Second moments related to Poisson hyperplane tessellations

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Abstract

It is well known that the vertex number of the typical cell of a stationary hyperplane tessellation in $\mathbb{R}^d$ has, under some mild conditions, an expectation equal to 2^d, independent of the underlying distribution. The variance of this vertex number can vary widely. Under Poisson assumptions, we give sharp bounds for this variance, showing, in particular, that its maximum is attained if and only if the tessellation is isotropic (that is, its distribution is rotation invariant) with respect to a suitable scalar product on $\mathbb{R}^d$.

The employed representation of the second moment of the vertex number is a special case of formulas providing the covariance matrix of the random vector $(\ell_0, \dots, \ell_d)$, where ℓ_k is the total k-face content of the typical cell of a stationary Poisson hyperplane mosaic. In the isotropic case, such formulas were first obtained by Miles. We give a more elementary proof and extend the formulas to general orientation distributions.

Key words. Hyperplane tessellation, typical cell, face content, second moment, isotropic process, covariance matrix

1 Introduction

Under suitable assumptions, a stationary random hyperplane process $\widehat{X}$ in $\mathbb{R}^d$ induces a tessellation of $\mathbb{R}^d$ into bounded polytopes. The typical cell of such a tessellation, intuitively and heuristically speaking, is obtained by choosing at random, with equal chances, one of the d-dimensional polytopes of the tessellation within a 'large' region of $\mathbb{R}^d$; a precise definition is recalled in the next section. It is well known that a number of combinatorial quantities connected with this typical cell have expectations that are essentially independent of the distribution of the underlying hyperplane process (see, e.g., Theorem 10.3.1 in [8]). For example, the expected vertex number of the typical cell is equal to 2^d, which is the obvious value for a *parallel process*, where the hyperplanes belong to only d translation classes. In the latter case, the typical cell is a parallelepiped and thus its vertex number is constant, hence it has variance zero. In this note, we determine the stationary Poisson hyperplane processes for which the variance of the vertex number of the typical cell attains its maximum.

Let $\widehat{X}$ be a nondegenerate stationary Poisson hyperplane process in $\mathbb{R}^d$, and let X be its induced mosaic. We use the terminology of [8], in particular, X is considered as the particle process defined by the cells (the d-dimensional polytopes) of the tessellation defined by $\widehat{X}$. The typical grain of this particle process is, by definition, the typical cell of X. We denote it by Z. The vertex number of a convex polytope P is denoted by $f_0(P)$. If $\mathbb{R}^d$ is equipped with a scalar product, then $\widehat{X}$ and X are called *isotropic* if their distribution is invariant under rotations.

Theorem 1. *Let $\widehat{X}$ be a nondegenerate stationary Poisson hyperplane process in $\mathbb{R}^d$, and let Z be the typical cell of its induced mosaic. The variance of the vertex number $f_0(Z)$ satisfies*

$$0 \leq \operatorname{Var} f_0(Z) \leq \left(2^d d! \sum_{j=0}^{d} \frac{\kappa_j^2}{4^j (d-j)!} \right) - 2^{2d}. \tag{1}$$

Here κ_j is the volume of the j-dimensional unit ball. Equality on the left side of (1) holds if and only if $\widehat{X}$ is a parallel process, and on the right side if and only if $\widehat{X}$ is isotropic with respect to a suitable scalar product on $\mathbb{R}^d$.

The vertex number $f_0 = L_0$ is just the first polytope functional in the series $L_0, \dots, L_d$ of total k-face contents. These are defined by

$$L_k(P) = \sum_{F \in \mathcal{F}_k(P)} \mathcal{H}^k(F),$$

for convex polytopes $P \subset \mathbb{R}^d$ and for $k \in \{0, \dots, d\}$. Here $\mathcal{F}_k(P)$ is the set of k-dimensional faces of P and $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. In particular, L_d is the volume, L_{d-1} the surface area, L_1 the total edge length, and $L_0 = f_0$ the number of vertices. For the typical cell Z of a stationary, isotropic Poisson hyperplane mosaic, Miles [4] has determined all expectations $\mathbb{E}\,(L_r L_s)(Z)$, $r, s \in \{0, \dots, d\}$. The result is reproduced, without proof, in [5], formula (63). The proof given by Miles in [4] makes heavy use of ergodic theory and is not explicitly carried out in all details. Below, we give a short proof, based only on the Slivnyak–Mecke formula, and extend the result to not necessarily isotropic mosaics (Theorem 2 in Section 4). This is in contrast to a remark of Miles, who after the treatment of the isotropic case in [4], Sec. 11.7, wrote: “It does not seem at all practicable to generalise the theory of §§3–7 to the case of a general orientation distribution”. Of course, the general result cannot be so explicit as in the isotropic case, since the second moments heavily depend on the directional distribution of the underlying hyperplane process. The result makes this dependence as explicit as possible, in terms of the associated zonoid. The special case of $\mathbb{E}\,(L_0 L_0)(Z)$ is the foundation for the proof of Theorem 1.

After collecting some preparations in the next section, we describe in Section 3 the approach of Favis and Weiss [3] to k-face-content weighted typical cells. This is used in Section 4 to prove Theorem 2. The proof of Theorem 1 is then completed in Section 5.

2 Preliminaries

We fix some notation and recall some definitions. We work in the d-dimensional real vector space $\mathbb{R}^d$ ($d \geq 2$) and use its standard scalar product $\langle \cdot, \cdot \rangle$ to define, for example, its unit sphere $\mathbb{S}^{d-1}$. Lebesgue measure on $\mathbb{R}^d$ is denoted by λ. The space of hyperplanes in $\mathbb{R}^d$ is denoted by $A(d, d-1)$; it is equipped with its usual topology. Hyperplanes are often parametrized in the form

$$H(u, \tau) = \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau\}$$

with $u \in \mathbb{S}^{d-1}$ and $\tau \in \mathbb{R}$. The hyperplane through 0 orthogonal to u is denoted by $u^\perp$, so that we can also write $H(u, \tau) = u^\perp + \tau u$.

By $\mathcal{P}^d$ we denote the space of (nonempty, compact, convex) polytopes in $\mathbb{R}^d$, endowed with the topology induced by the Hausdorff metric. For a polytope $P \in \mathcal{P}^d$ and for $k \in \{0, \dots, d\}$, we denote, as already mentioned, by $\mathcal{F}_k(P)$ the set of its k-dimensional faces.

Probabilities are denoted by $\mathbb{P}$ and expectations by $\mathbb{E}$. Let T be a topological space. Then $\mathcal{B}(T)$ is the σ-algebra of its Borel sets. By $\mathsf{N}_s(T)$ we denote the set of simple counting measures on T; it is equipped with its usual σ-algebra (see [8], Sec. 3.1). It is convenient to identify a simple counting measure η with its support; with this identification made, $\eta(\{x\}) = 1$ and $x \in \eta$ are used synonymously, and so are $\eta \cup \eta'$ and $\eta + \eta'$, as long as the latter is simple.

Let $\widehat{X}$ be a stationary Poisson hyperplane process in $\mathbb{R}^d$, with intensity measure $\widehat{\Theta}$ and intensity $\widehat{\gamma} > 0$. Its spherical directional distribution is denoted by $\widehat{\varphi}$. We refer to [8], in particular Sections 4.4 and 10.3, for notions that are not explained here. We assume that $\widehat{X}$ is nondegenerate, which means that $\widehat{\varphi}$ is not concentrated on some great subsphere of $\mathbb{S}^{d-1}$. It follows from this assumption that $\widehat{X}$ almost surely induces a tessellation of $\mathbb{R}^d$ into bounded polytopes. For $k = 0, \dots, d$, we denote by $\mathcal{F}_k(\widehat{X})$ the set of k-faces of the tessellation induced by $\widehat{X}$. In particular, the random system $\mathcal{F}_d(\widehat{X})$ of its d-dimensional polytopes or *cells* is the *Poisson mosaic* induced by $\widehat{X}$; it is interpreted as a particle process and denoted by X. The intensity measure of X is denoted by Θ and its intensity by γ. The definition of the typical cell involves a centre function $c : \mathcal{P}^d \to \mathbb{R}^d$, for which we choose (as in [8], Sec. 4.1) the circumcentre, that is, $c(P)$ is the centre of the smallest ball containing P. We write $\mathcal{P}^d_0 := \{P \in \mathcal{P}^d : c(P) = 0\}$. The *typical cell* of X, denoted by Z, is by definition a random polytope whose distribution, denoted by $\mathbb{Q}$, is the grain distribution of X. This distribution $\mathbb{Q}$ is obtained by decomposing the intensity measure of the particle process X in the form

$$\int_{\mathcal{P}^d} g \, \mathrm{d}\Theta = \gamma \int_{\mathcal{P}^d_0} \int_{\mathbb{R}^d} g(K + x) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}K), \tag{2}$$

for any nonnegative, measurable function $g : \mathcal{P}^d \to \mathbb{R}$ (see [8], (4.3)). Explicitly, for $A \in \mathcal{B}(\mathcal{P}^d_0)$ we have

$$\mathbb{Q}(A) = \frac{1}{\gamma} \mathbb{E} \sum_{K \in X} \mathbf{1}_A(K - c(K)) \mathbf{1}_B(c(K)), \tag{3}$$

where $B \subset \mathbb{R}^d$ is an arbitrary Borel set with $\lambda(B) = 1$, and also

$$\mathbb{Q}(A) = \lim_{r \to \infty} \frac{\mathbb{E} \sum_{K \in X, \, K \subset rW} \mathbf{1}_A(K - c(K))}{\mathbb{E} \sum_{K \in X, \, K \subset rW} 1},$$

where $W \subset \mathbb{R}^d$ is an arbitrary convex body with interior points; this follows from Theorem 4.1.3 in [8].

A very useful tool for the treatment of Poisson processes is the Slivnyak–Mecke formula (e.g., [8], Corollary 3.2.3), which we formulate here for the stationary Poisson hyperplane process $\widehat{X}$. By $\widehat{X}^m_{\neq}$ we denote the set of ordered m-tuples of pairwise distinct elements from $\widehat{X}$, for $m \in \mathbb{N}$. Let

$$f : \mathsf{N}_s(A(d, d-1)) \times A(d, d-1)^m \to [0, \infty)$$

be measurable. Then the Slivnyak–Mecke formula states that

$$\mathbb{E} \sum_{(H_1, \dots, H_m) \in \widehat{X}^m_{\neq}} f(\widehat{X}, H_1, \dots, H_m) \tag{4}$$

$$= \int_{A(d,d-1)^m} \mathbb{E} f\left(\widehat{X} \cup \{H_1, \dots, H_m\}, H_1, \dots, H_m\right) \widehat{\Theta}^m(\mathrm{d}(H_1, \dots, H_m)).$$

We use this together with the decomposition of the intensity measure, given by

$$\int_{A(d,d-1)} g \,\mathrm{d}\widehat{\Theta} = \widehat{\gamma} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} g(H(u,\tau)) \,\mathrm{d}\tau \,\widehat{\varphi}(\mathrm{d}u), \tag{5}$$

for any nonnegative, measurable function $g : A(d, d-1) \to \mathbb{R}$ (see [8], (4.33)).

We shall make essential use of the *associated zonoid* of the hyperplane process $\widehat{X}$ (see [8], p. 156). This is the convex body $\Pi_{\widehat{X}}$ with support function given by

$$h(\Pi_{\widehat{X}}, u) = \frac{\widehat{\gamma}}{2} \int_{\mathbb{S}^{d-1}} |\langle u, v\rangle| \,\widehat{\varphi}(\mathrm{d}v), \quad u \in \mathbb{R}^d. \tag{6}$$

Let $L \subset \mathbb{R}^d$ be a linear subspace of dimension at least 1. Then the section process $\widehat{X} \cap L$ is a stationary Poisson hyperplane process with respect to L, and its associated zonoid is given by

$$\Pi_{\widehat{X}\cap L} = \Pi_{\widehat{X}}|L,$$

where $(\cdot)|L$ denotes the orthogonal projection to L.

The intrinsic volumes of the associated zonoids will play an important role. For a convex body K, we denote by $V_j(K)$ its jth intrinsic volume. If K is centrally symmetric with respect to 0 (as it holds for $\Pi_{\widehat{X}}$ and its projections), then K° denotes the polar body of K, constructed within the linear hull of K.

3 Face-content weighted cells

Following the approach of Favis and Weiss [3], we first introduce a random polytope which can be considered as the $(d-k)$-face-content weighted typical cell of the stationary Poisson random mosaic X.

Let $h : \mathcal{P}_0^d \to [0, \infty)$ be a Borel measurable function, and let $B \in \mathcal{B}(\mathbb{R}^d)$ be a set with $\lambda(B) = 1$. Using the Campbell theorem ([8], Thm. 3.1.2), the decomposition (2) and Fubini's theorem, we get

$$\begin{aligned}
&\mathbb{E} \sum_{K\in X} h(K - c(K)) \sum_{F\in\mathcal{F}_{d-k}(K)} \mathcal{H}^{d-k}(F \cap B) \\
&= \int_{\mathcal{P}^d} h(K - c(K)) \sum_{F\in\mathcal{F}_{d-k}(K)} \mathcal{H}^{d-k}(F \cap B) \,\Theta(\mathrm{d}K) \\
&= \gamma \int_{\mathcal{P}_0^d} h(K) L_{d-k}(K) \,\mathbb{Q}(\mathrm{d}K). \qquad (7)
\end{aligned}$$

Since

$$\int_{\mathcal{P}_0^d} L_{d-k}(K) \,\mathbb{Q}(\mathrm{d}K) = \mathbb{E}\, L_{d-k}(Z)$$

is finite (by (16) below), the measure

$$A \mapsto \frac{1}{\mathbb{E}\, L_{d-k}(Z)} \int_{\mathcal{P}_0^d} \mathbf{1}_A(K) L_{d-k}(K) \,\mathbb{Q}(\mathrm{d}K), \quad A \in \mathcal{B}(\mathcal{P}_0^d),$$

is a probability measure. Let D_k be a random polytope with this distribution. Then

$$\mathbb{E}\, h(D_k) = \frac{1}{\mathbb{E}\, L_{d-k}(Z)} \int_{\mathcal{P}_0^d} h(K) L_{d-k}(K)\, \mathbb{Q}(\mathrm{d}K) = \frac{\mathbb{E}\,(hL_{d-k})(Z)}{\mathbb{E}\, L_{d-k}(Z)}. \tag{8}$$

Thus, the random polytope D_k can be considered as the $(d-k)$-face-content weighted typical cell of X.

We assume now that the function h is translation invariant. We need a more explicit expression for $\mathbb{E}\, h(D_k)$ if $k \geq 1$, which is given by (10) below. This corresponds to [3], Proposition 4.1. For the reader's convenience, we present a slightly modified version of the proof, which avoids the use of d-fold Palm distributions and is simplified at the end. From (8) and (7),

$$\mathbb{E}\, h(D_k) = \frac{1}{\gamma\, \mathbb{E}\, L_{d-k}(Z)}\, \mathbb{E} \sum_{K\in X} h(K) \sum_{F\in\mathcal{F}_{d-k}(K)} \mathcal{H}^{d-k}(F\cap B). \tag{9}$$

First we assume now that $k \geq 1$. Since the $(d-k)$-faces of $K \in X$ are precisely the $(d-k)$-faces in $\mathcal{F}(\widehat{X})$ that are contained in K, we may rewrite the expectation on the right side of (9) as

$$\begin{aligned}
&\mathbb{E} \sum_{K\in X} h(K) \sum_{F\in\mathcal{F}_{d-k}(\widehat{X})} \mathbf{1}\{F\subset K\} \int_F \mathbf{1}_B(x)\, \mathcal{H}^{d-k}(\mathrm{d}x) \\
&= \mathbb{E} \sum_{F\in\mathcal{F}_{d-k}(\widehat{X})} \int_F \mathbf{1}_B(x) \sum_{K\in X} \mathbf{1}\{F\subset K\} h(K)\, \mathcal{H}^{d-k}(\mathrm{d}x) \\
&= \frac{1}{k!}\, \mathbb{E} \sum_{(H_1,\dots,H_k)\in\widehat{X}^k_{\neq}} \int_{H_1\cap\cdots\cap H_k} \mathbf{1}_B(x) \sum_{K\in X} \mathbf{1}\{x\in K\} h(K)\, \mathcal{H}^{d-k}(\mathrm{d}x).
\end{aligned}$$

Here we have used that $\mathcal{H}^{d-k}$-almost every $x \in H_1 \cap \cdots \cap H_k$ is contained in the relative interior of a unique face $F \in \mathcal{F}_{d-k}(\widehat{X})$, and if this holds, then the relations $F \subset K$ and $x \in K$ are equivalent for all $K \in X$. Now we apply the Slivnyak–Mecke formula (4), Fubini's theorem and the decomposition (5) of the intensity measure, to obtain

$$\begin{aligned}
&(\gamma k!\, \mathbb{E}\, L_{d-k}(Z))\, \mathbb{E}\, h(D_k) \\
&= \int_{A(d,d-1)^k} \mathbb{E} \int_{H_1\cap\cdots\cap H_k} \mathbf{1}_B(x) \\
&\quad\times \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{H_1,\dots,H_k\})} \mathbf{1}\{x\in K\} h(K)\, \mathcal{H}^{d-k}(\mathrm{d}x)\, \widehat{\Theta}^k(\mathrm{d}(H_1,\dots,H_k)) \\
&= \int_{A(d,d-1)^k} \int_{H_1\cap\cdots\cap H_k} \mathbf{1}_B(x) \\
&\quad\times \mathbb{E} \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{H_1,\dots,H_k\})} \mathbf{1}\{x\in K\} h(K)\, \mathcal{H}^{d-k}(\mathrm{d}x)\, \widehat{\Theta}^k(\mathrm{d}(H_1,\dots,H_k))
\end{aligned}$$

$$= \widehat{\gamma}^k \int_{(\mathbb{S}^{d-1})^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Bigg[\int_{H(u_1,\tau_1)\cap\cdots\cap H(u_k,\tau_k)} \mathbf{1}_B(x)$$

$$\times \mathbb{E} \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{H(u_1,\tau_1),\ldots,H(u_k,\tau_k)\})} \mathbf{1}\{x \in K\} h(K)\, \mathcal{H}^{d-k}(\mathrm{d}x) \Bigg] \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k$$

$$\times\, \widehat{\varphi}^k(\mathrm{d}(u_1,\ldots,u_k)).$$

To simplify the bracket, we may assume that $u_1,\ldots,u_k$ are linearly independent. Let $s_{u,\tau} \in \mathrm{lin}\{u_1,\ldots,u_k\}$ be the vector for which

$$H(u_i,\tau_i) = u_i^{\perp} + s_{u,\tau} \quad \text{for } i = 1,\ldots,k.$$

Inserting this in the integral in the bracket and using, in this order, the translation invariance of $\mathcal{H}^{d-k}$, the stationarity of $\widehat{X}$ and the translation invariance of h, we obtain

$$\int_{H(u_1,\tau_1)\cap\cdots\cap H(u_k,\tau_k)} \mathbf{1}_B(x)\, \mathbb{E} \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{H(u_1,\tau_1),\ldots,H(u_k,\tau_k)\})} \mathbf{1}\{x \in K\} h(K)\, \mathcal{H}^{d-k}(\mathrm{d}x)$$

$$= \int_{u_1^{\perp}\cap\cdots\cap u_k^{\perp}} \mathbf{1}_B(x + s_{u,\tau})\, \mathbb{E} \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{u_1^{\perp},\ldots,u_k^{\perp}\})} \mathbf{1}\{x \in K\} h(K)\, \mathcal{H}^{d-k}(\mathrm{d}x).$$

Again by the stationarity of $\widehat{X}$ and the translation invariance of h, the expectation

$$\mathbb{E} \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{u_1^{\perp},\ldots,u_k^{\perp}\})} \mathbf{1}\{x \in K\} h(K),$$

where $x \in u_1^{\perp} \cap \cdots \cap u_k^{\perp}$, does not change if we replace x by 0. Since $\lambda(B) = 1$, we have

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{u_1^{\perp}\cap\cdots\cap u_k^{\perp}} \mathbf{1}_B\left(x + s_{u,\tau}\right)\, \mathcal{H}^{d-k}(\mathrm{d}x)\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k = \nabla_k(u_1,\ldots,u_k),$$

where $\nabla_k(u_1,\ldots,u_k)$ denotes the k-dimensional volume of the parallelepiped spanned by the vectors $u_1,\ldots,u_k$ (cf. [8], p. 135). Thus, we get

$$\mathbb{E}\, h(D_k) = \frac{\widehat{\gamma}^k}{\gamma k! \mathbb{E}\, L_{d-k}(Z)} \int_{(\mathbb{S}^{d-1})^k} \mathbb{E} \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{u_1^{\perp},\ldots,u_k^{\perp}\})} \mathbf{1}\{0 \in K\} h(K) \qquad (10)$$

$$\times\, \nabla_k(u_1,\ldots,u_k)\, \widehat{\varphi}^k(\mathrm{d}(u_1,\ldots,u_k)).$$

From (8) and (10), we obtain

$$\mathbb{E}\,(hL_{d-k})(Z) = \frac{\widehat{\gamma}^k}{\gamma k!} \int_{(\mathbb{S}^{d-1})^k} \mathbb{E} \sum_{K\in\mathcal{F}_d(\widehat{X}\cup\{u_1^{\perp},\ldots,u_k^{\perp}\})} \mathbf{1}\{0 \in K\} h(K) \qquad (11)$$

$$\times\, \nabla_k(u_1,\ldots,u_k)\, \widehat{\varphi}^k(\mathrm{d}(u_1,\ldots,u_k)),$$

for $k = 1,\ldots,d$.

4 A covariance matrix

In this section we determine $\mathbb{E}\,(L_r L_s)(Z)$. First we consider the case $s = d$ (and recall that $L_d = V_d$ is the volume). By [8], Thm. 10.4.1 (where $\gamma^{(d)}$ is what we here denote by γ), we have

$$\mathbb{E}\,(L_r L_d)(Z) = \frac{1}{\gamma}\mathbb{E}\,L_r(Z_0).$$

Here Z_0 is the zero cell of X, that is, the almost surely unique cell of X that contains the origin. Together with [8], (10.50) and (10.51), this yields

$$\mathbb{E}\,(L_r L_d)(Z) = \frac{d!}{2^d\gamma} V_{d-r}(\Pi_{\widehat{X}}) V_d(\Pi^\circ_{\widehat{X}}). \tag{12}$$

Now let $s = d - k$ with $k \geq 1$. We apply (11) with $h = L_r$, where $r \in \{0, \dots, d\}$. To transform the sum $\sum h(K)$ in (11), which extends over the cells induced by $\widehat{X} \cup \{u_1^\perp, \dots, u_k^\perp\}$ and containing 0, we use an idea of Miles [4], Section 11.6.

Let $H_1, \dots, H_k$ be fixed hyperplanes through 0 in general position (hyperplanes in $\mathbb{R}^d$ are said to be *in general position* if any $m \leq d$ of them have an intersection of dimension $d-m$). Almost surely, $\widehat{X}$ and $H_1, \dots, H_k$ are in general position, and this is assumed for the realisations of $\widehat{X}$ considered in the following. We define

$$\mathcal{C}_0 := \big\{K \in \mathcal{F}_d(\widehat{X} \cup \{H_1, \dots, H_k\}) : 0 \in K\big\}.$$

For $j \in \{r, \dots, d\}$ with $d - j \leq k$, let

$$\mathcal{F}_{r,j} := \big\{F \in \mathcal{F}_r(K) : K \in \mathcal{C}_0,\ F \subset H_i \text{ for precisely } d-j \text{ indices } i \in \{1, \dots, k\}\big\}.$$

We have

$$\sum_{K\in\mathcal{C}_0} L_r(K) = \sum_{K\in\mathcal{C}_0} \sum_{F\in\mathcal{F}_r(K)} \mathcal{H}^r(F) = \sum_{j=\max\{r,d-k\}}^{d} 2^{d-j} \sum_{F\in\mathcal{F}_{r,j}} \mathcal{H}^r(F),$$

since each $F \in \mathcal{F}_{r,j}$ belongs to precisely 2^{d-j} polytopes $K \in \mathcal{C}_0$.

Let

$$\mathcal{Z}_j := \big\{Z_0 \cap H_{i_1} \cap \dots \cap H_{i_{d-j}} : 1 \leq i_1 < \dots < i_{d-j} \leq k\big\}$$

with $\mathcal{Z}_d = \{Z_0\}$. Thus, $\mathcal{Z}_j$ is a set of j-dimensional polytopes, containing the origin. Each r-face $F \in \mathcal{F}_{r,j}$ satisfies $F \subset G \in \mathcal{F}_r(M)$ for a unique $M \in \mathcal{Z}_j$ and a unique $G \in \mathcal{F}_r(M)$. Conversely, for $M \in \mathcal{Z}_j$ and $G \in \mathcal{F}_r(M)$, the r-face G is the union of r-faces from $\mathcal{F}_{r,j}$, which pairwise have no relatively interior points in common. It follows that

$$\sum_{F\in\mathcal{F}_{r,j}} \mathcal{H}^r(F) = \sum_{M\in\mathcal{Z}_j} \sum_{F\in\mathcal{F}_r(M)} \mathcal{H}^r(F).$$

We conclude that

$$\begin{aligned}\sum_{K\in\mathcal{C}_0} L_r(K) &= \sum_{j=\max\{r,d-k\}}^{d} 2^{d-j} \sum_{M\in\mathcal{Z}_j} \sum_{F\in\mathcal{F}_r(M)} \mathcal{H}^r(F)\\ &= \sum_{j=\max\{r,d-k\}}^{d} 2^{d-j} \sum_{M\in\mathcal{Z}_j} L_r(M).\end{aligned}$$

Inserting this in (11) with $h = L_r$, we obtain

$$
\begin{aligned}
&\mathbb{E}\,(L_r L_{d-k})(Z)\\
&= \frac{\widehat{\gamma}^k}{\gamma k!} \sum_{j=\max\{r,d-k\}}^{d} 2^{d-j} \int_{(\mathbb{S}^{d-1})^k} \mathbb{E} \sum_{1\le i_1<\cdots<i_{d-j}\le k} L_r(Z_0 \cap u_{i_1}^\perp \cap \cdots \cap u_{i_{d-j}}^\perp)\\
&\quad \times \nabla_k(u_1,\dots,u_k)\, \widehat{\varphi}^k(\mathrm{d}(u_1,\dots,u_k))\\
&= \frac{\widehat{\gamma}^k}{\gamma k!} \sum_{j=\max\{r,d-k\}}^{d} 2^{d-j} \binom{k}{d-j} \int_{(\mathbb{S}^{d-1})^k} \mathbb{E}\, L_r(Z_0 \cap u_1^\perp \cap \cdots \cap u_{d-j}^\perp) \qquad (13)\\
&\quad \times \nabla_k(u_1,\dots,u_k)\, \widehat{\varphi}^k(\mathrm{d}(u_1,\dots,u_k)).
\end{aligned}
$$

For given linearly independent vectors $u_1,\dots,u_{d-j} \in \mathbb{S}^{d-1}$ in general position, let

$$L := u_1^\perp \cap \cdots \cap u_{d-j}^\perp.$$

We can now argue as in [6], p. 690: the intersection $Z_0 \cap L$ is the zero cell of the intersection process $\widehat{X} \cap L$ (see (4.61) in [8]), and it is known (see [8], Thm. 10.4.9) that

$$\mathbb{E}\, L_r(Z_0 \cap L) = 2^{-j} j! V_{j-r}(\Pi_{\widehat{X}}|L) V_j((\Pi_{\widehat{X}}|L)^\circ).$$

Thus, the expectation in the integrand of (13) can be expressed in terms of the associated zonoid. This holds also for the intensity γ appearing in front of the sum. Indeed, we have

$$\gamma = V_d(\Pi_{\widehat{X}}). \qquad (14)$$

This is formula (10.44) in [8] for $k = d$ (note that $\gamma^{(d)}$ appearing there is the intensity of the process of d-dimensional cells of X and hence is what we have here denoted by γ).

We have obtained the following result. It includes also the case $s = d$, given by (12), where in (15) the projections and integration have to be deleted.

Theorem 2. *The face contents of the typical cell of the stationary Poisson random mosaic X satisfy*

$$
\begin{aligned}
&\mathbb{E}\,(L_r L_s)(Z) \qquad (15)\\
&= V_d(\Pi_{\widehat{X}})^{-1} \sum_{j=\max\{r,s\}}^{d} 2^{d-2j} j! \binom{d-s}{d-j}\\
&\quad \times \frac{\widehat{\gamma}^{d-s}}{(d-s)!} \int_{(\mathbb{S}^{d-1})^{d-s}} V_{j-r}(\Pi_{\widehat{X}}|u_1^\perp \cap \cdots \cap u_{d-j}^\perp) V_j((\Pi_{\widehat{X}}|u_1^\perp \cap \cdots \cap u_{d-j}^\perp)^\circ)\\
&\quad \times \nabla_{d-s}(u_1,\dots,u_{d-s})\, \widehat{\varphi}^{d-s}(\mathrm{d}(u_1,\dots,u_{d-s}))
\end{aligned}
$$

for $r, s \in \{0,\dots,d\}$.

How the associated zonoid $\Pi_{\widehat{X}}$ depends on the spherical directional distribution $\widehat{\varphi}$, is seen from (6).

Together with the known relations

$$\mathbb{E}\, L_r(Z) = \frac{2^{d-r}}{\gamma}\, V_{d-r}(\Pi_{\widehat{X}}), \quad r = 0, \dots, d, \tag{16}$$

Theorem 2 provides an explicit expression for the covariance matrix of the random vector $(L_0(Z), \dots, L_d(Z))$. Concerning (16), we notice that it follows from [8], Theorem 10.1.2 and (10.9), that

$$\gamma \mathbb{E}\, L_r(Z) = 2^{d-r} d_r^{(r)} \quad \text{with} \quad d_r^{(r)} := \gamma \mathbb{E}\, V_r(Z),$$

and by [8], (10.43), we have $d_r^{(r)} = V_{d-r}(\Pi_{\widehat{X}})$.

Assume that $\widehat{X}$ is isotropic. Then the associated zonoid $\Pi_{\widehat{X}}$ is a ball of radius

$$R = \widehat{\gamma}\, \frac{\kappa_{d-1}}{d\kappa_d}, \tag{17}$$

see [8], p. 490. The kth intrinsic volume of the unit ball B^d is given by

$$V_k(B^d) = \binom{d}{k} \frac{\kappa_d}{\kappa_{d-k}}$$

([8], (14.8)). It follows that

$$\begin{aligned} V_{j-r}(\Pi_{\widehat{X}} | u_1^\perp \cap \dots \cap u_{d-j}^\perp) &= R^{j-r} \binom{j}{r} \frac{\kappa_j}{\kappa_r}, \\ V_j((\Pi_{\widehat{X}} | u_1^\perp \cap \dots \cap u_{d-j}^\perp)^\circ) &= R^{-j} \kappa_j. \end{aligned}$$

The remaining integral is known (also in the non-isotropic case), namely

$$\frac{\widehat{\gamma}^{d-s}}{(d-s)!} \int_{(\mathbb{S}^{d-1})^{d-s}} \nabla_{d-s}(u_1, \dots, u_{d-s})\, \widehat{\varphi}^{d-s}(\mathrm{d}(u_1, \dots, u_{d-s})) = V_{d-s}(\Pi_{\widehat{X}}), \tag{18}$$

by [8], formula (14.35) with $\rho = \widehat{\gamma}\widehat{\varphi}/2$. Since

$$V_{d-s}(\Pi_{\widehat{X}}) = R^{d-s} \binom{d}{s} \frac{\kappa_d}{\kappa_s}, \qquad V_d(\Pi_{\widehat{X}}) = R^d \kappa_d,$$

we obtain

$$\mathbb{E}\, (L_r L_s)(Z) = \frac{2^d d!}{\kappa_r \kappa_s} \left(\frac{d\kappa_d}{\kappa_{d-1}\widehat{\gamma}} \right)^{r+s} \sum_{j=\max\{r,s\}}^{d} \frac{\kappa_j^2}{4^j (d-j)!} \binom{j}{r} \binom{j}{s}. \tag{19}$$

Since we can also write

$$\frac{d\kappa_d}{\kappa_{d-1}} = \frac{2\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2}[d+1]\right)}{\Gamma\left(\frac{1}{2}d\right)}, \qquad \kappa_j = \frac{\pi^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} = \frac{2^j \pi^{\frac{j-1}{2}} \Gamma\left(\frac{1}{2}[j+1]\right)}{j!},$$

(19) is the same as

$$\begin{aligned} \mathbb{E}\, (L_r L_s)(Z) \quad = \quad & \frac{2^d \pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}[r+1]\right) \Gamma\left(\frac{1}{2}[s+1]\right)} \left\{ \frac{\Gamma\left(\frac{1}{2}[d+1]\right)}{\Gamma\left(\frac{1}{2}d\right)\widehat{\gamma}} \right\}^{r+s} \\ & \times \sum_{j=\max\{r,s\}}^{d} \binom{d}{j} \left(\frac{\pi}{2}\right)^j \frac{\Gamma\left(\frac{1}{2}[j+1]\right)}{\Gamma\left(\frac{1}{2}j+1\right)} (j)_r (j)_s \end{aligned}$$

with $(j)_r = j!/(j-r)!$. This is formula (63) of Miles [5].

Another simple case where (15) can be evaluated explicitly is that of a cuboid process $\widehat{X}$, where the hyperplanes of the process belong to d pairwise orthogonal translation classes. Cuboid processes were studied by Favis [2]. We consider only the case of a quasi-isotropic cuboid process, where the directional distribution $\widehat{\varphi}$ is concentrated in $\pm e_1, \dots, \pm e_n$ for an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^d$ and assigns the same value to each of these points. In this case, the associated zonoid $\widehat{X}$ is a cube of edge length $2^{-d}\widehat{\gamma}$. Moreover, for $(u_1, \dots, u_{d-j})$ in the support of the measure $\widehat{\varphi}^{d-s}$, the orthogonal projection $\Pi_{\widehat{X}}|u_1^\perp \cap \dots \cap u_{d-j}^\perp$ is a j-dimensional cube of edge length $2^{-d}\widehat{\gamma}$. For a d-dimensional cube C of edge length a and for $j \in \{0, \dots, d\}$ we have

$$V_j(C) = \binom{d}{j} a^j$$

by formula (4.23) in [7], together with $f_j(C) = 2^{d-j}\binom{d}{j}$. From the well-known relation $V_d(C)V_d(C^\circ) = 4^d/d!$, if C is centred at 0, it follows that

$$V_d(C^\circ) = \frac{4^d}{d!a^d}.$$

Now we conclude from (15), observing (18), that

$$\mathbb{E}\,(L_r L_s)(Z) = \frac{2^{d(r+s+1)}}{\widehat{\gamma}^{r+s}} \sum_{j=\max\{r,s\}}^{d} \binom{d}{j}\binom{j}{r}\binom{j}{s} \tag{20}$$

for $r, s \in \{0, \dots, d\}$.

5 Proof of Theorem 1

For $r = 0$, the functional appearing in the integrand of (15) is a volume product. For a 0-symmetric convex body $K \subset \mathbb{R}^d$ of dimension $j \in \{0, \dots, d\}$, the *volume product* is defined by

$$\mathrm{vp}(K) := V_j(K)V_j(K^\circ).$$

Specializing (15) to $r = s = 0$, we obtain

$$\begin{aligned} &\mathbb{E}\, f_0^2(Z) \qquad (21)\\ &= \sum_{j=0}^{d} \frac{2^{d-2j}d!}{(d-j)!} \cdot \frac{\widehat{\gamma}^d}{\gamma\, d!} \int_{(\mathbb{S}^{d-1})^d} \mathrm{vp}(\Pi_{\widehat{X}}|u_1^\perp \cap \dots \cap u_{d-j}^\perp) \nabla_d(u_1, \dots, u_d)\, \widehat{\varphi}^d(\mathrm{d}(u_1, \dots, u_d)). \end{aligned}$$

We notice that for the integral appearing here we have

$$\frac{\widehat{\gamma}^d}{\gamma\, d!} \int_{(\mathbb{S}^{d-1})^d} \nabla_d(u_1, \dots, u_d)\, \widehat{\varphi}^d(\mathrm{d}(u_1, \dots, u_d)) = 1.$$

This follows from (18) for $s = 0$, together with (14).

For a j-dimensional zonoid K, such as $\Pi_{\widehat{X}}|u_1^\perp \cap \dots \cap u_{d-j}^\perp$, the inequalities

$$\frac{4^j}{j!} \leq \mathrm{vp}(K) \leq \kappa_j^2$$

are valid. The right-hand inequality is known as the Blaschke–Santaló inequality (for references, see [8], Chap. 14). We conclude that

$$2^{2d} \leq \mathbb{E}\, f_0^2(Z) \leq \sum_{j=0}^{d} \frac{2^{d-2j} d!}{(d-j)!} \kappa_j^2.$$

Since $\mathbb{E}\, f_0(Z) = 2^d$, this yields the inequalities (1).

If equality holds in one of the inequalities (1), then it holds in the corresponding one of the inequalities

$$\frac{4^d}{d!} \leq \mathrm{vp}(\Pi_{\widehat{X}}) \leq \kappa_d^2. \tag{22}$$

Equality on the left implies that $\Pi_{\widehat{X}}$ is a parallelepiped, and inequality on the right implies that $\Pi_{\widehat{X}}$ is an ellipsoid. Choosing a suitable scalar product on $\mathbb{R}^d$, we can assume that $\Pi_{\widehat{X}}$ is a ball in the second case. Since the associated zonoid $\Pi_{\widehat{X}}$ determines the distribution of $\widehat{X}$ uniquely ([8], Theorem 4.6.4), $\widehat{X}$ is a parallel process in the first case and is isotropic (with respect to the new scalar product) in the second case. Conversely, if $\widehat{X}$ is a parallel process, then trivially $\mathrm{Var}\, f_0(Z) = 0$. If $\widehat{X}$ is isotropic, then all orthogonal projections $\Pi_{\widehat{X}}|L$ are balls in L, hence equality holds on the right side of (1). This completes the proof of Theorem 1.

We mention that the just proved characterizations of the processes for which $\mathrm{Var}\, f_0(Z)$ attains its extreme values, can be strengthened in the form of stability assertions, following the example set out by Böröczky and Hug [1]. For this, we write

$$\mathrm{Var}\, f_0(Z) = 2^{-d} d! \left[\mathrm{vp}(\Pi_{\widehat{X}}) + \Phi(\widehat{X}) \right]$$

with

$$\begin{aligned} \Phi(\widehat{X}) \quad = \quad & \sum_{j=0}^{d-1} \frac{2^{2(d-j)}}{(d-j)!} \cdot \frac{\widehat{\gamma}^d}{\gamma\, d!} \int_{(\mathbb{S}^{d-1})^d} \\ & \times \mathrm{vp}(\Pi_{\widehat{X}} | u_1^\perp \cap \cdots \cap u_{d-j}^\perp) \nabla_d(u_1, \ldots, u_d)\, \widehat{\varphi}^d(\mathrm{d}(u_1, \ldots, u_d)) - \frac{2^{3d}}{d!}. \end{aligned}$$

Then (22) holds together with

$$-\frac{4^d}{d!} =: c_d \leq \Phi(\widehat{X}) \leq C^d := \sum_{j=0}^{d-1} \frac{2^{2(d-j)}}{(d-j)!} \kappa_j^2 - \frac{2^{3d}}{d!}.$$

Suppose that $\mathrm{Var}\, f_0(Z)$ is close to its minimal value 0, say

$$\mathrm{Var}\, f_0(Z) \leq \varepsilon$$

with some $\varepsilon > 0$. Then

$$2^{-d} d! [\mathrm{vp}(\Pi_{\widehat{X}}) + c_d] \leq 2^{-d} d! [\mathrm{vp}(\Pi_{\widehat{X}}) + \Phi(\widehat{X})] = \mathrm{Var}\, f_0(Z) \leq \varepsilon$$

and hence

$$\mathrm{vp}(\Pi_{\widehat{X}}) \leq (1 + 2^{-d}\varepsilon) \frac{4^d}{d!}. \tag{23}$$

If (23) holds with sufficiently small $\varepsilon > 0$, then it is shown in [1] how a suitable distance (e.g., Wasserstein or Prokhorov) of the directional distribution $\widehat{\varphi}$ of $\widehat{X}$ from the distribution of a suitable parallel process can be estimated from above in terms of ε.

Now suppose that $\mathrm{Var}\, f_0(Z)$ is close to its maximal value, say

$$\mathrm{Var}\, f_0(Z) \geq (1-\varepsilon)2^{-d}d![\kappa_d^2 + C_d]$$

with some $\varepsilon \in (0,1)$. Then

$$2^{-d}d![\mathrm{vp}(\Pi_{\widehat{X}}) + C_d] \geq 2^{-d}d![\mathrm{vp}(\Pi_{\widehat{X}}) + \Phi(\widehat{X})] = \mathrm{Var}\, f_0(Z) \geq (1-\varepsilon)2^{-d}d!(\kappa_d^2 + C_d)$$

and hence

$$\mathrm{vp}(\Pi_{\widehat{X}}) \geq (1 - a_d\varepsilon)\kappa_d^2 \tag{24}$$

with

$$a_d = 1 + C_d/\kappa_d^2 > 0.$$

If (24) holds with sufficiently small $\varepsilon > 0$, then one can see from [1] how a suitable distance of the directional distribution $\widehat{\varphi}$ of $\widehat{X}$ from the isotropic distribution can be estimated from above in terms of ε.

Finally we remark that also for the volume $V_d = L_d$, estimates of $\mathrm{Var}\, V_d(Z)$ can be obtained. But since $V_d(Z)$ is not homogeneous of degree 0, no absolute inequalities are possible, but only inequalities of isoperimetric type. It follows from (12), (14), (16) that for each stationary Poisson hyperplane process $\widehat{X}$ we have

$$\frac{\mathrm{Var}\, V_d(Z)}{(\mathbb{E}V_d(Z))^2} = 2^{-d}d!\mathrm{vp}(\Pi_{\widehat{X}}) - 1.$$

Hence, for given expectation $\mathbb{E}V_d(Z)$, the variance $\mathrm{Var}\, V_d(Z)$ is maximal for isotropic processes and minimal for parallel processes.

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