Simplices

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These two lectures are about extremal properties of simplices in the affine geometry of convex bodies. I begin with the well-known fact that the space of affine equivalence classes of convex bodies in $\mathbb{R}^n$ is compact. As a consequence, every affine-invariant continuous functional on $\mathcal{K}^n$ (the space of convex bodies with interior points in $\mathbb{R}^n$) attains a maximum and a minimum. For the proof of the compactness, often John's theorem is invoked. There is, however, an easier way. Let $K \in \mathcal{K}^n$. There exists a simplex $T \subseteq K$ of maximal volume. Let F be a facet of T, v the opposite vertex, and H the hyperplane through v parallel to F. Then H supports K, since otherwise one would obviously obtain a contradiction to the maximality of T. Since F was an arbitrary facet of T, we see that K is contained in the simplex $-n(T-c)+c$, where c is the centroid of T. Let Δ be a regular simplex with centroid 0, and let $\Delta' := -n\Delta$. There exists $\alpha \in \mathrm{Aff}(n)$ (the group of regular affine maps from $\mathbb{R}^n$ to $\mathbb{R}^n$) with $\alpha T = \Delta$. Then $\Delta \subseteq \alpha K \subseteq \Delta'$. Hence, every convex body has an affine transform in the set $\{M \in \mathcal{K}^n : \Delta \subseteq M \subseteq \Delta'\}$. The latter set is clearly compact. From this, the assertion follows.

In the following, we denote by T^n an n-dimensional simplex and by B^n an n-dimensional ball. Only the affine equivalence classes of T^n and B^n will play a role. By V we denote volume.

The geometry of convex bodies has produced a wealth of natural (that is, geometrically meaningful) functionals on convex bodies which are continuous and invariant under affine transformations. Each of these functionals gives rise to the question for the bodies on which the functional attains its maximum and its minimum. There are not too many cases where both extrema are known and are non-trivial. There are several cases where symmetrization has been applied successfully to show that one extremum is attained by ellipsoids, but the other extremum is often unknown. Generally, simplices are strong candidates for extremal bodies. It is this role of the simplices which is the theme of these lectures. I intend to discuss the following points, though not necessarily in this order:

- The main examples of cases where both extrema are known.
- If an extremum is known, there arises immediately the question for stability improvements: if a convex body attains the extremum up to ϵ, is it $f(\epsilon)$-close to some simplex, with explicit $f(\epsilon)$? Recent new results of this type are mentioned, and one or two new proofs will be given.

- Some open extremum problems, where simplices have been conjectured to be extremal, with varying degrees of confidence.
- Two examples from applications, where extremal properties of simplices have been observed, will be mentioned.

We begin with some affine-invariant functionals of a simple structure, which are known as so-called **measures of symmetry** (see Grünbaum [16] for a survey).

The Minkowski measure of symmetry

For a convex body $K \in \mathcal{K}^n$, the **Minkowski measure of symmetry** can be defined by

$$q(K) := \min\{\lambda > 0 : \exists\, x \in K : -(K - x) \subseteq \lambda(K - x)\}.$$

It is known that

$$1 \leq q(K) \leq n. \tag{1}$$

Equality on the left holds trivially if and only if K is centrally symmetric, and on the right it holds if and only if K is a simplex (references are in [16]).

The extremal property of the simplex can be improved in the form of a stability version. For this, we need an appropriate notion of distance for affine equivalence classes of convex bodies. The extended Banach-Mazur distance of not necessarily symmetric convex bodies $K, L \in \mathcal{K}^n$ is defined by

$$d_{BM}(K, L) := \min\{\lambda \geq 1 : \exists\, \alpha \in \mathrm{Aff}(n)\, \exists\, x \in \mathbb{R}^n : L \subseteq \alpha K \subseteq \lambda L + x\}$$

Recently, two papers independently gave stability estimates for the right-hand side of (1), Böröczky [8] and Guo [17]. Böröczky's result is stronger, he shows:

$$q(K) \geq n - \epsilon \quad \text{with} \quad 0 < \epsilon < \frac{1}{4n} \quad \Rightarrow \quad d_{BM}(K, T^n) < 1 + 4n\epsilon.$$

Since the order of ϵ is optimal, it is of some interest to find good constants. Böröczky's result can still be improved:

Theorem 1.

$$q(K) \geq n - \epsilon \quad \text{with} \quad 0 < \epsilon < \frac{1}{n} \quad \Rightarrow \quad d_{BM}(K, T^n) < 1 + \frac{(n+1)\epsilon}{1 - n\epsilon}.$$

Proof. I give the proof here, since it has one interesting aspect. If we want to show that some convex body is close to a simplex, we must construct this simplex, and thus its vertices. In the present case, the vertices are found by an application of Helly's theorem.

For $0 \leq q \leq n$ and $x \in K$, define

$$K(x, q) := \frac{q}{q+1}(K - x) + x.$$

Lemma.

$$c \in \bigcap_{x \in K} K(x,q) \Leftrightarrow -(K-c) \subseteq q(K-c). \tag{2}$$

Proof. We have

$$\begin{aligned} c \in K(x,q) \quad &\Leftrightarrow \quad \exists k \in K : c = \frac{q}{q+1}(k-x)+x \\ &\Leftrightarrow \quad \exists k \in K : -(x-c) = q(k-c) \\ &\Leftrightarrow \quad -(x-c) \in q(K-c), \end{aligned}$$

from which the lemma follows.

Now let $K \subset \mathbb{R}^n$ be a convex body with $q(K) > n-\epsilon$, where $0 < \epsilon < 1/n$, and put $q := n-\epsilon$. Since $q < q(K)$, no point $c \in K$ satisfies the right-hand side of (2). By the Lemma and by Helly's theorem, there must exist $n+1$ points $e_0, e_1, \ldots, e_n \in K$ such that

$$\bigcap_{i=0}^{n} K(e_i, q) = \emptyset. \tag{3}$$

Since the set of all $(n+1)$-tuples $(e_0, \ldots, e_n)$ satisfying (3) is open in K^{n+1}, we can assume that $e_0, \ldots, e_n$ are affinely independent. Then $\Delta := \operatorname{conv}\{e_0, \ldots, e_n\}$ is an n-simplex contained in K. We set

$$r := \frac{q}{q+1} = \frac{n-\epsilon}{n+1-\epsilon}$$

and

$$z := [1-n(1-r)]e_0 + (1-r)\sum_{i=1}^{n} e_i.$$

With

$$\alpha_0 := \frac{1-n(1-r)}{r} = \frac{1-\epsilon}{n-\epsilon}, \quad \alpha_1 := 0, \quad \alpha_j := \frac{1-r}{r} = \frac{1}{n-\epsilon}$$

for $j = 2, \ldots, n$ we have $\alpha_i \geq 0$ and $\sum_{i=0}^{n} \alpha_i = 1$, hence

$$\begin{aligned} z &= \sum_{i=0}^{n} \alpha_i [r e_i + (1-r)e_1] \in \operatorname{conv}\{r(e_i - e_1) + e_1 : i = 0, \ldots, n\} \\ &= r(\Delta - e_1) + e_1 = \Delta(e_1, q). \end{aligned}$$

Similarly, $z \in \Delta(e_i, q)$ for $i = 1, \ldots, n$. Since $\Delta(e_i, q) \subset K(e_i, q)$, it follows from (3) that

$$z \notin K(e_0, q).$$

Now we assume, without loss of generality, that

$$\sum_{i=0}^{n} e_i = 0, \tag{4}$$

so that $z = [1-(n+1)(1-r)]e_0$. From $z \notin r(K-e_0)+e_0$ we get that the point

$$z_0 := -\tau e_0 \quad \text{with} \quad \tau := \frac{1+\epsilon}{n-\epsilon}$$

satisfies $z_0 \notin K$. By (4),

$$z_0 = \tau(e_1 + \cdots + e_n). \tag{5}$$

Now we set

$$\lambda := 1 + \frac{(n+1)\epsilon}{1-n\epsilon}$$

and assert that

$$K \subseteq \operatorname{int} \lambda\Delta. \tag{6}$$

Suppose that (6) were false. Then some facet of $\lambda\Delta$, say the one opposite to λe_0, contains a point $p \in K$. There is a unique representation

$$p = \sum_{i=1}^{n} \gamma_i \lambda e_i \quad \text{with} \quad \gamma_i \geq 0, \quad \sum_{i=1}^{n} \gamma_i = 1.$$

Further, there is a unique affine representation

$$z_0 = \sum_{i=1}^{n} \beta_i e_i + \beta_{n+1} p \quad \text{with} \quad \sum_{i=1}^{n+1} \beta_i = 1,$$

thus

$$z_0 = \sum_{i=1}^{n} [\beta_i + \beta_{n+1}\gamma_i\lambda] e_i.$$

Comparing this with (5), we get

$$\beta_i + \beta_{n+1}\gamma_i\lambda = \tau \quad \text{for } i = 1, \dots, n.$$

By addition,

$$\sum_{i=1}^{n} \beta_i + \lambda\beta_{n+1} = n\tau,$$

hence

$$\beta_{n+1} = \frac{n\tau - 1}{\lambda - 1} = \frac{1-n\epsilon}{n-\epsilon} \geq 0.$$

For $i = 1, \dots, n$, we get

$$\beta_i = \frac{1+\epsilon}{n-\epsilon}(1-\gamma_i) \geq 0.$$

Thus

$$z_0 \in \operatorname{conv}\{e_1, \dots, e_n, p\} \subseteq K,$$

a contradiction. This shows that (6) holds, which implies that $d_{BM}(K,T) < \lambda$. $\square$

The survey of Grünbaum [16] mentions several other measures of symmetry where the extremal property of simplices is either proved or conjectured (and still not proved today). We mention only one here, which is quite well known but still offers new developments and open questions.

The difference body measure of symmetry

The difference body of a convex body $K \in \mathcal{K}^n$ is the body $\mathrm{D}K := K - K$. The affine invariant $V(\mathrm{D}K)/V(K)$ is estimated by

$$2^n \leq \frac{V(\mathrm{D}K)}{V(K)} \leq \binom{2n}{n}. \tag{7}$$

Equality on the left holds if and only if K is centrally symmetric, and on the right if and only if K is a simplex. The right-hand inequality is known as the Rogers-Shephard inequality (1957). It would be implied by a beautiful inequality conjectured by Godbersen (1938), namely

$$\frac{V(K[i], -K[n-i])}{V(K)} \leq \binom{n}{i}. \tag{8}$$

Here the numerator is a mixed volume, and $i \in \{1, \dots, n-1\}$. It is conjectured that (8) holds and equality characterizes simplices. For $i = 1$ and $i = n-1$, this is true.

An interesting recent development concerning the Rogers-Shephard inequality is the proof of a stability estimate by Böröczky [8]. He showed that

$$\frac{V(\mathrm{D}K)}{V(K)} \geq (1-\epsilon)\binom{2n}{n} \quad \Rightarrow \quad d_{BM}(K, T^n) \leq 1 + n^{50n^2}\epsilon.$$

The dependence on ϵ cannot be improved. The proof has to quantify the characterization of the simplex as it is used in the equality case of the Rogers-Shephard inequality. The crucial point is here the following. Let $x \in \mathrm{D}K \setminus \{0\}$. There is a unique point $y \in \mathrm{bd}\,\mathrm{D}K$ and a unique $\lambda \in (0, 1]$ such that $y = \lambda x$. It is easy to see that

$$(1-\lambda)K + \lambda y \subseteq K \cap (K + x).$$

The resulting inequality

$$V((1-\lambda)K) \leq V(K \cap (K + x))$$

leads, with integration tricks, to the Rogers-Shephard inequality. If equality holds in the Rogers-Shephard inequality, then we must have

$$(1-\lambda)K + \lambda y = K \cap (K + x)$$

for all $x \in \mathrm{D}K$. Hence, any nonempty intersection of K with a translate of K is homothetic to K. This property characterizes simplices. The principal difficulty lies in a stability version of this simplex characterization. Böröczky's proof is certainly a remarkable achievement.

I will now give two examples showing how extremal properties of simplices related to variants of the difference body inequality appear in applications. The examples are taken from stochastic geometry. The first one concerns continuum percolation in the plane. Let X_λ be a stationary Poisson point process of intensity λ in the plane, and let K be a convex body of area $A(K) = 1$. We translate K by the vectors of X_λ and form the union:

$$S := \bigcup_{x \in X_\lambda} (K + x).$$

The fundamental question in continuum percolation (formulated for this special model) asks whether S has some unbounded connected component. It is known that there exists a critical threshold λ_c: if $\lambda < \lambda_c$, then S a.s. contains no unbounded connected component, and if $\lambda > \lambda_c$, then S a.s. contains an unbounded component. This critical value λ_c depends on the convex body K (of given area 1), and one may ask for its extrema. It was proved by Jonasson [19] that the critical value λ_c becomes minimal if K is a triangle. This follows from a strengthened version of the difference body inequality in the plane. The latter says that

$$A(\mathrm{D}K) \leq A(\mathrm{D}T) \quad \text{if} \quad A(K) = A(T)$$

for any convex body K and triangle T. Jonasson obtained the following remarkable improvement: To any convex body K in the plane there exists a triangle T such that

$$\mathrm{D}K \subseteq \mathrm{D}T \qquad \text{and} \qquad A(K) = A(T).$$

By $A(\mathrm{D}K) \leq A(\mathrm{D}T) = 6A(T) = 6A(K)$, this implies the difference body inequality.

The application to continuum percolation follows from the observation that

$$\begin{aligned} (K + x) \cap (K + y) \neq \emptyset \quad &\Leftrightarrow \quad x - y \in \mathrm{D}K \\ &\Rightarrow \quad x - y \in \mathrm{D}T \Leftrightarrow (T + x) \cap (T + y) \neq \emptyset. \end{aligned}$$

Hence, if $\bigcup_{x \in X_\lambda} (K + x)$ contains an unbounded connected component, then so does $\bigcup_{x \in X_\lambda} (T + x)$. An extension to higher dimensions would be very interesting, but seems rather hopeless.

The next 'application' we want to describe concerns random coverings by translates of a convex body. Let $A, B \subset \mathbb{R}^n$ be convex bodies (for simplicity) with $A \subset \operatorname{int} B$. Further, a convex body K is given. Let $(x_i)_{i \in \mathbb{N}}$ be sequence of independent, identically distributed random points with uniform distribution in B. Janson [18] studied the random variable

$$N_r := \min\{m \in \mathbb{N} : A \subseteq \bigcup_{i=1}^{m} (rK + x_i)\}$$

for $r > 0$, the smallest number m so that B is covered by the first m translates of rK in the sequence $(rK + x_i)_{i \in \mathbb{N}}$. He was able to determine the asymptotic distribution of N_r for $r \to 0$. What interests us here is how the shape of K influences this asymptotic distribution. Janson found that the expression for the asymptotic distribution involves a certain functional $\beta(K)$, and later he observed that this functional attains its maximum on simplices. I will explain what this has to do with an extension of the difference body inequality.

Since $x \in \mathrm{D}K \Leftrightarrow K \cap (K+x) \neq \emptyset$, we can write

$$V(\mathrm{D}K) = \int_{\mathbb{R}^n} \mathbf{1}\{K \cap (K+x) \neq \emptyset\}\,\mathrm{d}x.$$

We generalize this, for $p \in \mathbb{N}$, to

$$\mathrm{D}_p K := \{(x_1,\dots,x_p) \in (\mathbb{R}^n)^p : K \cap (K+x_1) \cap \dots \cap (K+x_p) \neq \emptyset\}$$

and

$$V_{pn}(\mathrm{D}_p K) = \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \mathbf{1}\{K \cap (K+x_1) \cap \dots \cap (K+x_p) \neq \emptyset\}\,\mathrm{d}x_1 \dots \mathrm{d}x_p.$$

In [29], the difference body inequality was generalized to

$$\frac{V_{pn}(\mathrm{D}_p K)}{V(K)^p} \leq \binom{pn+n}{n},$$

with equality if and only if K is a simplex. (A side remark: The other extremum is unknown for $n > 2$ and $p > 1$. For $p = 2$ and $n = 3$, the functional attains different values on the class of centrally symmetric bodies.)

The multiple integral defining $V_{pn}(\mathrm{D}_p K)$ can be generalized and leads to a polynomial expansion analogous to the expansion leading to mixed volumes. For $K_1,\dots,K_k \in \mathcal{K}^n$ and $\lambda_1,\dots,\lambda_k \geq 0$, we have

$$\int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \mathbf{1}\{\lambda_1 K_1 \cap (\lambda_2 K_2 + x_2) \cap \dots \cap (\lambda_k K_k + x_k) \neq \emptyset\}\,\mathrm{d}x_2 \cdots \mathrm{d}x_k$$
$$= \sum_{\substack{m_1,\dots,m_k=0 \\ m_1+\dots+m_k=(k-1)n}}^{n} \lambda_1^{m_1} \cdots \lambda_k^{m_k} V^{(0)}_{m_1,\dots,m_k}(K_1,\dots,K_k).$$

This defines functionals $V^{(0)}_{m_1,\dots,m_k}$, with properties similar to those of mixed volumes: $V^{(0)}_{m_1,\dots,m_k}$ is continuous, and as a function of its ith argument, it is translation invariant, homogeneous of degree m_i, and a valuation (see [30] for a general approach and more information). The functional

$$\frac{V^{(0)}_{m_1,\dots,m_k}(K,\dots,K)}{V(K)^{k-1}}$$

is affine invariant, so it attains a minimum and a maximum, but these are generally unknown. To indicate the difficulty of the problem, we mention that it includes two longstanding unsolved problems. We have

$$V^{(0)}_{i,n-i}(K,K) = \binom{n}{i} V(K[i], -K[n-i]),$$

so that the question for the maximum in this case is the Godbersen problem mentioned earlier. If $K_1,\dots,K_n$ are centrally symmetric, then

$$V^{(0)}_{n-1,\dots,n-1}(K_1,\dots,K_n) = \frac{n!}{2^n} V(\Pi K_1,\dots,\Pi K_n),$$

where V is the mixed volume and Π denotes the projection body operator. In particular, the question for the minimum of

$$\frac{V^{(0)}_{n-1,\dots,n-1}(K,\dots,K)}{V(K)^{n-1}} = \frac{n!}{2^n}\frac{V(\Pi K)}{V(K)^{n-1}}$$

is nothing but Petty's unsolved problem on the volume of projection bodies.

Coming back to Janson's functional $\beta(K)$ appearing in the covering problem, we can now state that it is given by

$$\beta(K) = V^{(0)}_{n-1,\dots,n-1}(K,\dots,K).$$

From the polynomial expansion above, one can deduce that, for $k \in \mathbb{N}$,

$$\frac{V_{(k-1)n}(\mathrm{D}_{k-1}K)}{V(K)^{k-1}} = \frac{k^n}{n!}\frac{\beta(K)}{V(K)^{n-1}} + O(k^{n-1})$$

(where $O(k^{n-1})$ refers to $k \to \infty$). The generalized difference body inequality gives

$$\frac{V_{(k-1)n}(\mathrm{D}_{k-1}K)}{V(K)^{k-1}} \le \binom{kn}{n} = \frac{(kn)^n}{n!} + O(k^{n-1}).$$

Letting $k \to \infty$, we deduce that

$$\frac{\beta(K)}{V(K)^{n-1}} \le n^n.$$

Here equality holds for simplices. It follows that in Janson's covering problem, among convex bodies of the same volume, simplices have the worst covering property. It is unknown whether simplices are the only extremal bodies.

The preceding two examples were connected to variants of the difference body inequality. We mention a third variant, giving another characterization of simplices. Recall that the volume of the difference body of the convex body K can be defined by

$$\int_{\mathbb{R}^n} \mathbf{1}\{K \cap (K+x) \neq \emptyset\}\,\mathrm{d}x = V(\mathrm{D}K).$$

In analogy to this, we can define a convex body $\mathrm{M}K$ by

$$\int_{\mathbb{R}^n} K \cap (K+x)\,\mathrm{d}x =: \mathrm{M}K.$$

(The set valued integral can be defined via support functions.) Then, with a suitable translation vector t, the inclusion

$$V(\mathrm{D}K)K \subseteq (n+1)\mathrm{M}K + t$$

holds. Here equality holds if and only if K is a simplex. A more general version of this inequality was proved in [31].

We continue with our general theme, affine inequalities, with special regard to simplices as either established or conjectured extremal bodies. References not given here are found in the excellent Handbook article by Lutwak [22]. Also the book of Leichtweiß [21] should be consulted.

Already a 'classic' is the two-sided inequality

$$\frac{1}{n^n}\binom{2n}{n} \le V(\Pi^* K)V(K)^{n-1} \le \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n$$

(κ_n = volume of the n-dimensional Euclidean unit ball) for the volume of the polar projection body $\Pi^* K$ of a convex body K. Equality on the right side, which is due to Petty, characterizes ellipsoids, and equality on the left side, due to Zhang, characterizes simplices.

Gardner and Zhang [14] have introduced a remarkable array of convex bodies connecting the difference body with the polar projection body. Let $K \in \mathcal{K}^n$ and $p > -1$, $p \neq 0$. The radial pth mean body $R_p K$ of K is defined by

$$\rho_{R_p K}(u) := \left(\frac{1}{V(K)}\int_K \rho_K(x,u)^p \,\mathrm{d}x\right)^{1/p},$$

where $\rho_L(x,\cdot)$ is the radial function of L with respect to x. With the constant

$$c_{n,p} := \left(n\int_0^1 (1-t)^p t^{n-1}\,\mathrm{d}t\right)^{-1/p},$$

Gardner and Zhang proved that, for $-1 < p < q$,

$$\mathrm{D}K \subseteq c_{n,q} R_q K \subseteq c_{n,p} R_p K \subseteq nV(K)\Pi^* K.$$

In each inclusion, equality holds if and only if K is a simplex. Consequently,

$$\frac{V(\mathrm{D}K)}{V(K)} \le c_{n,q}^n \frac{V(R_q K)}{V(K)} \le c_{n,p}^n \frac{V(R_p K)}{V(K)} \le n^n V(\Pi^* K)V(K)^{n-1}.$$

In each inequality, equality holds if and only if K is a simplex. Since

$$\frac{V(R_n K)}{V(K)} = 1,$$

these inequalities include the difference body inequality and the Zhang projection inequality.

A new development to be reported is that Böröczky [8], in his work on the difference body inequality, also has stability results for these later inequalities. So he proves that, with $\epsilon > 0$,

$$V(\Pi^* K)V(K)^{n-1} \ge (1+\epsilon)\frac{1}{n^n}\binom{2n}{n}$$

implies

$$d_{BM}(K,T^n) \le 1 + n^{88n}\epsilon^{1/n},$$

and if $-1 \le p < q$, then

$$V(c_{n,p}R_pK) \ge (1+\epsilon)V(c_{n,q}R_qK)$$

implies

$$d_{BM}(K,T^n) \le 1 + c\epsilon^{1/n}$$

with a positive constant c depending on p,q,n.

Maximal ellipsoids

Perhaps the most interesting extremal properties of simplices, found in the last 15 years, are related to inscribed or circumscribed ellipsoids of extremal volume. For $K \in \mathcal{K}^n$, one denotes by $\mathcal{E}_J(K)$ the ellipsoid of maximal volume contained in K. By a result of John, the concentric homothetic ellipsoid $n(\mathcal{E}_J(K)-c)+c$, where c denotes the centre of $\mathcal{E}_J(K)$, contains the body K. If K is a simplex, then the factor n cannot be decreased, but the simplex is not characterized by this extremal property. This changes if shifts are allowed. John's result implies that

$$d_{BM}(K,B^n) \le n \tag{9}$$

for $K \in \mathcal{K}^n$. Here, equality holds if and only if K is a simplex. This was proved by Leichtweiß [20] and was rediscovered by Palmon [27]. As soon as one has uniqueness, the question for a stability improvement of the inequality can be raised. For the inequality (9), such a stability result seems to be unknown.

The number $d_{BM}(K,B^n)$ can be interpreted as the result of a general procedure to obtain affine invariants. Let $r(K)$, $R(K)$ denote the (Euclidean) inradius and circumradius, respectively, of K. Then

$$d_{BM}(K,B^n) = \inf_{\alpha \in \mathrm{Aff}(n)} \frac{R(\alpha K)}{r(\alpha K)}.$$

It often happens that a continuous similarity invariant function of convex bodies attains one extremum (as here, where $R/r \ge 1$ trivially), but not the other, since the function is unbounded. By taking the infimum or the supremum over all affine transforms (also called 'positions') of a convex body, one obtains an affine invariant functional which may be bounded and hence attains both extrema. This procedure, which leads to interesting new extremal problems, was pioneered by Behrend [9]. He already gave neat proofs for the reverse isoperimetric inequality in two dimensions, with equality characterizations both in the general and the centrally symmetric case. Generally, the affine-invariant isoperimetric quotient is defined by

$$I(K) := \inf_{\alpha \in \mathrm{Aff}(n)} \frac{S(\alpha K)^n}{V(\alpha K)^{n-1}},$$

where S denotes the surface area. It was shown by Ball [1] that $I(K)$ attains its maximum when K is a simplex. He first proved that the *volume ratio*

$$\mathrm{vr}(K) := \left(\frac{V(K)}{V(\mathcal{E}_J(K))}\right)^{1/n}$$

attains its maximum for simplices. The reverse isoperimetric inequality is then easily obtained. The convex body K is said to be *in John position* if its John ellipsoid $\mathcal{E}_J(K)$ is the unit ball B^n. Every convex body has an affine transform that is in John position. Let Δ^n denote a regular simplex circumscribed to B^n. Now let K be in John position. Then Ball's volume ratio result states that

$$V(K) \leq V(\Delta^n). \tag{10}$$

We have $S(K) \leq nV(K)$, since $B^n \subseteq K$, and $S(\Delta^n) = nV(\Delta^n)$. Together with (10), this gives

$$I(K) \leq \frac{S(K)^n}{V(K)^{n-1}} \leq n^n V(K) \leq n^n V(\Delta^n) = \frac{S(\Delta^n)^n}{V(\Delta^n)^{n-1}} = I(\Delta^n),$$

where the last equality is classical. If equality holds in this reverse isoperimetric inequality, then it holds in (10), hence K has maximal volume ratio. In that case, K must be a simplex. A proof of this fact has only become possible when Barthe [3, 4] found a new proof of the Brascamp–Lieb inequality (and its reverse).

If K is in John position, then (10) also gives

$$S(K) \leq S(\Delta^n).$$

Barthe posed the question whether corresponding results hold also for the other intrinsic volumes, and he proved that the mean width W indeed satisfies

$$W(K) \leq W(\Delta^n),$$

with equality if and only if K is a rotation image of the simplex Δ^n (see [4, 7], and also the summary [6]).

Similar extremal properties of simplices exist in relation to the Löwner ellipsoid $\mathcal{E}_L(K)$, the ellipsoid of smallest volume containing K. The *exterior volume ratio*

$$\mathrm{evr}(K) = \left(\frac{V(K)}{V(\mathcal{E}_L(K))}\right)^{1/n}$$

attains its minimum on the simplices (see Barthe [4, 6]). If $\mathcal{E}_L(K) = B^n$, then the mean width of K is not smaller than the mean width of a regular simplex inscribed to B^n (Schmuckenschläger [28]).

The preceding characterizations are beautiful and deep. Since already the uniqueness proofs are delicate, there is perhaps little hope for improving them in the form of stability estimates.

In the following, we will sketch the proof of a stability result for a consequence of the inequality (9). Let

$$\mathrm{vq}(K) := \left(\frac{V(\mathcal{E}_L(K))}{V(\mathcal{E}_J(K))}\right)^{1/n},$$

(vq stands for 'volume quotient'). It follows from (9) that

$$\mathrm{vq}(K) \leq n, \tag{11}$$

with equality precisely for simplices. In joint work with Daniel Hug, we obtained the following stability version.

Theorem 2. *There exist constants $c(n), \epsilon_0(n) > 0$ depending only on the dimension n such that the following holds. If $0 \le \epsilon \le \epsilon_0(n)$ and*

$$\mathrm{vq}(K) \ge (1-\epsilon)n,$$

then

$$d_{BM}(K, T^n) \le 1 + c_0(n)\epsilon^{1/4}.$$

A rough estimate for $c_0(n)$ shows that that it can be assumed to be of order $n^{13/2}$.

The sketch of the proof is not reproduced here. The proof will be published elsewhere.

Finally, we recall some affine inequalities where the extremal property of simplices has been established in the two-dimensional case, but seems difficult, or even doubtful, in higher dimensions.

The Blaschke-Santaló inequality

Let $K \in \mathcal{K}^n$, and let K^* denote the polar body of K with respect to the Santaló point, which is the point with respect to which the volume of the polar body becomes minimal. Then

$$? \le V(K)V(K^*) \le \kappa_n^2.$$

On the right-hand side, equality holds if and only if K is an ellipsoid. It has been conjectured that simplices give the minimal value, that is,

$$\frac{(n+1)^{n+1}}{(n!)^2} \le V(K)V(K^*).$$

For $n = 2$, this was proved by Mahler [24]. A new proof was given by Meyer [25], who also showed that only the triangles are extremal. Recently Meyer and Reisner have proved that the conjecture holds for n-polytopes with at most $n + 3$ vertices.

The L^p-Busemann-Petty centroid inequality

For $K \in \mathcal{K}^n$ and $p \ge 1$, the L^p-centroid body of K is the convex body $\Gamma_p K$ with support function

$$h_{\Gamma_p K}(u) := \left(\frac{1}{a_{n,p} V(K)} \int_K |\langle u, x\rangle|^p \mathrm{d}x \right)^{1/p}, \quad u \in \mathbb{R}^n,$$

where

$$a_{n,p} := \frac{\kappa_{n+p}}{\kappa_2 \kappa_n \kappa_{p-1}}.$$

Up to normalizing factors, $\Gamma_1 K$ is the ordinary centroid body of K, and $\Gamma_2 K$ is the Legendre ellipsoid of K. The inequality

$$\frac{V(\Gamma_p K)}{V(K)} \geq 1,$$

with equality precisely if K is a centered ellipsoid, was proved by Lutwak, Yang and Zhang [23], and with a different proof by Campi and Gronchi [11]. The cases $p = 1$ (Busemann–Petty) and $p = 2$ (Blaschke, John) are older.

The function $V(\Gamma_p K)/V(K)$ is invariant under linear transformations, but not under translations. The modified functional defined by $C_p(K) := V(\Gamma_p(K - c_K))/V(K)$, where c_K is the centroid of K, is affine invariant; hence, the question for its maximum becomes meaningful. Campi and Gronchi [11] have proved that for $n = 2$ this maximum is attained precisely by the triangles.

Random polytopes

For $K \in \mathcal{K}^n$ and $m \geq n + 1$, define

$$M(K, m) := \frac{1}{V(K)^m} \int_K \cdots \int_K V(\mathrm{conv}\{x_1, \ldots, x_m\})\, \mathrm{d}x_1 \cdots \mathrm{d}x_m.$$

Thus, $M(K, m)$ is the expected volume of the convex hull of m independent, uniformly distributed random points in K. We have

$$\frac{M(B^n, m)}{V(B^n)} \leq \frac{M(K, m)}{V(K)} \leq \ ?,$$

with equality on the left if and only if K is an ellipsoid. In special cases, this goes back to Blaschke; in an even more general version, for higher moments of the volume, it was proved by Groemer. A major open problem is the maximum on the right-hand side, in particular for $m = n + 1$, the expected volume of a random simplex in K. For $n = 2$ and $m = 3$, it was proved by Blaschke that the maximum is attained precisely by the triangles. That in the plane the triangles are also extremal for $m > 3$, was proved by Dalla and Larman [13], and that only triangles are extremal was shown by Giannopoulos [15]. In $\mathbb{R}^n$ for $n > 2$, only the following hints to the conjectured extremal property of the simplices are known. Dalla and Larman have shown that among polytopes with at most $n + 2$ vertices, precisely the simplices yield the maximum. This was extended to higher moments by Campi, Colesanti and Gronchi [10]. These authors have also obtained some restrictions for the possible maximizers. Bárány and Buchta [2] define $E(K, m) := V(K) - M(K, m)$ for bodies K with $V(K) = 1$ and prove that

$$\liminf_{m \to \infty} \frac{E(K, m)}{E(T^n, m)} \geq 1 + \frac{1}{n + 1}$$

unless K is a simplex.

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