:

Let C, D be cones. Let D undergo a uniform random rotation Θ.

What is the probability that C and ΘD have a nontrivial intersection, that is, $C \cap \Theta D \neq \{o\}$?



The answer is immediate if C and D happen to be subspaces:

$$\mathbb{P}(C \cap \Theta D \neq \{o\}) = \begin{cases} 0 & \text{if } \dim C + \dim D \leq d, \\ 1 & \text{if } \dim C + \dim D > d. \end{cases}$$

Thus, the probability depends only on the sum of the **dimensions** of C and D.

The great surprise is that for cones one can define a number, the statistical dimension, such that a similar result holds approximately.

Where does this come from?

The use of convex optimization for signal demixing under a certain random model.

Michael B. McCoy: *A geometric analysis of convex demixing.* PhD Thesis, California Institute of Technology, 2013.

Quotation:

“Our framework includes a random orientation model for the constituent signals that ensures the structures are incoherent. This work introduces a summary parameter, the **statistical dimension**, that reflects the intrinsic complexity of a signal.

. . . demixing succeeds with high probability when the sum of the complexities is less than the ambient dimension; otherwise, it fails with high probability.

The fact that a **phase transition** between success and failure occurs in demixing is a consequence of a new inequality in conic integral geometry. Roughly speaking, this inequality asserts that a convex cone behaves like a subspace whose dimension is equal to the statistical dimension of the cone.”

This was continued and expanded in

D. Amelunxen, M. Lotz, M.B. McCoy, J.A. Tropp: Living on the edge: phase transitions in convex programs with random data. *Information and Inference* **3** (2014), 224–294.

(Winner of the “Information and Inference Best Paper Prize” 2015).

Quotation: “This paper introduces a summary parameter, called the statistical dimension, that canonically extends the dimension of a linear subspace to the class of convex cones.

The main technical result demonstrates that the sequence of intrinsic volumes of a convex cone concentrates sharply around the statistical dimension.

This fact leads to accurate bounds on the probability that a randomly rotated cone shares a ray with a fixed cone.”

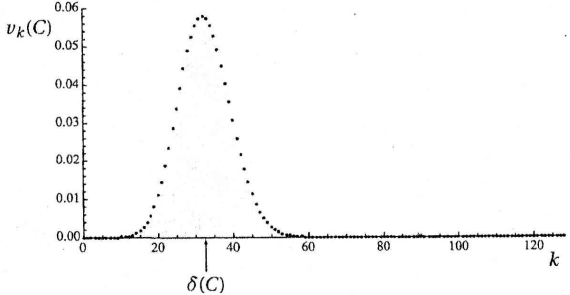


FIG. 3. Concentration of conic intrinsic volumes. This plot displays the conic intrinsic volumes $v_k(C)$ of a circular cone $C \subset \mathbb{R}^{128}$ with angle $\pi/6$. The distribution concentrates sharply around the statistical dimension $\delta(C) \approx 32.5$. See Section 3.4 for further discussion of this example.

Amelunxen, Lotz, McCoy, Tropp 2014

Applications described in that paper: Phase transitions in random convex optimization problems

• compressed sensing; ℓ_1 minimization for identifying a sparse vector from random linear measurements

• regularized linear inverse problems with random measurements

• demixing problems under a random incoherence model

• cone programs with random affine constraints

The core from the viewpoint of convex geometry

C, D are cones, not both subspaces, and Θ is a uniform random rotation.

We ask for the probability

$$\mathbb{P}(C \cap \Theta D \neq \{o\}).$$

In principle, the answer is known for a long time:

Spherical integral geometry (Santaló 1976, Glasauer 1995), translated into the conical setting, yields the following.

First, one needs to define the conic intrinsic volumes $v_1(C), \dots, v_d(C)$ of a closed convex cone $C \subseteq \mathbb{R}^d$.

Second, the conic kinematic formula provides the expectation

$$\mathbb{E} v_k(C \cap \Theta D) = \sum_{i=k}^{d} v_i(C) v_{d+k-i}(D)$$

for $k = 1, \dots, d$.

Third, a version of the spherical Gauss–Bonnet theorem says that

$$2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2k+1}(C) = 1 \quad \text{if } C \text{ is not a subspace.}$$

Since $C \cap \Theta D$ is, with probability one, either $\{o\}$ (in which case $v_k(C \cap \Theta D) = 0$ for $k \geq 1$) or not a subspace, this implies that

$$\mathbb{1}\{C \cap \Theta D \neq \{o\}\} = 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2k+1}(C \cap \Theta D) \quad \text{almost surely}$$

and hence

$$\mathbb{P}(C \cap \Theta D \neq \{o\}) = 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{i=2k+1}^{d} v_i(C) v_{d+2k+1-i}(D).$$

This is the promised explicit answer.

However, it is useless for most applications, since one cannot compute the conic intrinsic volumes.

Fortunately, one can prove concentration of the conic intrinsic volumes around the statistical dimension, and this allows valuable conclusions.

The following gives an introduction to the conic intrinsic volumes, the conic kinematic formula, and the concentration result.

Convex cones

$\mathcal{C}^d$ denotes the set of closed convex cones in $\mathbb{R}^d$.

Let $C \in \mathcal{C}^d$.

The dual or polar cone of C is

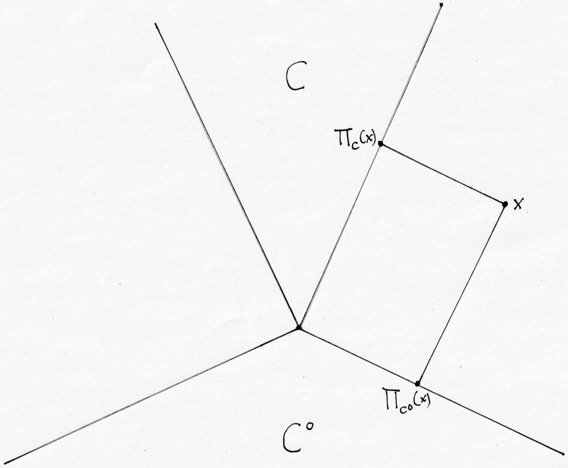
$$C^\circ := \{x \in R^d : \langle x, y \rangle \leq 0 \text{ for all } y \in C\}.$$

We have $C^{\circ\circ} = C$.

For each $x \in \mathbb{R}^d$, there is a unique point $\Pi_C(x) \in C$ such that

$$\|x - \Pi_C(x)\| = \min\{\|x - y\| : y \in C\}.$$

This defines the metric projection or nearest-point map Π_C of C.


C
$\Pi_C(x)$
x
$\Pi_{C^\circ}(x)$
C°

Lemma. (Moreau decomposition)

For all $x \in \mathbb{R}^d$,

$$x = \Pi_C(x) + \Pi_{C^\circ}(x),$$

where

$$\langle \Pi_C(x), \Pi_{C^\circ}(x) \rangle = 0.$$

Proof.

W.l.o.g., $x \in \mathbb{R}^d \setminus (C \cup C^\circ)$ (otherwise trivial).

$u := x - \Pi_C(x)$ is an outer normal vector of a supporting hyperplane H of C through $\Pi_C(x)$. We have $o \in H$.

$\Rightarrow u \in C^\circ$

$o \in H \Rightarrow \Pi_C(x) \perp u$

The hyperplane $H' \perp \Pi_C(x)$ through o supports C°.

$\Rightarrow \Pi_{C^\circ}(x)$, the point in C° nearest to x, is also the point in H' nearest to x.

$\Rightarrow \Pi_{C^\circ}(x) = u$

The points $x, \Pi_C(x), \Pi_{C^\circ}(x), o$ are the vertices of a rectangle.

$\Rightarrow$ assertion

Conic intrinsic volumes of polyhedral cones

First, we consider only polyhedral cones.

For these, the quickest definition of the conic intrinsic volumes requires the k-skeleton

$$\mathrm{skel}_k(C) := \bigcup_{F \in \mathcal{F}_k(C)} \mathrm{relint}\, F,$$

where $\mathcal{F}_k$ denotes the set of k-faces of C.

Let $\mathbf{g}$ be a standard Gaussian random vector in $\mathbb{R}^d$.

Definition. For $k = 0, \ldots, d$, the kth conic intrinsic volume of C is defined by

$$v_k(C) := \mathbb{P}(\Pi_C(\mathbf{g}) \in \mathrm{skel}_k(C)).$$

Some explanations:

The distribution of **g** is the standard Gaussian measure,

$$\gamma_d(A) = \frac{1}{\sqrt{2\pi}^d} \int_A e^{-\frac{1}{2}\|x\|^2} \mathrm{d}x.$$

The Gaussian measure on a subspace L is denoted by γ_L.

The relation between the spherical Lebesgue measure σ_{d-1} of a Borel set $A \subseteq \mathbb{S}^{d-1}$ (unit sphere) and the Gaussian measure of the spanned cone $A^\vee := \{\lambda a : a \in A,\ \lambda \geq 0\}$ is given by

$$\gamma_d(A^\vee) = \frac{\sigma_{d-1}(A)}{\sigma_{d-1}(\mathbb{S}^{d-1})}.$$

(Hence, ‘angles’ in the following are spherical volumes.)

We define the internal angle of C at o by

$$\beta(o, C) := \gamma_{\langle C \rangle}(C),$$

where $\langle C \rangle$ is the linear hull of C, and the external angle of C at its face F by

$$\gamma(F, C) := \gamma_{\langle F \rangle^\perp}(N(C, F)),$$

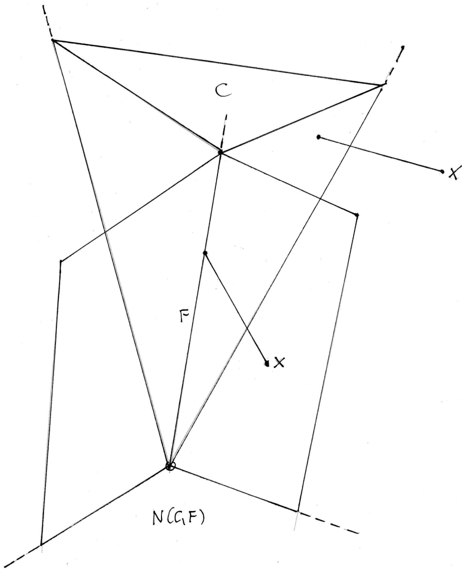
where $N(C, F)$ is the normal cone of C at F.

Recall that

$$v_k(C) := \mathbb{P}(\Pi_C(\mathbf{g}) \in \mathrm{skel}_k(C)).$$

For a face $F \in \mathcal{F}_k(C)$ we have

$$\Pi_C(x) \in \mathrm{relint}\, F \Leftrightarrow x \in (\mathrm{relint}\, F) + N(C, F).$$


C
X
F
X
N(C,F)

Therefore,

$$\mathbb{P}(\Pi_C(\mathbf{g}) \in \operatorname{relint} F) = \gamma_d(F + N(C, F)) = \beta(o, F)\gamma(F, C).$$

The latter holds since the Gauss measure of a direct sum is the product of the Gauss measures of the summands.

We conclude that

$$\begin{aligned} v_k(C) &= \mathbb{P}(\Pi_C(\mathbf{g}) \in \operatorname{skel}_k(C)) \\ &= \sum_{F \in \mathcal{F}_k(C)} \gamma_d(F + N(C, F)) \\ &= \sum_{F \in \mathcal{F}_k(C)} \beta(o, F)\gamma(F, C). \end{aligned}$$

Relations between conic intrinsic volumes

A duality relation

Let $F \in \mathcal{F}_k(C)$, then $G := N(C,F) \in \mathcal{F}_{d-k}(C^\circ)$ and $N(C^\circ, G) = F$, hence $F + N(C,F) = G + N(C^\circ, G)$.
It follows that

$$v_k(C) = v_{d-k}(C^\circ) \quad \text{for } k = 0, \ldots, d.$$

A trivial relation

From the definition

$$v_k(C) := \mathbb{P}(\Pi_C(\mathbf{g}) \in \mathrm{skel}_k(C)),$$

it follow immediately that

$$v_0 + \cdots + v_d = 1.$$

A nontrivial relation

If C is not a subspace, then

$$v_0(C) - v_1(C) + v_2(C) - \cdots + (-1)^d v_d(C) = 0.$$

This follows from the identity

$$\sum_{F \in \mathcal{F}(C)} (-1)^{\dim F} \mathbb{1}_{F - N(C,F)}(x) = 0 \quad \text{for } x \in \mathbb{R}^d \setminus U,$$

where U is the union of all faces of dimension $< d - 1$ of all cones $F - N(C,F)$, $F \in \mathcal{F}(C)$.

The identity is due to P. McMullen (sketched 1975, proved 1981, reproduced in Schneider and Weil, Stochastic and Integral Geometry, 2008).

A side remark

The identity

$$(*) \qquad \sum_{F \in \mathcal{F}(C)} (-1)^{\dim F} \mathbb{1}_{F - N(C,F)}(x) = 0$$

was recently proved for all $x \in \mathbb{R}^d$, without an exceptional set.

More generally, the following identity for convex polyhedra (not necessarily bounded) holds.

Let P be a convex polyhedron,

- F a face of P,
- $A(F, P)$ the angle cone of P at F,
- $N(P, F)$ the normal cone of P at F.

Theorem 1. *Let $E \neq P$ be a face of P. Then*

$$\sum_{E \subseteq F \in \mathcal{F}(P)} (-1)^{\dim F} \mathbb{1}_{A(E,F)-N(P,F)} = 0.$$

The proof, in

R. Schneider, Combinatorial identities for polyhedral cones. *Algebra i Analiz* **29** (2017), 279–295

uses the generalized Brianchon–Gram–Sommerville relation (on the level of indicator functions) and employs the incidence algebra of the face lattice of P.

If P is a cone (not a subspace) and E is the lineality space of P, then $A(E,F) = F$, hence Theorem 1 gives (*).

The identity

$$(*) \qquad \sum_{F \in \mathcal{F}(C)} (-1)^{\dim F} \mathbb{1}_{F - N(C,F)} = 0$$

was extended to all polyhedra by Hug and Kabluchko 2018.

(End of the side remark)

Back to the identity

$$v_0(C) - v_1(C) + v_2(C) - \cdots + (-1)^d v_d(C) = 0.$$

Above, we had excluded subspaces. If C is a subspace, then

$$v_0(C) - v_1(C) + v_2(C) - \cdots + (-1)^d v_d(C) = (-1)^{\dim C}.$$

For a formula comprising both cases, we need the Euler characteristic.

More generally, for a closed convex set $K \subset \mathbb{R}^d$, define

$$\text{type}(K) := (k, \epsilon),$$

where k is the largest dimension of an affine subspace contained in K, and $\epsilon = 1$ or 0 according to whether the line-free kernel of K is bounded or not.

Theorem and Definition. *There is a unique real valuation χ on $U(\mathcal{CC}^d)$ (the set of all finite unions of closed convex sets), the Euler characteristic, with*

$$\chi(K) = (-1)^k \epsilon \quad \textit{if type}(K) = (k, \epsilon), \quad \textit{for } K \in \mathcal{CC}^d.$$

Hadwiger' (1955) elementary existence proof extends.

With this definition, we have for any polyhedral cone C

$$\sum_{k=0}^{d}(-1)^k v_k(C) = \chi(C).$$

Together with

$$\sum_{k=0}^{d} v_k(C) = 1,$$

this yields

$$2\sum_{k=0}^{\lfloor\frac{d-1}{2}\rfloor} v_{2k+1}(C) = 1 - \chi(C).$$

This is important for our initial question, since almost surely

$$2\sum_{k=0}^{\lfloor\frac{d-1}{2}\rfloor} v_{2k+1}(C \cap \Theta D) = \begin{cases} 1 & \text{if } C \cap \Theta D \neq \{o\}, \\ 0 & \text{if } C \cap \Theta D = \{o\}. \end{cases}$$

Remark. The relation

$$2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2k+1}(C) = 1 - \chi(C)$$

is a version of the spherical Gauss–Bonnet theorem.

For smooth submanifolds of the sphere, it appears in Santaló 1955, 1976.

Up to now, we have only considered polyhedral cones.

We use a Steiner formula to extend the conic intrinsic volumes to general closed convex cones.

Steiner formulas

M.B. McCoy, J.A.Tropp, From Steiner formulas for cones to concentration of intrinsic volumes. *Discrete Comput. Geom.* **51** (2014), 926–963:

Theorem 2. (Master Steiner formula) *For a polyhedral cone C and a measurable function f* : $\mathbb{R}_+^2 \to \mathbb{R}_+$ *define*

$$\varphi_f(C) := \mathbb{E} f\left(\|\Pi_C(\mathbf{g})\|^2, \|\Pi_{C^\circ}(\mathbf{g})\|^2 \right).$$

Then

$$\varphi_f(C) = \sum_{k=0}^{d} \mathcal{I}_k(f) \cdot v_k(C),$$

where

$$\mathcal{I}_k(f) = \varphi_f(L_k), \quad L_k \in G(d,k).$$

Explicitly,

$$\mathcal{I}_k(f) = \frac{\omega_k \omega_{d-k}}{\sqrt{2\pi}^d} \int_0^\infty \int_0^\infty f(r^2, s^2) e^{-\frac{1}{2}(r^2+s^2)} r^{k-1} s^{d-k-1} \, \mathrm{d}s \, \mathrm{d}r$$

for $k = 1, \ldots, d-1$ and

$$\begin{aligned} \mathcal{I}_0(f) &= \frac{\omega_d}{\sqrt{2\pi}^d} \int_0^\infty f(0, s^2) e^{-\frac{1}{2}s^2} s^{d-1} \, \mathrm{d}s, \\ \mathcal{I}_d(f) &= \frac{\omega_d}{\sqrt{2\pi}^d} \int_0^\infty f(r^2, 0) e^{-\frac{1}{2}r^2} r^{d-1} \, \mathrm{d}r. \end{aligned}$$

Essentials of the proof:

$$
\begin{aligned}
&\varphi_f(C) \\
&= \mathbb{E} f\left(\|\Pi_C(\mathbf{g})\|^2, \|\Pi_{C^\circ}(\mathbf{g})\|^2\right) \\
&= \sum_{k=0}^{d} \sum_{F \in \mathcal{F}_k(C)} \mathbb{E}\left[f\left(\|\Pi_C(\mathbf{g})\|^2, \|\Pi_{C^\circ}(\mathbf{g})\|^2\right) \mathbb{1}_{\operatorname{relint} F}(\Pi_C(\mathbf{g})\right].
\end{aligned}
$$

In the following, we use the Moreau decomposition.

Then we use the fact that the Gauss measure of a direct sum is the product of the Gauss measures of the summands.

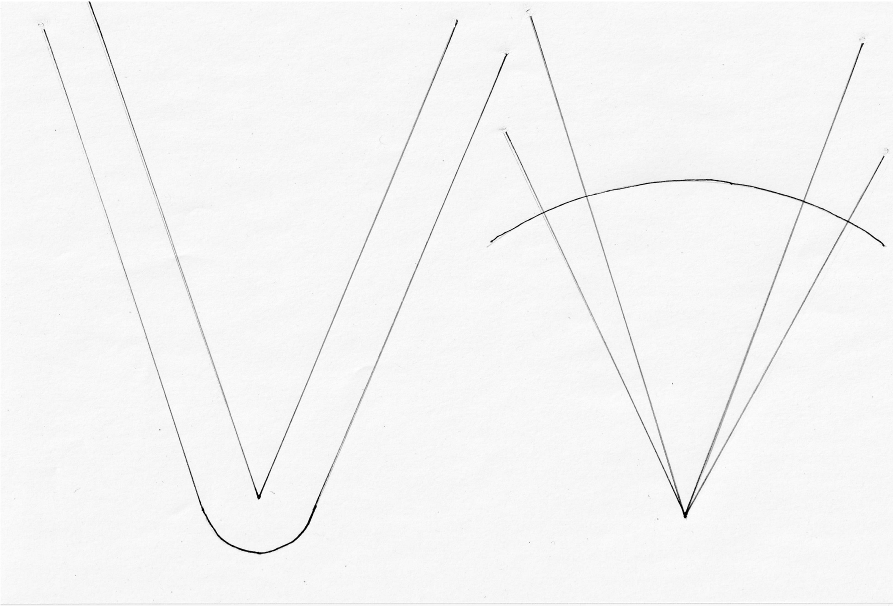
For $F \in \mathcal{F}_k(C)$,

$$\mathbb{E}\left[f\left(\|\Pi_C(\mathbf{g})\|^2, \|\Pi_{C^\circ}(\mathbf{g})\|^2\right) \mathbb{1}_{\mathrm{relint}\,F}(\Pi_C(\mathbf{g})\right]$$
$$= \frac{1}{\sqrt{2\pi}^d} \int_{F \oplus N(C,F)} f\left(\|\Pi_C(z)\|^2, \|\Pi_{C^\circ}(z)\|^2\right) e^{-\frac{1}{2}\|z\|^2} \lambda_d(\mathrm{d}z)$$
$$= \frac{1}{\sqrt{2\pi}^d} \int_F \int_{N(C,F)} f(\|x\|^2, \|y\|^2) e^{-\frac{1}{2}(\|x\|^2 + \|y\|^2)} \lambda_{d-k}(\mathrm{d}y) \lambda_k(\mathrm{d}x).$$

The rest is computation.

Specializations of the Master Steiner formula yield ‘usual’ Steiner formulas, for the volume of parallel sets.

But for cones, we replace ‘volume’ by Gauss measure, and ‘parallel sets’ can be interpreted differently.



(a) The Gaussian Steiner formula

Here we consider the usual parallel set of C at distance $\lambda \geq 0$,

$$\{x \in \mathbb{R}^d : \mathrm{dist}(x, C) \leq \lambda\} = C + \lambda B^d.$$

The choice

$$f(a, b) = \begin{cases} 1 & \text{if } b \leq \lambda^2, \\ 0 & \text{otherwise} \end{cases}$$

in the Master Steiner formula yields

Corollary 1. *For a polyhedral cone C, the Gaussian measure of the parallel set $C + \lambda B^d$ is given by*

$$\gamma_d(C + \lambda B^d) = \sum_{k=0}^{d} f_k(\lambda) \cdot v_k(C)$$

with

$$f_k(\lambda) = \frac{\omega_{d-k}}{\sqrt{2\pi}^{\,d-k}} \int_0^\lambda e^{-\frac{1}{2}s^2} s^{d-k-1}\, ds$$

for $k = 0, \ldots, d-1$ and $f_d \equiv 1$.

This allows an approach to the conic intrinsic volumes that parallels the approach to the classical intrinsic volumes of convex bodies.

(b) The spherical Steiner formula
(in a conic interpretation)

Let $d_a(x, y)$ denote the angle between the vectors $x, y \in \mathbb{R}^d$,

$$d_a(x, y) = \arccos \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle, \quad x, y \neq o.$$

The angular distance of $x \in \mathbb{R}^d \setminus C^\circ$ from a cone $C \neq \{o\}$ is

$$\begin{aligned} d_a(x, C) &= \min\{d_a(x, y) : y \in C \setminus \{o\}\} \\ &= \arccos \frac{\|\Pi_C(x)\|}{\|x\|}. \end{aligned}$$

The angular parallel set of $C \neq \{o\}$ at distance $\lambda \geq 0$ is

$$C^a_\lambda = \{x \in \mathbb{R}^d \setminus C^\circ : d_a(x, C) \leq \lambda\}.$$

The choice

$$f(a,b) = \begin{cases} 1 & \text{if } a \le b \tan^2 \lambda, \\ 0 & \text{otherwise} \end{cases}$$

in the Master Steiner formula yields:

Corollary 2. *For a polyhedral cone C, the Gaussian measure of the angular parallel set C_λ^a at distance $0 \le \lambda < \pi/2$ is given by*

$$\gamma_d(C_\lambda^a) = \sum_{k=0}^{d} g_k(\lambda) \cdot v_k(C)$$

with

$$g_k(\lambda) = \frac{\omega_k \omega_{d-k}}{\omega_d} \int_0^\lambda \cos^{k-1}\varphi \sin^{d-k-1}\varphi \, d\varphi$$

for $k = 1, \ldots, d-1$ and $g_d \equiv 1$.

McCoy and Tropp (2014) have elegant reformulations of the special Steiner formulas:

Corollary 3. *For* $\lambda \geq 0$*,*

$$\mathbb{P}(dist^2(\mathbf{g}, C) \leq \lambda) = \sum_{k=0}^{d} \mathbb{P}(\mathcal{X}_{d-k} \leq \lambda) \cdot v_k(C),$$

where $\mathcal{X}_{d-k}$ *is a random variable following the chi-square distribution with* $d - k$ *degrees of freedom.*

Corollary 4. *If* $\mathbf{u}$ *is a uniform random vector in the sphere* $\mathbb{S}^{d-1}$*, then, for* $\lambda \in [0, 1]$*,*

$$\mathbb{P}(dist^2(\mathbf{u}, C) \leq \lambda) = \sum_{k=0}^{d} \mathbb{P}(B_{d-k,k} \leq \lambda) \cdot v_k(C),$$

where $B_{d-k,k}$ *is a random variable following the beta distribution with parameters* $j/2$ *and* $k/2$*.*

We can now extend all this by continuity.

Let $\mathcal{C}_*^d$ denote the set of closed convex cones $\neq \{o\}$ in $\mathbb{R}^d$.

On $\mathcal{C}_*^d$, we define the angular Hausdorff metric by

$$\delta_a(C, D) := \min\{\varepsilon \geq 0 : C \subseteq D_\varepsilon^a,\ D \subseteq C_\varepsilon^a\}.$$

With respect to this metric, polarity is a local isometry:
If $C, D \in \mathcal{C}_*^d$ are cones $\neq \mathbb{R}^d$ with $\delta_a(C, D) < \pi/2$, then

$$\delta_a(C^\circ, D^\circ) = \delta_a(C, D)$$

(Glasauer 1995).

For $0 \leq \lambda < \pi/2$, let

$$\mu_\lambda(C) := \gamma_d(C_\lambda^a)$$

denote the Gaussian measure of the angular parallel set of C at angular distance λ.

If $C_i, C \in \mathcal{C}_*^d$ and $C_i \to C$ in the angular Hausdorff metric, then

$$\mu_\lambda((C_i)_\lambda^a) \to \mu_\lambda(C_\lambda^a).$$

Further, any cone $C \in \mathcal{C}_*^d$ can be approximated arbitrarily closely by polyhedral cones, with respect to the angular Hausdorff metric.

With this, the following theorem can be proved.

Theorem 3.
To every cone $C \in \mathcal{C}_*^d$*, there exist nonnegative numbers* $v_0(C), \dots, v_d(C)$ *such that, for every* λ *with* $0 \leq \lambda < \pi/2$*, the Gaussian measure of the angular parallel set* C_λ^a *is given by*

$$\gamma_d(C_\lambda^a) = \sum_{k=1}^{d} g_k(\lambda) \cdot v_k(C)$$

with

$$g_k(\lambda) = \frac{\omega_k \omega_{d-k}}{\omega_d} \int_0^\lambda \cos^{k-1} \varphi \sin^{d-k-1} \varphi \, d\varphi$$

for $k = 1, \dots, d-1$ *and* $g_d \equiv 1$*. Further,* $v_k(\{o\}) = \delta_{k,0}$*.*

Each mapping $v_k : \mathcal{C}^d \to \mathbb{R}$ *is a continuous valuation.*

$v_k(C)$ is called the kth conic intrinsic volume of the cone C.

Let us briefly resume where we are.

Our goal is to compute and estimate the probability

$$\mathbb{P}(C \cap \Theta D \neq \{o\}),$$

where C, D are given convex cones (not both subspaces) and Θ is a uniform random rotation.

By now, we know how to express the indicator function of the crucial event $C \cap \Theta D \neq \{o\}$ in terms of conic intrinsic volumes of $C \cap \Theta D$, namely (a.s.)

$$2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2k+1}(C \cap \Theta D) = \begin{cases} 1 & \text{if } C \cap \Theta D \neq \{o\}, \\ 0 & \text{if } C \cap \Theta D = \{o\}. \end{cases}$$

Recall that Θ is a uniform random rotation, that is, a random variable with values in the rotation group SO_d, with distribution given by the Haar probability measure ν on SO_d.

Hence, we have to compute

$$\mathbb{E} v_k(C \cap \Theta D) = \int_{SO_d} v_k(C \cap \vartheta D)\, \nu(\mathrm{d}\vartheta).$$

The computation of such integrals is a task of integral geometry. The relevant kinematic formulas are well known in Euclidean space, and moderately well known in spherical space.

Spherical integral geometry, from the conic viewpoint, received new interest when its applicability in convex programming was discovered.

Theorem 4. (Conic kinematic formula)
If $C, D \in \mathcal{C}^d$ are closed convex cones, then

$$\int_{SO_d} v_k(C \cap \vartheta D)\, \nu(d\vartheta) = \sum_{i=k}^{d} v_i(C) v_{d+k-i}(D)$$

for $k = 1, \ldots, d$, and

$$\int_{SO_d} v_0(C \cap \vartheta D)\, \nu(\mathrm{d}\vartheta) = \sum_{i=0}^{d} \sum_{j=0}^{d-i} v_i(C) v_j(D).$$

The second part of the theorem follows from the first, since $v_0 + \cdots + v_d = 1$.

Duality at work

Duality immediately yields the following:

Theorem 5.
If $C, D \in \mathcal{C}^d$ are closed convex cones, then

$$\int_{SO_d} v_{d-k}(C + \vartheta D)\,\nu(d\vartheta) = \sum_{i=0}^{d-k} v_i(C) v_{d-k-i}(D)$$

for $k = 1, \ldots, d$, and

$$\int_{SO_d} v_d(C + \vartheta D)\,\nu(\mathrm{d}\vartheta) = \sum_{i=0}^{d} \sum_{j=d-r}^{d} v_i(C) v_j(D).$$

In the following, we sketch a new proof, which is due to

D. Amelunxen, M. Lotz, Intrinsic volumes of polyhedral cones: a combinatorial perspective. *Discrete Comput. Geom.* **58** (2017), 371–409.

We restrict ourselves to polyhedral cones (the general case can be obtained by approximation) and to the principal ideas.

These consist in clever integrations over suitable subgroups of the rotation group and applying (of course) Fubini's theorem.

First we prove two special subcases of the two theorems above (which are used in the general proof).

Lemma 1. *Let* $C, D \subset \mathbb{R}^d$ *be polyhedral cones, where*

$$\dim C = i, \quad \dim D = j, \quad i + j = d + k > d.$$

Then

$$\int_{SO_d} v_k(C \cap \vartheta D)\,\nu(\mathrm{d}\vartheta) = v_i(C) v_j(D).$$

For the proof, recall that we use $\langle C \rangle$ for the linear hull of the cone C.

We also use $SO_{\langle C \rangle}$ for the rotation group of the subspace $\langle C \rangle$ and $\nu_{\langle C \rangle}$ for its Haar probability measure.

For ν-almost all $\vartheta \in SO_d$, we have $\dim(\langle C \rangle \cap \vartheta \langle D \rangle) = k$. Therefore,

$$\int_{SO_d} V_k(C \cap \vartheta D)\, \nu \mathrm{d}\vartheta)$$

$$= \int_{SO_d} \gamma_{\langle C \rangle \cap \vartheta \langle D \rangle}(C \cap \vartheta D)\, \nu(\mathrm{d}\vartheta)$$

$$= \int_{SO_d} \int_{\langle C \rangle \cap \vartheta \langle D \rangle} \mathbb{1}_C(x) \mathbb{1}_{\vartheta D}(x) \gamma_{\langle C \rangle \cap \vartheta \langle D \rangle}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

Here we replace ϑ by $\vartheta\rho$ with $\rho \in SO_{\langle D \rangle}$, which does not change the outer integral (and observe that $\rho \langle D \rangle = \langle D \rangle$).

Hence, we can integrate over all ρ with respect to $\nu_{\langle D \rangle}$.

This gives

$$\int_{SO_d} v_k(C \cap \vartheta D)\, \nu \mathrm{d}\vartheta)$$

$$= \int_{SO_{\langle D \rangle}} \int_{SO_d} \int_{\langle C \rangle \cap \vartheta \langle D \rangle} \mathbb{1}_C(x) \mathbb{1}_D(\rho^{-1}\vartheta^{-1}x)$$

$$\times \gamma_{\langle C \rangle \cap \vartheta \langle D \rangle}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)\, \nu_{\langle D \rangle}(\mathrm{d}\rho)$$

$$= \int_{SO_d} \int_{\langle C \rangle \cap \vartheta \langle D \rangle} \mathbb{1}_C(x) \left[\int_{SO_{\langle D \rangle}} \mathbb{1}_D(\rho^{-1}\vartheta^{-1}x)\, \nu_{\langle D \rangle}(\mathrm{d}\rho) \right]$$

$$\times \gamma_{\langle C \rangle \cap \vartheta \langle D \rangle}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

By properties of the Haar measure, the integral in brackets is equal to

$$\gamma_{\langle D \rangle}(D) = v_j(D).$$

Thus, we obtain:

$$\int_{SO_d} v_k(C \cap \vartheta D)\, \nu \mathrm{d}\vartheta)$$

$$= v_j(D) \int_{SO_d} \int_{\langle C \rangle \cap \vartheta \langle D \rangle} \mathbb{1}_C(x)\, \gamma_{\langle C \rangle \cap \vartheta \langle D \rangle}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

Now we play the same trick again.

The outer integral does not change if we replace ϑ by $\sigma\vartheta$ with $\sigma \in SO_{\langle C \rangle}$.

Integrating over all σ, we get

$$
\begin{aligned}
&\int_{SO_d} v_k(C \cap \vartheta D)\, \nu \mathrm{d}\vartheta) \\
&= v_j(D) \int_{SO_{\langle C \rangle}} \int_{SO_d} \int_{\sigma(\langle C \rangle \cap \vartheta \langle D \rangle)} \mathbb{1}_C(x) \\
&\quad \times \gamma_{\sigma(\langle C \rangle \cap \vartheta \langle D \rangle)}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)\, \nu_{\langle C \rangle}(\mathrm{d}\sigma) \\
&= v_j(D) \int_{SO_d} \int_{\langle C \rangle \cap \vartheta \langle D \rangle} \left[\int_{SO_{\langle C \rangle}} \mathbb{1}_C(\sigma x)\, \nu_{\langle C \rangle}(\mathrm{d}\sigma) \right] \\
&\quad \times \gamma_{\langle C \rangle \cap \vartheta \langle D \rangle}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \\
&= v_j(D) v_i(C),
\end{aligned}
$$

since $\gamma_{\langle C \rangle \cap \vartheta \langle D \rangle}$ and ν are probability measures.
This proves Lemma 1.

Lemma 2. *Let* $C, D \subset \mathbb{R}^d$ *be polyhedral cones, where*

$$\dim C = i, \quad \dim D = j, \quad i + j = d - k < d.$$

Then

$$\int_{SO_d} v_{d-k}(C + \vartheta D)\, \nu(\mathrm{d}\vartheta) = v_i(C) v_j(D).$$

For ν-almost all ϑ we have $\dim(\langle C\rangle + \vartheta\langle D\rangle) = i + j = d - k$. Therefore, using the product property of the Gaussian measure,

$$\begin{aligned} v_{d-k}(C + \vartheta D) &= \int_{\langle C\rangle + \vartheta\langle D\rangle} \mathbb{1}_{C+\vartheta D}(y) \gamma_{\langle C\rangle + \vartheta\langle D\rangle}(\mathrm{d}y) \\ &= \int_{\mathbb{R}^d} \mathbb{1}_{C+\vartheta D + (\langle C\rangle + \vartheta\langle D\rangle)^\perp}(x)\, \gamma_d(\mathrm{d}x). \end{aligned}$$

For ν-almost all ϑ, there is the unique ϑ-decomposition

$$x = x_{C,\vartheta} + x_{D,\vartheta} + x_\vartheta$$

with

$$x_{C,\vartheta} \in \langle C \rangle, \quad x_{D,\vartheta} \in \vartheta\langle D \rangle, \quad x_\vartheta \in (\langle C \rangle + \vartheta\langle D \rangle)^\perp.$$

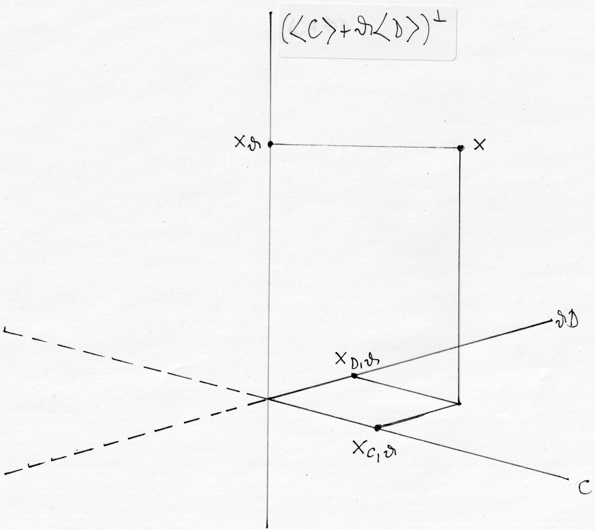
Apply $\rho \in SO_{\langle C \rangle}$, to get

$$\rho x = \rho x_{C,\vartheta} + \rho x_{D,\vartheta} + \rho x_\vartheta$$

with

$$\rho x_{C,\vartheta} \in \langle C \rangle, \quad \rho x_{D,\vartheta} \in \rho\vartheta\langle D \rangle, \quad \rho x_\vartheta \in (\langle C \rangle + \rho\vartheta\langle D \rangle)^\perp,$$

which is the $\rho\vartheta$-decomposition of ρx.



$(\langle C\rangle + \vartheta\langle D\rangle)^{\perp}$
x_ϑ
x
ϑD
$x_{D,\vartheta}$
$x_{C,\vartheta}$
C

Thus, we obtain

$$\int_{SO_d} v_{d-k}(C + \vartheta D)\,\nu(\mathrm{d}\vartheta)$$

as rewritten above

$$= \int_{SO_d} \int_{\mathbb{R}^d} \mathbb{1}_{C+\vartheta D+(\langle C\rangle+\vartheta\langle D\rangle)^\perp}(x)\,\gamma_d(\mathrm{d}x)\,\nu(\mathrm{d}\vartheta)$$

apply the ϑ-decomposition

$$= \int_{SO_d} \int_{\mathbb{R}^d} \mathbb{1}_C(x_{C,\vartheta})\mathbb{1}_{\vartheta D}(x_{D,\vartheta})\,\gamma_d(\mathrm{d}x)\,\nu(\mathrm{d}\vartheta)$$

replace ϑ by $\rho\vartheta$ and x by ρx with $\rho \in SO_{\langle C\rangle}$

$$= \int_{SO_d} \int_{\mathbb{R}^d} \mathbb{1}_C((\rho x)_{C,\rho\vartheta})\mathbb{1}_{\rho\vartheta D}((\rho x)_{D,\rho\vartheta})\,\gamma_d(\mathrm{d}x)\,\nu(\mathrm{d}\vartheta)$$

$(\rho\vartheta)$-decomposition of $\rho x = \rho(\vartheta$-decomposition of $x)$

$$= \int_{SO_d} \int_{\mathbb{R}^d} \mathbb{1}_C(\rho x_{C,\vartheta}) \mathbb{1}_{\rho\vartheta D}(\rho x_{D,\vartheta})\, \gamma_d(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$

integrate over all ρ

$$= \int_{SO_{\langle C\rangle}} \int_{SO_d} \int_{\mathbb{R}^d} \mathbb{1}_C(\rho x_{C,\vartheta}) \mathbb{1}_{\vartheta D}(x_{D,\vartheta})\, \gamma_d(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)\, \}\nu_{\langle C\rangle}(\mathrm{d}\rho)$$

apply Fubini

$$= \int_{SO_d} \int_{\mathbb{R}^d} \left[\int_{SO_{\langle C\rangle}} \mathbb{1}_C(\rho x_{C,\vartheta}) \nu_{\langle C\rangle}(\mathrm{d}\rho) \right] \mathbb{1}_D(\vartheta^{-1} x_{D,\vartheta})\, \gamma_d(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$

integral in brackets $= v_i(C)$

$$= v_i(C) \int_{SO_d} \int_{\mathbb{R}^d} \mathbb{1}_D(\vartheta^{-1} x_{D,\vartheta})\, \gamma_d(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$

replace ϑ by $\vartheta\sigma^{-1}$ with $\sigma \in SO_{\langle D\rangle}$, integrate over all σ

$= v_i(C) v_j(D).$ This proves Lemma 2.

Aiming at the proof of the conic kinematic formula for polyhedral cones, we recall that

$$\begin{aligned} v_k(C) &= \sum_{F \in \mathcal{F}_k(C)} \beta(o, F)\gamma(F, C) \\ &= \sum_{F \in \mathcal{F}_k(C)} v_k(F) v_{d-k}(N(C, F)). \end{aligned}$$

Here we introduce the abbreviation

$$\varphi_F(C) := v_k(F) v_{d-k}(N(C, F)) \quad \text{if } \dim F = k.$$

Hence, we have to integrate

$$v_k(C \cap \vartheta D) = \sum_{J \in \mathcal{F}_k(C \cap \vartheta D)} \varphi_J(C \cap \vartheta D).$$

Thus, we need to consider the k-faces of $C \cap \vartheta D$.

Definition. The cones F, G *intersect transversally*, written $F \pitchfork G$, if

$$\dim(F \cap G) = \dim F + \dim G - d \quad \text{and} \quad \operatorname{relint} F \cap \operatorname{relint} G \neq \emptyset.$$

Lemma. *Let C, D be polyhedral cones. For ν-almost all $\vartheta \in SO_d$, each k face J of $C \cap \vartheta D$ is of the form*

$$J = F \cap \vartheta G$$

with $F \in \mathcal{F}_i(C)$, $G \in F_j(D)$, where $i + j = d + k$ and $F \pitchfork \vartheta G$.

Therefore,

$$\int_{SO_d} v_k(C \cap \vartheta D)\, \nu(\mathrm{d}\vartheta)$$

$$= \sum_{i+j=k+d} \sum_{F \in \mathcal{F}_i(C)} \sum_{G \in \mathcal{F}_j(D)} \int_{SO_d} \varphi_{F \cap \vartheta D}(C \cap \vartheta D) \mathbb{1}\{F \pitchfork \vartheta G\}\, \nu(\mathrm{d}\vartheta).$$

We have to show that

$$\int_{SO_d} \varphi_{F \cap \vartheta D}(C \cap \vartheta D) \mathbb{1}\{F \pitchfork \vartheta G\}\, \nu(\mathrm{d}\vartheta) = \varphi_F(C) \varphi_G(D)$$

for $\dim F + \dim G = d + k$.

For ν-almost all ϑ,

$$\varphi_{F\cap\vartheta D}(C\cap\vartheta D) = v_k(F\cap\vartheta G)v_{d-k}(N(C,F)+\vartheta N(D,G)),$$

thus we have to prove that

$$\int_{SO_d} v_k(F\cap\vartheta G)v_{d-k}(N(C,F)+\vartheta N(D,G))\,\nu(\mathrm{d}\vartheta)$$
$$= \varphi_F(C)\varphi_G(D)$$

for $\dim F + \dim G = d + k$.

Lemmas 1 and 2 treated corresponding integrals where one of the factors in the integrand is missing.

The proof:

$$\int_{SO_d} v_k(F \cap \vartheta G) v_{d-k}(N(C,F) + \vartheta N(D,G))\, \nu(\mathrm{d}\vartheta)$$

replace ϑ by $\rho\vartheta$ with $\rho \in SO_{\langle F \rangle}$ and integrate over all ρ

$$= \int_{SO_{\langle F \rangle}} \int_{SO_d} v_k(F \cap \rho\vartheta G) v_{d-k}(N(C,F) + \rho\vartheta N(D,G))\, \nu(\mathrm{d}\vartheta)$$

$$\times \nu_{\langle F \rangle}(\mathrm{d}\rho)$$

Fubini; $N(C,F) = \rho N(C,F)$; SO_d-invariance of v_{d-k}

$$= \int_{SO_d} \left[\int_{SO_{\langle F \rangle}} v_k(F \cap \rho\vartheta G)\, \nu_{\langle F \rangle}(\mathrm{d}\rho) \right]$$

$$\times v_{d-k}(N(C,F) + \vartheta N(D,G))\, \nu(\mathrm{d}\vartheta)$$

apply Lemma 1 in $\langle F \rangle$ to the integral in brackets

$$= v_i(F) \int_{SO_d} v_k(\vartheta G \cap \langle F \rangle) v_{d-k}(N(C,F) + \vartheta N(D,G))\, \nu(\mathrm{d}\vartheta)$$

use SO_d-invariance of v_k

$$= v_i(F) \int_{SO_d} v_k(G \cap \vartheta^{-1} \langle F \rangle) v_{d-k}(N(C,F) + \vartheta N(D,G))\, \nu(\mathrm{d}\vartheta)$$

replace ϑ by $\vartheta\sigma$ with $\sigma \in SO_{\langle G \rangle}$ and integrate over all σ

$$= v_i(F) \int_{SO_{\langle G \rangle}} \int_{SO_d} v_k(\sigma G \cap \vartheta^{-1} \langle F \rangle) v_{d-k}(N(C,F) + \vartheta N(D,G))$$

$$\times \nu(\mathrm{d}\vartheta)\, \nu_{\langle G \rangle}(\mathrm{d}\sigma)$$

Fubini

$$= v_i(F) \int_{SO_d} \left[\int_{SO_{\langle G \rangle}} v_k(\sigma G \cap \vartheta^{-1} \langle F \rangle) \, \nu_{\langle G \rangle}(\mathrm{d}\sigma) \right]$$

$$\times v_{d-k}(N(C,F) + \vartheta N(D,G)) \, \nu(\mathrm{d}\vartheta)$$

apply Lemma 1 in $\langle G \rangle$ to the integral in brackets

$$= v_i(F) v_j(G) \int_{SO_d} v_{d-k}(N(C,F) + \vartheta N(D,G)) \, \nu(\mathrm{d}\vartheta)$$

apply Lemma 2

$$= v_i(F) v_j(G) v_{d-k}(N(C,F)) v_{d-j}(N(D,G)$$

$$= \varphi_F(C) \varphi_G(D).$$

This finishes the proof of the conic kinematic formula for polyhedral cones.
Approximation extends this to general convex cones.

A side remark

Much of this can be localized.

For $\eta \subset \mathbb{R}^d \times \mathbb{R}^d$, let

$$\eta^+ := \{(\lambda x, \mu y) : (x, y) \in \eta,\ \lambda, \mu > 0\}.$$

The biconic σ-algebra is defined by

$$\widehat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d) := \{\eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) : \eta^+ = \eta\}.$$

For a polyhedral cone C and for $\eta \in \widehat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d)$, let

$$\Omega_k(C, \eta) := \mathbb{P}\left(\Pi_C(\mathbf{g}) \in \operatorname{skel}(C),\ (\Pi_C(\mathbf{g}), \Pi_{C^\circ}(\mathbf{g})) \in \eta\right).$$

Then $\Omega_k(C, \cdot)$ is a measure, the kth conic support measure of C, and $\Omega_k(C, \mathbb{R}^d \times \mathbb{R}^d) = v_k(C)$.

Much of what we said about conic intrinsic volumes extends to conic support measures.

- Explicit representation:

$$\Omega_k(C,\eta) = \sum_{F\in\mathcal{F}_k} \int_F \int_{N(C,F)} \mathbb{1}_\eta(x,y)\,\gamma_{\langle F\rangle^\perp}(\mathrm{d}y)\,\gamma_{\langle F\rangle}(\mathrm{d}x).$$

- Duality:

$$\Omega_k(C,\eta) = \Omega_{d-k}(C^\circ,\eta*), \quad \eta* := \{(y,x) : (x,y)\in\eta\}.$$

- O_d-equivariance:

$$\Omega_k(\vartheta C,\vartheta\eta) = \Omega_k(C,\eta), \quad \vartheta\in O_d.$$

• Master Steiner formula: Set

$$\varphi_f(C,\eta) := \mathbb{E}\left[f\left(\|\Pi_C(\mathbf{g})\|^2, \|\Pi_{C^\circ}(\mathbf{g})\|^2\right) \mathbb{1}_\eta(\Pi_C(\mathbf{g}), \Pi_{C^\circ}(\mathbf{g}))\right].$$

Then

$$\varphi_f(C,\eta) = \sum_{k=0}^{d} \mathcal{I}_k(f) \cdot \Omega_k(C,\eta).$$

• Conic kinematic formula: Define the conic curvature measures by

$$\Phi_k(C,A) := \Omega_k(C, A \times \mathbb{R}^d), \quad A \in \widehat{\mathcal{B}}(\mathbb{R}^d).$$

Then, for $A, B \in \widehat{\mathcal{B}}(\mathbb{R}^d)$,

$$\int_{SO_d} \Phi_k(C \cap \vartheta D, A \cap \vartheta B)\, \nu(\mathrm{d}\vartheta) = \sum_{i=k}^{d} \Phi_i(C,A)\Phi_{d+k-i}(D,B).$$

By approximation and weak continuity, an extension to general closed convex cones is possible.

Some references:

R. Schneider: Intersection probabilities and kinematic formulas for polyhedral cones. *Acta Math. Hungar.* **155** (2018), 3–24.

R. Schneider: Conic support measures. arXiv:1807.03614v1

(end of side remark)

The following is based on:

D. Amelunxen, M. Lotz, M.B. McCoy, J.A. Tropp: Living on the edge: phase transitions in convex programs with random data. *Information and Inference* **3** (2014), 224–294.

M.B. McCoy, J.A. Tropp, From Steiner formulas for cones to concentration of intrinsic volumes. *Discrete Comput. Geom.* **51** (2014), 926–963:

Back to the initial question! It now has the following answer:

$$\begin{aligned}
\mathbb{P}(C \cap \Theta D \neq \{o\}) &= \mathbb{E}\mathbb{1}\{C \cap \Theta D \neq \{o\}\} \\
&= 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=2k+1}^{d} v_j(C) v_{d+2k+1-j}(D) \\
&= \sum_{i=1}^{d-1} (1 + (-1)^{i+1}) \sum_{j=i}^{d} v_j(C) v_{d+i-j}(D).
\end{aligned}$$

An explicit computation of the conic intrinsic volumes is only possible in very special cases.

One example: If C is the nonnegative orthant in $\mathbb{R}^d$, then

$$v_k(C) = 2^{-d}\binom{d}{k}.$$

Concentration of the conic intrinsic volumes

The dimension of a linear subspace L plays a particular role:

$$v_k(L) = \begin{cases} 1 & \text{if } \dim L = k, \\ 0 & \text{otherwise.} \end{cases}$$

If also C is a subspace, then

$$\mathbb{P}(C \cap \Theta L \neq \{o\}) = \begin{cases} 0 & \text{if } \dim C + \dim L \leq d, \\ 1 & \text{if } \dim C + \dim L > d. \end{cases}$$

The statistical dimension $\delta(C)$ of a general closed convex cone C has the property that the conic intrinsic volumes of C concentrate near $\delta(C)$, therefore:

$\mathbb{P}(C \cap \Theta L \neq \{o\})$ is close to 0 if $\delta(C) + \dim L$ is smaller than d,

$\mathbb{P}(C \cap \Theta L \neq \{o\})$ is close to 1 if $\delta(C) + \dim L$ is larger than d.

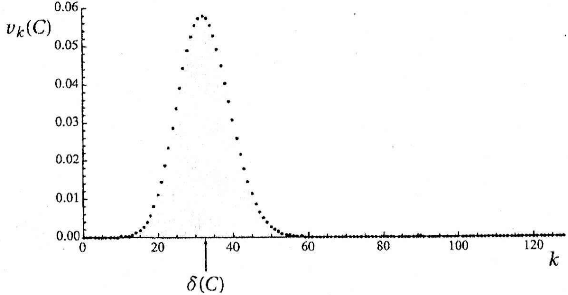


FIG. 3. Concentration of conic intrinsic volumes. This plot displays the conic intrinsic volumes $v_k(C)$ of a circular cone $C \subset \mathbb{R}^{128}$ with angle $\pi/6$. The distribution concentrates sharply around the statistical dimension $\delta(C) \approx 32.5$. See Section 3.4 for further discussion of this example.

Amelunxen, Lotz, McCoy, Tropp 2014

The statistical dimension

Recall that the conic intrinsic volumes of a cone C satisfy

$$v_k(C) \geq 0, \quad v_0(C) + \cdots + v_d(C) = 1.$$

Definition. The intrinsic volume random variable $\mathbf{V}_C$ of a cone $C \in \mathcal{C}^d$ is defined as a random variable with values in $\{0, \ldots, d\}$ and with distribution

$$\mathbb{P}(\mathbf{V}_C = k) = v_k(C), \quad k = 0, \ldots, d.$$

By duality,

$$\mathbb{P}(\mathbf{V}_{C^\circ} = k) = v_k(C^\circ) = v_{d-k}(C) = \mathbb{P}(\mathbf{V}_C = d - k) = \mathbb{P}(d - \mathbf{V}_C = k),$$

hence

$$\mathbf{V}_{C^\circ} \sim d - \mathbf{V}_C$$

(where $\sim$ means equality in distribution).

Definition and Theorem. *The* *statistical dimension* *of the cone* $C \in \mathcal{C}^d$ *is the number*

$$\delta(C) := \mathbb{E}\mathbf{V}_C = \sum_{k=0}^{d} k v_k(C) = \mathbb{E}\|\Pi_C(\mathbf{g})\|^2.$$

Proof of the last equality:

$$\begin{aligned}
\mathbb{E}\|\Pi_C(\mathbf{g})\|^2 &= \int_{\mathbb{R}^d} \|\Pi_C(\mathbf{g})\|^2 \, \gamma_d(\mathrm{d}x) \\
&= \int_0^\infty \gamma_d\left(\{x \in \mathbb{R}^d : \|\Pi_c(x)\|^2 > t\}\right) \mathrm{d}t \\
&= \int_0^\infty \mathbb{P}\left(\|\Pi_C(\mathbf{g})\|^2 > t\right) \mathrm{d}t.
\end{aligned}$$

Let C be polyhedral. By the Gaussian Steiner formula in the version of Corollary 1,

$$
\begin{aligned}
\mathbb{P}(\|\Pi_C(\mathbf{g})\|^2 > t) &= \mathbb{P}(\mathrm{dist}^2(\mathbf{g}, C^\circ) > t) \\
&= 1 - \mathbb{P}(\mathrm{dist}^2(\mathbf{g}, C^\circ) \le t) \\
&= 1 - \sum_{k=0}^{d} \mathbb{P}(\mathcal{X}_{d-k} \le t) v_k(C^\circ) \\
&= 1 - \sum_{k=0}^{d} \left[1 - \mathbb{P}(\mathcal{X}_k > t)\right] v_k(C) \\
&= \sum_{k=0}^{d} \mathbb{P}(\mathcal{X}_k > t) v_k(C).
\end{aligned}
$$

Since $\int_0^\infty \mathbb{P}(\mathcal{X}_k > t)\, \mathrm{d}t = \mathbb{E}\mathcal{X}_k = k$, the assertion follows (then approximation).

Properties of the statistical dimension:

- intrinsic
- rotation-invariant, continuous valuation
- increasing under set inclusion
- $\delta(C) + \delta(C^\circ) = d$
- $\delta(L) = \dim L$ for a subspace L
- $\delta(C) = \frac{1}{2}d$ if C is self-dual (i.e., $C^\circ = -C$)
- $\delta(C \times D) = \delta(C) + \delta(D)$

The last relation will be explained later.

A variance estimate

By the same argument that above yielded

$$\mathbb{E}\|\Pi_C(\mathbf{g})\|^2 = \sum_{k=0}^{d} (\mathbb{E}\mathcal{X}_k)\, v_k(C),$$

we can prove that

$$\mathbb{E}\|\Pi_C(\mathbf{g})\|^4 = \sum_{k=0}^{d} \left(\mathbb{E}\mathcal{X}_k^2\right) v_k(C).$$

Since $\mathbb{E}\mathcal{X}_k^2 = k^2 + 2k$, we obtain

$$\mathbb{E}\|\Pi_C(\mathbf{g})\|^4 = \mathbb{E}\mathbf{V}_C^2 + 2\delta(C).$$

This yields an expression for the variance of the intrinsic volume random variable $\mathbf{V}_C$:

$$
\begin{aligned}
\operatorname{Var} \mathbf{V}_C &= \mathbb{E}\mathbf{V}_C^2 - (\mathbb{E}\mathbf{V}_C)^2 \\
&= \mathbb{E}\|\Pi_C(\mathbf{g})\|^4 - 2\delta(C) - \delta(C)^2 \\
&= \mathbb{E}\|\Pi_C(\mathbf{g})\|^4 - (\mathbb{E}\|\Pi_C(\mathbf{g})\|^2)^2 - 2\delta(C) \\
&= \operatorname{Var}(\|\Pi_C(\mathbf{g})\|^2) - 2\delta(C).
\end{aligned}
$$

We apply the Gaussian Poincaré inequality

$$
\operatorname{Var} f(\mathbf{g}) \leq \mathbb{E}(\|\nabla f(\mathbf{g})\|^2)
$$

to the function $f(x) = \|\Pi_C(x)\|^2$ and use that

$$
\nabla \|\Pi_C(x)\|^2 = 2\Pi_C(x).
$$

The result is that

$$
\operatorname{Var} \mathbf{V}_C \leq 2\delta(C).
$$

Possibly an improvement: Since $\mathbf{V}_{C^\circ} \sim d - \mathbf{V}_C$, we have

$$\operatorname{Var} \mathbf{V}_C = \operatorname{Var} \mathbf{V}_{C^\circ}$$

and hence

$$\operatorname{Var} \mathbf{V}_C \leq 2(\delta(C) \wedge \delta(C^\circ)),$$

where $a \wedge b := \min\{a, b\}$.

Tschebyscheff's inequality yields a first concentration result:

$$\mathbb{P}\left(|\mathbf{V}_C - \delta(C)| \geq \lambda\sqrt{\delta(C)}\right) \leq \frac{\operatorname{Var} \mathbf{V}_C}{\lambda^2 \delta(C)} \leq \frac{2}{\lambda^2}.$$

The following gives sharper concentration.

Theorem 6. *For a closed convex cone C, define*

$$\omega(C)^2 := \delta(C) \wedge \delta(C^\circ)$$

and

$$p_C(\lambda) := \exp\left(\frac{-\lambda^2/4}{\omega(C)^2 + \lambda/3}\right) \quad \textit{for } \lambda \geq 0.$$

Then

$$\mathbb{P}(|\mathbf{V}_C - \delta(C)| \geq \lambda) \leq p_C(\lambda)$$

for $\lambda \geq 0$.

Before saying a few words about the proof, we indicate how this leads to estimates of

$$\mathbb{P}(C \cap \Theta D \neq \{o\}).$$

We consider first the special case where D is a subspace L.

Theorem 7. *Let $C \in \mathcal{C}^d$, and let L be a subspace. Then the following holds for $\lambda \geq 0$:*

$$\delta(C) + \dim L \leq d - \lambda \quad \Rightarrow \quad \mathbb{P}(C \cap \Theta L \neq \{o\}) \leq p_C(\lambda),$$
$$\delta(C) + \dim L \geq d + \lambda \quad \Rightarrow \quad \mathbb{P}(C \cap \Theta L \neq \{o\}) \geq 1 - p_C(\lambda).$$

Proof. We write $\dim L = d - m$, then

$$\delta(C) + \dim L \leq d - \lambda \Leftrightarrow m \geq \delta(C) + \lambda,$$

and similarly in the second case.

We recall the general formula (C not a subspace)

$$\mathbb{P}\left(C \cap \Theta D \neq \{o\}\right) = 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=2k+1}^{d} v_j(C) v_{d+2k+1-j}(D)$$

and observe that now $v_k(D) = \delta_{k,d-m}$ (Kronecker symbol).

This gives

$$\mathbb{P}(C \cap \Theta L \neq \{o\}) = 2h_{m+1}(C)$$

with the half-tail function

$$h_m(C) := v_m(C) + v_{m+2}(C) + \dots$$

Define also the tail function

$$t_m(C) := v_m(C) + v_{m+1}(C) + \dots$$

Then, with a subspace $E \supset L$ of dimension $d - m + 1$,

$$2h_{m+1}(C) = \mathbb{P}(C \cap \Theta L \neq \{o\}) \leq \mathbb{P}(C \cap \Theta E \neq \{o\}) = 2h_m(C)$$

and hence $2h_{m+1}(C) \leq h_{m+1}(C) + h_m(C) = t_m(C)$, thus

$$\mathbb{P}(C \cap \Theta L \neq \{o\}) \leq t_m(C).$$

Since, by the definition of $\mathbf{V}_C$,

$$\mathbb{P}(\mathbf{V}_C - \delta(C) \geq \lambda) = \sum_{k \geq \delta(C)+\lambda} v_k(C)$$

assumption $m \geq \delta(C) + \lambda$

$$\geq \sum_{k \geq m} v_k(C) = t_m(C),$$

we get

$$\mathbb{P}(C \cap \Theta L \neq \{o\}) \leq t_m(C) \leq \mathbb{P}(\mathbf{V}_C - \delta(C) \geq \lambda) \leq p_C(\lambda)$$

by Theorem 6.

This is the first assertion of Theorem 7.

The second assertion is obtained similarly.

Some remarks about the crucial

Theorem 6. *For a closed convex cone C, define*

$$\omega(C)^2 := \delta(C) \wedge \delta(C^\circ)$$

and

$$p_C(\lambda) := \exp\left(\frac{-\lambda^2/4}{\omega(C)^2 + \lambda/3}\right) \quad \textit{for } \lambda \geq 0.$$

Then

$$\mathbb{P}(\mathbf{V}_C - \delta(C) \geq \lambda) \leq p_C(\lambda),$$
$$\mathbb{P}(\mathbf{V}_C - \delta(C) \leq -\lambda) \leq p_C(\lambda)$$

for $\lambda \geq 0$.

McCoy and Tropp derive it from the stronger

Theorem 8.
Let $C \in \mathcal{C}^d$. Define the function ψ by

$$\psi(u) := (u+1)\log(u+1) - u \quad \text{for } u \geq -1,$$

and $\psi(u) = \infty$ for $u < -1$. Then, for all $\lambda \geq 0$,

$$\mathbb{P}\{\mathbf{V}_C - \delta(C) \geq \lambda\} \leq \exp\left(-\frac{1}{2}\max\left\{\delta(C)\psi\left(\frac{\lambda}{\delta(C)}\right), \delta(C^\circ)\psi\left(\frac{-\lambda}{\delta(C^\circ)}\right)\right\}\right)$$

and

$$\mathbb{P}\{\mathbf{V}_C - \delta(C) \leq -\lambda\} \leq \exp\left(-\frac{1}{2}\max\left\{\delta(C)\psi\left(\frac{-\lambda}{\delta(C)}\right), \delta(C^\circ)\psi\left(\frac{\lambda}{\delta(C^\circ)}\right)\right\}\right).$$

Their proof first expresses the moment generating function of the intrinsic volume random variable as

$$\mathbb{E} e^{\eta \mathbf{V}_C} = \mathbb{E} e^{\xi \|\Pi_C(\mathbf{g})\|^2} \quad \text{with } \xi = \frac{1}{2}\left(1 - e^{-2\eta}\right).$$

The proof uses the Master Steiner formula and the moment generating functions of chi-square random variables.

The main ingredient of the subsequent estimations is the Gaussian logarithmic Sobolev inequality.

The rest is analysis.

After the special case C, L, where L is a subspace, it remains to consider the case of two general cones C, D.

It is convenient to consider their product $C \times D$.

Convention. For two Euclidean spaces $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$, their Cartesian product $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is always equipped with the scalar product given by

$$\langle (x, y), (x', y') \rangle := \langle x, x' \rangle_1 + \langle y, y' \rangle_2, \quad (x, y), (x', y') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},$$

where $\langle \cdot\,, \cdot \rangle_r$ denotes the scalar product in $\mathbb{R}^{d_r}$, $r = 1, 2$.

Then $C \times D$ can be considered as a direct orthogonal sum $C \oplus D$.

Some remarks about $C \times D$

First remark:

$$\Pi_{C \times D}(x, y) = (\Pi_C(x), \Pi_D(y)) \quad \text{for } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

The proof makes use of the Moreau decomposition.

Second remark:

$$\delta(C \times D) = \delta(C) + \delta(D).$$

Proof:

Assume $\mathbb{R}^{2d} = L_1 \oplus L_2$, $\dim L_r = d$, $C \subset L_1$, $D \subset L_2$;
identify $C \times D$ with $C \oplus D$.

$$\begin{aligned} \delta(C \oplus D) &= \mathbb{E}\|\Pi_{C\oplus D}(\mathbf{g})\|^2 = \int_{\mathbb{R}^{2d}} \|\Pi_{C\oplus D}(x)\|^2 \, \gamma_{2d}(\mathrm{d}x) \\ &= \int_{L_1}\int_{L_2} \left(\|\Pi_C(y)\|^2 + \|\Pi_D(z)\|^2\right) \gamma_{L_2}(\mathrm{d}z)\, \gamma_{L_1}(\mathrm{d}y)) \\ &= \delta(C) + \delta(D). \end{aligned}$$

Third remark:

$$v_k(C \times D) = \sum_{i+j=k} v_i(C) v_j(D).$$

To prove

$$v_k(C \times D) = \sum_{i+j=k} v_i(C) v_j(D),$$

we can assume that $C, D \subset \mathbb{R}^d \times \mathbb{R}^d$; $C \subset \mathbb{R}^d \times \{o\}$; $D \subset \{o\} \times R^d$ and $C \times D$ is the direct orthogonal sum $C \oplus D$.

Each face $F \in \mathcal{F}_k(C \oplus D)$ is of the form $F = F_i \oplus G_j$ with $F_i \in \mathcal{F}_i(C)$, $G_j \in \mathcal{F}_j(D)$, $i + j = k$.

We have

$$N(C \oplus D, F_i \oplus G_j) = N(C, F_i) \oplus N(D, G_j).$$

The assertion then follows after computing the internal and external angles.

Theorem 9. *For closed convex cones* C, D*, define*

$$\sigma(C,D)^2 := (\delta(C) \wedge \delta(C^\circ)) + (\delta(D) + \delta(D^\circ))$$

and

$$p_{C,D}(\lambda) := \exp\left(\frac{-\lambda^2/4}{\sigma(C,D)^2 + \lambda/3}\right) \quad \textit{for } \lambda \geq 0.$$

Then

$$\mathbb{P}(|\mathbf{V}_{C \times D} - \delta(C \times D)| \geq \lambda) \leq p_{C,D}(\lambda)$$

for $\lambda \geq 0$*.*

The proof uses the previous estimates, and in addition that

$$v_k(C \times D) = \sum_{i+j=k} v_i(C) v_j(D).$$

With this, we can write (if C, D are not both subspaces)

$$\begin{aligned}\mathbb{P}(C \cap \Theta D \neq \{o\}) &= 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{i+j=d+2k+1} v_i(C) v_j(D) \\ &= 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{d+2k+1}(C \times D) \\ &= 2 h_{d+1}(C \times D).\end{aligned}$$

This is convenient for the proof of the following theorem.

Theorem 10. *Let* $C, D \in \mathcal{C}^d$ *be cones. Then, for* $\lambda \geq 0$*,*

$$\delta(C) + \delta(D) \leq d - \lambda \quad \Rightarrow \quad \mathbb{P}(C \cap \Theta D \neq \{o\}) \leq p_{C,D}(\lambda),$$

$$\delta(C) + \delta(D) \geq d + \lambda \quad \Rightarrow \quad \mathbb{P}(C \cap \Theta D \neq \{o\}) \geq 1 - p_{C,D}(\lambda).$$

Proof.

We can assume that D, C are not both subspaces. Then, as shown above,

$$\mathbb{P}(C \cap \Theta D \neq \{o\}) = 2h_{d+1}(C \times D).$$

As shown before,

$$2h_{d+1}(C \times D) \leq t_d(C \times D).$$

The assumption $\delta(C) + \delta(D) \leq d - \lambda$ gives

$$d \geq \delta(C) + \delta(D) + \lambda = \delta(C \times D) + \lambda$$

and hence

$$\begin{aligned} &\mathbb{P}(\mathbf{V}_{C\times D} - \delta(C \times D) \geq \lambda) \\ &= \sum_{k \geq \delta(C\times D)+\lambda} v_k(C \times D) \geq \sum_{k \geq d} v_k(C \times D) = t_d(C \times D). \end{aligned}$$

Thus,

$$\mathbb{P}(C \cap \Theta D \neq \{o\}) \leq \mathbb{P}(\mathbf{V}_{C\times D} - \delta(C \times D) \geq \lambda) \leq p_{C,D}(\lambda),$$

by Theorem 9.

The second estimate is obtained similarly.