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Local tensor valuations on convex polytopes

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Abstract Local versions of the Minkowski tensors of convex bodies in n-dimensional Euclidean space are introduced. An extension of Hadwiger's characterization theorem for the intrinsic volumes, due to Alesker, states that the continuous, isometry covariant valuations on the space of convex bodies with values in the vector space of symmetric p-tensors are linear combinations of modified Minkowski tensors. We ask for a local analogue of this characterization, and we prove a classification result for local tensor valuations on polytopes, without a continuity assumption.

Keywords Tensor valuation · Minkowski tensor · convex polytope · isometry covariance · characterization theorem

Mathematics Subject Classification (2000) MSC 52A20

1 Introduction

The Minkowski tensors are the natural tensor-valued generalizations of the classical quermassintegrals or intrinsic volumes, which are central functionals in the Brunn–Minkowski theory of convex bodies. Let $\mathcal{K}^n$ denote the space of convex bodies (nonempty, compact, convex subsets) in n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$), equipped with the Hausdorff metric. For $p \in \mathbb{N}_0$, let $\mathbb{T}^p$ denote the vector space of symmetric tensors of rank p on $\mathbb{R}^n$. If a, b are symmetric tensors on $\mathbb{R}^n$ and $\odot$ denotes the symmetric tensor product, we use the abbreviations

$$a \odot b =: ab, \qquad \underbrace{a \odot \cdots \odot a}_{r} =: a^r.$$

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For $K \in \mathcal{K}^n$, the Minkowski tensors are defined by

$$\Psi_r(K) = \Phi_n^{r,0}(K) := \frac{1}{r!} \int_K x^r \mathcal{H}^n(\mathrm{d}x)$$

and

$$\Phi_k^{r,s}(K) := \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_\Sigma x^r u^s \Lambda_k(K, \mathrm{d}(x,u)),$$

for $r, s, k \in \mathbb{N}_0$ with $k \leq n-1$. Here, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure, $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$, and $\Lambda_k(K,\cdot)$ is a Borel measure on the space $\Sigma = \mathbb{R}^n \times \mathbb{S}^{n-1}$, the kth support measure of K (more explanations are given in Section 2). Thus, $\Psi_r(K) \in \mathbb{T}^r$ and $\Phi_k^{r,s}(K) \in \mathbb{T}^{r+s}$. (The normalizing factors have turned out to be convenient.)

In recent years, Minkowski tensors, for small dimensions and ranks, have found increasing interest in applied sciences; see [4], [5], [17], [18], [19], [20], for example. Also in stochastic geometry, densities of tensor valuations for Boolean models are under investigation. It seems, therefore, to be the right time to initiate a study of local versions of the Minkowski tensors.

The Minkowski tensors are valuations on $\mathcal{K}^n$. (A function φ from $\mathcal{K}^n$ to an abelian group is a valuation if $\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$ for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$.) Their special cases $\Phi_k^{0,0} = V_k$, $k = 0, \dots, n$, the intrinsic volumes, are the prototypes and most important examples of valuations on convex bodies. By the famous characterization theorem of Hadwiger, every continuous, isometry invariant real valuation on $\mathcal{K}^n$ is a linear combination, with constant coefficients, of the intrinsic volumes. (The nice and shorter proof of this theorem given by Klain [10] is reproduced in [11, Th. 9.1.1] and [16, Th. 14.4.6].) This characterization theorem has been extended to Minkowski tensors. For tensor functions on $\mathcal{K}^n$, the notion of isometry invariance has a natural extension to isometry covariance (which is defined in Section 2). The Minkowski tensors have this property, but also the constant metric tensor Q. Therefore, the Minkowski tensors must be slightly modified, and the proper extension of Hadwiger's characterization theorem then reads as follows.

Alesker's Characterization Theorem *Every continuous, isometry covariant valuation on $\mathcal{K}^n$ with values in $\mathbb{T}^p$ is a linear combination, with constant coefficients, of the tensor valuations $Q^m \Phi_k^{r,s}$, with $m, r, s, k \in \mathbb{N}_0$ satisfying $2m + r + s = p$.*

This was proved by Alesker [1], [3], based on his work in [2]. The dimensions of the vector spaces of tensor valuations addressed in this theorem were determined in [8]; this was cumbersome due to the existence of linear relations between the Minkowski tensors, which had been discovered by McMullen [12]. The system of integral-geometric formulae of Crofton and kinematic type, which is well known for the intrinsic volumes, has been extended to Minkowski tensors in [9].

The intrinsic volumes have local versions, the support measures (or generalized curvature measures), which are Borel measures on $\Sigma = \mathbb{R}^n \times \mathbb{S}^{n-1}$, and their marginal measures, namely the curvature measures on $\mathbb{R}^n$ and the surface area measures on $\mathbb{S}^{n-1}$ (see [15, Sec. 4.2]). All these are measure valued valuations on $\mathcal{K}^n$, with certain isometry covariance and continuity properties. Corresponding characterization theorems, in analogy to Hadwiger's theorem, were proved for the surface area measures in [13], for the curvature measures in [14], and for the support measures in [6]; see also [21]. In the case of the support measures, Glasauer [6] noted that the assumption that they are measures on sets of support elements is strong enough to allow a characterization theorem without assuming the valuation property; this property turns out to be a consequence.

In analogy to the support measures, the natural local versions of the Minkowski tensors are defined by

$$\psi_r(K,A) = \phi_n^{r,0}(K,A) := \frac{1}{r!}\int_{K\cap A} x^r \mathcal{H}^n(\mathrm{d}x) \tag{1}$$

for $A \in \mathcal{B}(\mathbb{R}^n)$ and

$$\phi_k^{r,s}(K,\eta) := \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_\eta x^r u^s \Lambda_k(K, \mathrm{d}(x,u)), \tag{2}$$

for $\eta \in \mathcal{B}(\Sigma)$ and $r,s,k \in \mathbb{N}_0$ with $k \le n-1$. Here, $\mathcal{B}(T)$ denotes the σ-algebra of Borel sets of a topological space T. For each convex body $K \in \mathcal{K}^n$, the function $\phi_n^{r,0}(K,\cdot)$ is a $\mathbb{T}^r$-valued measure on $\mathcal{B}(\mathbb{R}^n)$, and $\phi_k^{r,s}(K,\cdot)$ with $k \le n-1$ is a $\mathbb{T}^{r+s}$-valued measure on $\mathcal{B}(\Sigma)$, which is concentrated on $\operatorname{Nor} K$, the generalized normal bundle of K. Each function $\phi_k^{r,s}$ is a mapping from $\mathcal{K}^n \times \mathcal{B}(\Sigma)$ to $\mathbb{T}^{r+s}$ which is isometry covariant (see Definition 1), locally defined (see Definition 2), and weakly continuous (see Definition 3). The same properties are shared by the mappings $Q^m\phi_k^{r,s}$ (with values in $\mathbb{T}^{2m+r+s}$), where $m \in \mathbb{N}_0$. In analogy to Alesker's theorem quoted above (and excluding the tensor valuations (1)), the following question may be asked. Let $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^p$ be a mapping such that $\Gamma(K,\cdot)$ is a $\mathbb{T}^p$-valued measure, for each $K \in \mathcal{K}^n$, and that Γ is isometry covariant, locally defined, and weakly continuous; must Γ be a linear combination of the measure-valued tensor valuations $Q^m\phi_k^{r,s}$ with $k \le n-1$ and $2m+r+s = p$? This question is open.

In the following, we consider a variant of this question. Let $\mathcal{P}^n \subset \mathcal{K}^n$ denote the set of convex polytopes. We study mappings $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^p$ with the property that $\Gamma(P,\cdot)$ is, for each $P \in \mathcal{P}^n$, a $\mathbb{T}^p$-valued measure concentrated on $\operatorname{Nor} P$ and such that Γ is isometry covariant and locally defined. Thus, we assume neither any continuity nor the valuation property. The mappings with these properties can completely be classified and turn out to be valuations. In analogy to Alesker's theorem, they are linear combinations of modified versions of the valuations $\phi_k^{r,s}$, but the modifications involving the metric tensor are now more subtle.

To explain the modification, let L be a linear subspace of $\mathbb{R}^n$. By $\pi_L : \mathbb{R}^n \to L$ we denote the orthogonal projection, and by Q_L the metric tensor on L. For a tensor T on L, its pull-back to a tensor on $\mathbb{R}^n$, with respect to the orthogonal projection π_L, is denoted by $\pi_L^* T$ (see Section 2). For $m \in \mathbb{N}_0$ and $l \in \{0, \dots, m\}$ we define

$$Q_{m,l}(L) := (\pi_L^* Q_L^l)(\pi_{L^\perp}^* Q_{L^\perp}^{m-l}), \tag{3}$$

thus $Q_{m,l}(L) \in \mathbb{T}^{2m}$. This tensor satisfies $Q_{m,l}(\vartheta L) = \vartheta Q_{m,l}(L)$ for $\vartheta \in \mathrm{O}(n)$.

For a polytope $P \in \mathcal{P}^n$, the support measures have simple explicit expressions. From (2), [15, p. 201] and (4) below it follows that

$$\phi_k^{r,s}(P, \eta) = C_{n,k}^{r,s} \sum_{F \in \mathcal{F}_k(P)} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x, u) x^r u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \, \mathcal{H}^k(\mathrm{d}x)$$

with

$$C_{n,k}^{r,s} = \frac{1}{r!s!} \frac{\binom{n}{k}}{\omega_{n-k+s}}.$$

Here $\mathcal{F}_k(P)$ denotes the set of k-dimensional faces of the polytope P, and $\nu(P, F)$ is the set of outer unit normal vectors of the polytope P at its face F. As usual, $\mathbf{1}_\eta$ denotes the characteristic function of the set η.

For a face F of P, we denote by $D(F) := \operatorname{aff} F - \operatorname{aff} F$ its direction space, that is, the linear subspace parallel to the affine hull of F. The necessary modification of the local Minkowski tensors (2) is now defined by

$$\begin{aligned} &\phi_{k,m,l}^{r,s}(P, \eta) \\ &:= C_{n,k}^{r,s} \sum_{F \in \mathcal{F}_k(P)} Q_{m,l}(D(F)) \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x, u) x^r u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \, \mathcal{H}^k(\mathrm{d}x), \end{aligned}$$

where $m \in \mathbb{N}_0$ and $l \in \{0, \dots, m\}$. Thus, $\phi_{k,m,l}^{r,s}(P, \eta) \in \mathbb{T}^{2m+r+s}$.

Now we can formulate our main result.

Theorem 1 *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^p$ be a mapping with the following properties:*

(a) *For each $P \in \mathcal{P}^n$, $\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure, concentrated on* $\operatorname{Nor} P$.

(b) *Γ is isometry covariant.*

(c) *Γ is locally defined.*

Then Γ is a linear combination, with constant coefficients, of the mappings $\phi_{k,m,l}^{r,s}$, with $m, l, r, s, k \in \mathbb{N}_0$ satisfying $0 \le l \le m$, $0 \le k \le n-1$ and $2m + r + s = p$.

Conversely, it is easy to check that each mapping $\phi_{k,m,l}^{r,s}$ has the properties (a), (b), (c) of the theorem.

The theorem will be proved in Section 4. The next section collects some explanations and preparations, and Section 3 provides four lemmas on which the proof of the theorem is based.

2 Preliminaries

The standard scalar product of $\mathbb{R}^n$ is denoted by $\langle\cdot,\cdot\rangle$ and its induced norm by $\|\cdot\|$. The unit ball of $\mathbb{R}^n$ is denoted by B^n, and the unit sphere by $\mathbb{S}^{n-1}$. The product $\Sigma := \mathbb{R}^n \times \mathbb{S}^{n-1}$ is equipped with the product topology. We denote the orthogonal group of $\mathbb{R}^n$ by $\mathrm{O}(n)$ and call its elements *rotations*, even if they are not orientation preserving. The scalar product is used to identify $\mathbb{R}^n$ with its dual space. Thus, each vector $x \in \mathbb{R}^n$ is identified with a linear functional, by $x(y) := \langle x, y\rangle$ for $y \in \mathbb{R}^n$. By $\mathbb{T}^p$ we denote the vector space of symmetric tensors of rank $p \in \mathbb{N}_0$ on $\mathbb{R}^n$, where $\mathbb{T}^0 = \mathbb{R}$ and $\mathbb{T}^1$ is identified with $\mathbb{R}^n$. We view $\mathbb{T}^p$ as the vector space of symmetric p-linear functionals on $\mathbb{R}^n$. For $A \in \mathbb{T}^r$ and $B \in \mathbb{T}^s$, the symmetric tensor product of A and B is written as AB. This is the tensor in $\mathbb{T}^{r+s}$ given by

$$\begin{aligned}&(AB)(x_1,\dots,x_{r+s})\\ &= \frac{1}{(r+s)!}\sum_{\sigma\in\mathcal{S}_{r+s}} A(x_{\sigma(1)},\dots,x_{\sigma(r)})B(x_{\sigma(r+1)},\dots,x_{\sigma(r+s)})\end{aligned}$$

for $x_1,\dots,x_{r+s} \in \mathbb{R}^n$; here $\mathcal{S}_m$ is the group of permutations of the numbers $1,\dots,m$. For $x \in \mathbb{R}^n$ we write $x^r := x\cdots x$ (r factors) if $r \geq 1$, and we put $x^0 := 1$. We have

$$x^r(y_1,\dots,y_r) = \langle x, y_1\rangle\cdots\langle x, y_r\rangle \qquad \text{for } y_1,\dots,y_r \in \mathbb{R}^n.$$

Let $(e_1,\dots,e_n)$ be an orthonormal basis of $\mathbb{R}^n$, and for $T \in \mathbb{T}^r$ let

$$t_{i_1\dots i_r} := T(e_{i_1},\dots,e_{i_r})$$

for $i_1,\dots,i_r \in \{1,\dots,n\}$. A basis of $\mathbb{T}^r$ is given by the symmetric tensors $e_{i_1}\cdots e_{i_r}$ with $1 \leq i_1 \leq \cdots \leq i_r \leq n$, and the corresponding coordinate representation of T is

$$T = \sum_{1\leq i_1\leq\cdots\leq i_r\leq n} t_{i_1\dots i_r} e_{i_1}\cdots e_{i_r}.$$

On $\mathbb{T}^r$ ($r \geq 2$), we use the scalar product given by

$$\langle e_{i_1}\cdots e_{i_r}, e_{j_1}\cdots e_{j_r}\rangle = \prod_{k=1}^{r}\langle e_{i_k}, e_{j_k}\rangle$$

and bilinear extension, or, in coordinates, by

$$\langle S, T\rangle := \sum_{1\leq i_1\leq\cdots\leq i_r\leq n} s_{i_1\dots i_r} t_{i_1\dots i_r}.$$

This scalar product is invariant under the standard operation of the orthogonal group $\mathrm{O}(n)$ on $\mathbb{T}^r$, which is given by

$$(\vartheta T)(x_1,\dots,x_r) = T(\vartheta^{-1}x_1,\dots,\vartheta^{-1}x_r), \qquad x_1,\dots,x_r \in \mathbb{R}^n,$$

for $\vartheta \in \mathrm{O}(n)$. The norm induced by the scalar product on $\mathbb{T}^r$ is also denoted by $\|\cdot\|$.

If $L \subset \mathbb{R}^n$ is a linear subspace, we denote by $\mathbb{T}^r(L)$ the vector space of symmetric tensors of rank r on L and by $\pi_L : \mathbb{R}^n \to L$ the orthogonal projection to L. For a tensor $T \in \mathbb{T}^r(L)$, the tensor $\pi_L^* T \in \mathbb{T}^r$ is defined by

$$(\pi_L^* T)(x_1, \dots, x_r) := T(\pi_L x_1, \dots, \pi_L x_r), \qquad x_1, \dots, x_r \in \mathbb{R}^n.$$

The metric tensor on $\mathbb{R}^n$ can be defined by $Q = e_1^2 + \cdots + e_n^2$, thus $Q(a,b) = \langle a, b\rangle$ for $a, b \in \mathbb{R}^n$, which shows the independence of the basis. The metric tensor satisfies $\vartheta Q = Q$ for all $\vartheta \in \mathrm{O}(n)$. If L is a linear subspace of $\mathbb{R}^n$, the metric tensor on L is denoted by Q_L.

Let μ be a (countably additive) vector measure on some measurable space $(\Omega, \mathcal{A})$ with values in $\mathbb{T}^r$. The *variation measure* of μ is the measure $|\mu|$ defined by

$$|\mu|(A) := \sup \left\{ \sum_{B \in \mathcal{Z}} \|\mu(B)\| : \mathcal{Z} \text{ is a finite partition of } A \text{ in } \mathcal{A} \right\}$$

for $A \in \mathcal{A}$. Each coordinate of μ defines a real-valued signed measure. Its variation measure is finite, and it follows that also the variation measure of μ is a finite measure.

Let $K \in \mathcal{K}^n$ be a convex body. A *support element* of K is a pair (x,u) consisting of a boundary point x of K and an outer unit normal vector u of K at x. The set of all support elements of K is denoted by $\mathrm{Nor}\, K$ and is called the *generalized normal bundle* of K. It is a closed subset of the space $\Sigma = \mathbb{R}^n \times \mathbb{S}^{n-1}$. For $x \in \mathbb{R}^n \setminus K$, let $p(K,x)$ denote the unique point in K nearest to x, and put $u(K,x) := (x - p(K,x))/\|x - p(K,x)\|$. Then $(p(K,x), u(K,x)) \in \mathrm{Nor}\, K$. For $\epsilon > 0$ and $\eta \in \mathcal{B}(\Sigma)$, the n-dimensional Hausdorff measure of the local parallel set

$$M_\epsilon(K, \eta) := \{x \in (K + \epsilon B^n) \setminus K : (p(K,x), u(K,x)) \in \eta\}$$

is a polynomial in ϵ,

$$\mathcal{H}^n(M_\epsilon(K,\eta)) = \sum_{k=0}^{n-1} \epsilon^{n-k} \kappa_{n-k} \Lambda_k(K, \eta),$$

where κ_n denotes the volume of the unit ball B^n. This defines the support measures (or generalized curvature measures) $\Lambda_0(K,\cdot), \dots, \Lambda_{n-1}(K,\cdot)$ of K. They are a re-normalized version of the measures Θ_i used in [15, Sec. 4.2], namely,

$$n\kappa_{n-k} \Lambda_k(K, \cdot) = \binom{n}{k} \Theta_k(K, \cdot). \tag{4}$$

The measures $\Lambda_k(K,\cdot)$ are concentrated on $\mathrm{Nor}\, K$.

For $\eta \subset \Sigma$, we write $\eta + t := \{(x+t, u) : (x,u) \in \eta\}$ for $t \in \mathbb{R}^n$ and $\vartheta\eta := \{(\vartheta x, \vartheta u) : (x,u) \in \eta\}$ for $\vartheta \in \mathrm{O}(n)$.

If $P : \mathbb{R}^n \to \mathbb{T}^r$ is a function of the form

$$P(t) = \sum_{j=0}^{r} C^{(r-j)} t^j, \qquad t \in \mathbb{R}^n,$$

with fixed tensors $C^{(r-j)} \in \mathbb{T}^{r-j}$, $j = 0, \ldots, r$, then we say that P is an r-*tensor polynomial* in t.

For mappings

$$\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^r$$

we consider the properties described in Definitions 1 to 3.

Definition 1 The mapping Γ is *isometry covariant* if $\Gamma(K + t, \eta + t)$ is an r-tensor polynomial in t, for $t \in \mathbb{R}^n$, and $\Gamma(\vartheta K, \vartheta\eta) = \vartheta\Gamma(K, \eta)$ for $\vartheta \in \mathrm{O}(n)$, $K \in \mathcal{K}^n$ and $\eta \in \mathcal{B}(\Sigma)$.

Definition 2 The mapping Γ is *locally defined* if for $\eta \in \mathcal{B}(\Sigma)$ and $K, K' \in \mathcal{K}^n$ with $\eta \cap \mathrm{Nor}\, K = \eta \cap \mathrm{Nor}\, K'$ the equality $\Gamma(K, \eta) = \Gamma(K', \eta)$ holds.

Similar definitions are used with $\mathcal{K}^n$ replaced by $\mathcal{P}^n$.

Definition 3 The mapping Γ is *weakly continuous* if $\lim_{i\to\infty} K_i = K$ in $\mathcal{K}^n$ (with the Hausdorff metric) implies

$$\lim_{i\to\infty} \int_\Sigma f \,\mathrm{d}\Gamma(K_i, \cdot) = \int_\Sigma f \,\mathrm{d}\Gamma(K, \cdot)$$

for all continuous functions $\Gamma : \Sigma \to \mathbb{R}$ with compact support (the integral is defined coordinate-wise).

Each mapping $(K, \eta) \mapsto \phi_k^{r,s}(K, \eta)$, for $k = 0, \ldots, n-1$, is isometry covariant, locally defined, and weakly continuous. This follows from the corresponding properties of the support measures (see [15, Sec. 4.2]) and the fact that

$$(x + t)^r u^s = \sum_{i=0}^{r} \binom{r}{i} x^{r-i} u^s t^i,$$

since we are dealing with symmetric tensor products.

3 Auxiliary results

First we prove a lemma that allows us to deduce properties of the coefficient tensors of a tensor polynomial from properties of the polynomial itself. We assume that an orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ and a function $F : \mathrm{lin}\{e_n\} \to \mathbb{T}^r$ ($r \in \mathbb{N}$) are given with

$$F(t) = \sum_{j=0}^{r} C^{(r-j)} t^j, \qquad t \in \mathrm{lin}\{e_n\}, \tag{5}$$

where $r \in \mathbb{N}$ and $C^{(r-j)} \in \mathbb{T}^{r-j}$, $j = 0, \dots, r$. (Trivially, $C^{(r)} = F(0)$.) In the following lemma, we denote by $f_{i_1 \dots i_r}(t)$ and $c^{(r-j)}_{i_1 \dots i_{r-j}}$ the coordinates of $F(t)$ and $C^{(r-j)}$, respectively.

The following lemma shows, in particular, that a tensor polynomial determines its coefficient tensors uniquely.

Lemma 1 *Let $j \in \{1, \dots, r\}$ and $1 \le i_1 \le \dots \le i_{r-j} \le n$. Let $q \in \{0, \dots, r-j\}$ be the number of indices $k \in \{1, \dots, r-j\}$ for which $i_k < n$. Then*

$$c^{(r-j)}_{i_1 \dots i_{r-j}} = \frac{r!(r-q-j)!}{(r-q)!(r-j)!} \sum_{m=1}^{r+1} b_{jm} f_{i_1 \dots i_{r-j} n \dots n}(m e_n),$$

where the coefficients b_{jm} depend only on r, j, m.

Proof For $\lambda \in \mathbb{R}$, equation (5) gives

$$F(\lambda t) = \sum_{j=0}^{r} \lambda^j C^{(r-j)} t^j. \tag{6}$$

If we have a real polynomial

$$p(\lambda) = \sum_{j=0}^{r} a_j \lambda^j,$$

we can insert $\lambda = 1, \dots, r+1$ and solve the resulting system of linear equations (which has a Vandermonde determinant, different from zero) for $a_0, \dots, a_r$, to obtain representations

$$a_j = \sum_{m=1}^{r+1} b_{jm} p(m), \qquad j = 0, \dots, r.$$

The coefficients b_{jm} depend only on r, j, m. Applying this to (6) coordinatewise, we get

$$C^{(r-j)} t^j = \sum_{m=1}^{r+1} b_{jm} F(mt) \tag{7}$$

for $t \in \operatorname{lin}\{e_n\}$. We put $t = e_n$ in (7) and apply this tensor to the r-tuple

$$(e_{i_1}, \dots, e_{i_{r-j}}, \underbrace{e_n, \dots, e_n}_{j}) = (e_{i_1}, \dots, e_{i_q}, \underbrace{e_n, \dots, e_n}_{r-q}).$$

We have $i_s = n$ for $s = q+1, \dots, r-j$, and we define in addition $i_s := n$ for $s = r-j+1, \dots, r$. Then

$$\begin{aligned} &(C^{(r-j)} e_n^j)(e_{i_1}, \dots, e_{i_{r-j}}, e_n, \dots, e_n) = (C^{(r-j)} e_n^j)(e_{i_1}, \dots, e_{i_r}) \\ &= \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} C^{(r-j)}(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(r-j)}}) \langle e_n, e_{i_{\sigma(r-j+1)}} \rangle \cdots \langle e_n, e_{i_{\sigma(r)}} \rangle. \end{aligned}$$

The factor $\langle e_n, e_{i_{\sigma(r-j+1)}}\rangle \cdots \langle e_n, e_{i_{\sigma(r)}}\rangle$ is either 0 or 1, and it is 1 precisely if

$$\sigma(r-j+1), \ldots, \sigma(r) \geq q+1,$$

since $e_{i_p} \neq n$ for $p \leq q$. There are precisely $(r-j)!(r-q)!/(r-q-j)!$ such permutations σ. By the symmetry of $C^{(r-j)}$, each of them leads to the summand

$$\begin{aligned} C^{(r-j)}(e_{i_{\sigma(1)}}, \ldots, e_{i_{\sigma(r-j)}}) &= C^{(r-j)}(e_{i_1}, \ldots, e_{i_q}, \underbrace{n, \ldots, n}_{r-j-q}) \\ &= C^{(r-j)}(e_{i_1}, \ldots, e_{i_{r-j}}). \end{aligned}$$

This gives

$$(C^{r-j)} e_n^j)(e_{i_1}, \ldots, e_{i_{r-j}}, e_n, \ldots, e_n) = \frac{(r-q)!(r-j)!}{r!(r-q-j)!} C^{(r-j)}(e_{i_1}, \ldots, e_{i_{r-j}}).$$

Together with (7) this yields

$$c^{(r-j)}_{i_1 \ldots i_{r-j}} = \frac{r!(r-q-j)!}{(r-q)!(r-j)!} \sum_{m=1}^{r+1} b_{jm} f_{i_1 \ldots i_{r-j} n \ldots n}(m e_{i_n})$$

and thus the assertion of the lemma. □

For a linear subspace $L \subset \mathbb{R}^n$, we denote by $\mathcal{B}_b(L)$ the set of bounded Borel sets in L. We say that a mapping $F : \mathcal{B}_b(L) \to \mathbb{T}^r$ is a *local tensor measure* on L if for each compact set $C \subset L$, the restriction of F to $\mathcal{B}(C)$ is a $\mathbb{T}^r$-valued measure.

If $L \subset \mathbb{R}^n$ is a k-dimensional subspace and $r \in \mathbb{N}_0$, we define

$$\Psi_r^L(B) := \frac{1}{r!} \int_B x^r \, \mathcal{H}^k(\mathrm{d}x), \qquad B \in \mathcal{B}_b(L).$$

Then Ψ_r^L is a local tensor measure on L, and we have

$$\Psi_r^L(B+t) = \sum_{j=0}^{r} \Psi_{r-j}^L(B) \frac{1}{j!} t^j, \qquad t \in L, \tag{8}$$

for each $B \in \mathcal{B}_b(L)$.

The following lemma extends the characterization of Lebesgue measure by its translation invariance to tensor-valued measures which have polynomial behaviour under translations.

Lemma 2 *Let $L \subset \mathbb{R}^n$ be a linear subspace and $r \in \mathbb{N}_0$. Let $F_r : \mathcal{B}_b(L) \to \mathbb{T}^r$ be a local tensor measure on L with the property that $F_r(B+t)$ is an r-tensor polynomial in t, $t \in L$, for each $B \in \mathcal{B}_b(L)$. Then*

$$F_r(B) = \sum_{j=0}^{r} a^{(j)} \Psi_{r-j}^L(B), \qquad B \in \mathcal{B}_b(L),$$

with constant tensors $a^{(j)} \in \mathbb{T}^j$, $j = 0, \ldots, r$.

Proof By assumption, for $B \in \mathcal{B}_b(L)$ we have

$$F_r(B+t) = \sum_{j=0}^{r} F_{r-j}(B)\frac{1}{j!}t^j, \qquad t \in L, \tag{9}$$

with tensors $F_{r-j}(B) \in \mathbb{T}^{r-j}$ (the normalizing factor $1/j!$ is convenient). From Lemma 1 we deduce that each F_{r-j} is a local tensor measure. Computing $F_r((B+t)+z) = F(B+(t+z))$ for $B \in \mathcal{B}_b(L)$ and $t, z \in L$ in two different ways and using the fact that a tensor polynomial determines its coefficient tensors uniquely, we obtain

$$F_{r-j}(B+t) = \sum_{m=0}^{r-j} F_{r-j-m}(B)\frac{1}{m!}t^m \tag{10}$$

for $t \in L$ and $j = 0, \dots, r$.

Now we prove the assertion by induction over r, starting with $r = 0$. The function F_0 is real-valued, and restricted to $\mathcal{B}(C)$ for compact $C \subset L$ it is a signed measure. Applying the Jordan decomposition in each such C, we see that there are positive measures μ^+, μ^- on $\mathcal{B}(L)$ such that $F_0(B) = \mu^+(B) - \mu^-(B)$ for each $B \in \mathcal{B}_b(L)$. These measures are locally finite and invariant under translations in L, hence they are multiples of Lebesgue measure on L. It follows that there is a real constant $a^{(0)}$ such that $F_0(B) = a^{(0)}\Psi_0^L(B)$ for $B \in \mathcal{B}_b(L)$. This is the assertion for $r = 0$. Now let $r > 0$ and assume that the assertion is proved for tensor functions of rank less than r. Then the assertion applies to F_{r-1}, and we obtain that

$$F_{r-1}(B) = \sum_{j=0}^{r-1} c^{(j)}\Psi_{r-1-j}^L(B) \qquad \text{for } B \in \mathcal{B}_b(L), \tag{11}$$

with constant tensors $c^{(j)} \in \mathbb{T}^j$, $j = 0, \dots, r-1$. From (11) and (8) we get, for $t \in L$,

$$F_{r-1}(B+t) = \sum_{j=0}^{r-1} c^{(j)}\Psi_{r-1-j}^L(B+t) = \sum_{m=0}^{r-1}\left(\sum_{j=0}^{r-1-m} c^{(j)}\Psi_{r-1-j-m}^L(B)\right)\frac{1}{m!}t^m,$$

hence comparison with (10) (for $j = 1$) yields

$$F_{r-k} = \sum_{j=0}^{r-k} c^{(j)}\Psi_{r-k-j}^L \qquad \text{for } k = 1, \dots, r. \tag{12}$$

Now we define

$$G := F_r - \sum_{j=0}^{r-1} c^{(j)}\Psi_{r-j}^L$$

and obtain from (9), (8) and (12) that

$$G(B+t)=G(B)+\sum_{k=1}^{r}F_{r-k}(B)\frac{1}{k!}t^k-\sum_{k=1}^{r}\left(\sum_{j=0}^{r-k}c^{(j)}\Psi^L_{r-k-j}(B)\right)\frac{1}{k!}t^k=G(B)$$

for $B\in\mathcal{B}_b(L)$. Thus, the local tensor measure G is translation invariant. For each of its coordinates we obtain, by the same argument as used above for F_0, that it is a constant multiple of Lebesgue measure on L. Therefore, $G(B)=c^{(r)}\Psi^L_0(B)$ for $B\in\mathcal{B}_b(L)$, with a fixed tensor $c^{(r)}\in\mathbb{T}^r$, and we obtain the assertion of Lemma 2 for the rank r. This finishes the induction and thus the proof. □

Given an orthonormal basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$, we associate with $T\in\mathbb{T}^r$, represented in coordinates by

$$T=\sum_{1\le i_1\le\cdots\le i_r\le n}t_{i_1\dots i_r}e_{i_1}\cdots e_{i_r},$$

the polynomial on $\mathbb{R}^n$ defined by

$$p_T(y)=\sum_{1\le i_1\le\cdots\le i_r\le n}t_{i_1\dots i_r}y_{i_1}\cdots y_{i_r},\qquad y=\sum_{i=1}^{n}y_ie_i. \tag{13}$$

The mapping $T\mapsto p_T$ is a vector space isomorphism between $\mathbb{T}^r$ and the vector space of homogeneous polynomials of degree r on $\mathbb{R}^n$. It is compatible with the operation of the orthogonal group, that is, it satisfies $p_{\vartheta T}(y)=p_T(\vartheta^{-1}y)$ for $y\in\mathbb{R}^n$ and $\vartheta\in\mathrm{O}(n)$.

Lemma 3 *Let $L\subset\mathbb{R}^n$ be a linear subspace. Let $r\in\mathbb{N}_0$, let $T\in\mathbb{T}^r$ be a tensor satisfying $\vartheta T=T$ for each $\vartheta\in\mathrm{SO}(n)$ that fixes $L^\perp$ pointwise. Then*

$$T=\sum_{j=0}^{\lfloor r/2\rfloor}(\pi_L^*Q_L^j)\pi_{L^\perp}^*T^{(r-2j)}$$

with tensors $T^{(r-2j)}\in\mathbb{T}^{r-2j}(L^\perp)$, $j=0,\dots,\lfloor r/2\rfloor$.

Proof Let $\dim L=k>0$, without loss of generality. We choose the orthonormal basis $(e_1,\dots,e_n)$ in such a way that $e_1,\dots,e_k$ span the subspace L and $e_{k+1},\dots,e_n$ span its orthogonal complement $L^\perp$. Then the polynomial p_T defined by (13) satisfies $p_T(\vartheta^{-1}y)=p_{\vartheta T}(y)=p_T(y)$ for each rotation ϑ fixing $L^\perp$ pointwise. For $\rho>0$ and $\zeta_{k+1},\dots,\zeta_n\in\mathbb{R}$, the group of such rotations is transitive on the set

$$\left\{y=y_1e_1+\cdots+y_ne_n\in\mathbb{R}^n: y_1^2+\cdots+y_k^2=\rho^2,\ y_{k+1}=\zeta_{k+1},\dots,y_n=\zeta_n\right\}.$$

Hence, there is a function g of $n-k+1$ variables such that

$$p_T(y)=g\left(\sqrt{y_1^2+\cdots+y_k^2},\ y_{k+1},\dots,y_n\right).$$

Since this is a polynomial of degree at most r in $y_{k+1}, \dots, y_n$, we have

$$p_T(y) = \sum_{m=0}^{r} \sum_{k+1 \le i_1 \le \dots \le i_{r-m} \le n} q_{i_1 \dots i_{r-m}} \left(\sqrt{y_1^2 + \dots + y_k^2} \right) y_{i_1} \cdots y_{i_{r-m}}$$

with functions $q_{i_1 \dots i_{r-m}}$ of one variable. Each such function is positively homogeneous of degree m, which yields

$$q_{i_1 \dots i_{r-m}} \left(\sqrt{y_1^2 + \dots + y_k^2} \right) = q_{i_1 \dots i_{r-m}}(1)(y_1^2 + \dots + y_k^2)^{m/2}.$$

Since p_T is a polynomial, we have $q_{i_1 \dots i_{r-m}}(1) = 0$ for odd m and thus

$$p_T(y) = \sum_{j=0}^{\lfloor r/2 \rfloor} (y_1^2 + \dots + y_k^2)^j \sum_{k+1 \le i_1 \le \dots \le i_{r-2j} \le n} q_{i_1 \dots i_{r-2j}}(1) y_{i_1} \cdots y_{i_{r-2j}}.$$

We define a tensor $T^{(r-2j)} \in \mathbb{T}^{r-2j}(L^\perp)$ by

$$T^{(r-2j)} := \sum_{k+1 \le i_1 \le \dots \le i_{r-2j} \le n} q_{i_1 \dots i_{r-2j}}(1) e_{i_1} \cdots e_{i_{r-2j}}$$

and note that $e_1^2 + \dots + e_k^2 = Q_L$ is the metric tensor on L. Further we note that $\pi_{L^\perp}^* T^{(r-2j)}$ and $\pi_L^* Q_L$ are tensors on $\mathbb{R}^n$. Since the mapping $T \mapsto p_T$ induces an algebra isomorphism between the algebra of symmetric tensors on $\mathbb{R}^n$ and the polynomial algebra on $\mathbb{R}^n$ (see [7, Ch. VIII, §3] and recall that we have identified $\mathbb{R}^n$ with its dual space, via the scalar product), the assertion of the lemma follows. □

For a $\mathbb{T}^r$-valued Borel measure F on $\mathbb{S}^{n-1}$ we say that it *intertwines rotations* if $F(\vartheta B) = (\vartheta F)(B)$ for all $B \in \mathcal{B}(\mathbb{S}^{n-1})$ and all rotations $\vartheta \in \mathrm{O}(n)$.

Lemma 4 *Let $r \in \mathbb{N}_0$, and let $F : \mathcal{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^r$ be a $\mathbb{T}^r$-valued measure which intertwines rotations. Then*

$$F(B) = \sum_{j=0}^{\lfloor r/2 \rfloor} a_j Q^j \int_B u^{r-2j} \, \mathcal{H}^{n-1}(\mathrm{d}u), \qquad B \in \mathcal{B}(\mathbb{S}^{n-1}),$$

with real constants a_j, $j = 0, \dots, \lfloor r/2 \rfloor$.

Proof First we note that the variation measure $|F|$ is invariant under rotations, as follows immediately from the definition of the variation measure and the rotation invariance of the norm used in its definition. Since $|F|$ is finite, it is a constant multiple of spherical Lebesgue measure σ_{n-1}. Each coordinate of F is a finite signed measure which is absolutely continuous with respect to $|F|$ and hence with respect to σ_{n-1}. By the Radon–Nikodym theorem, applied to the finitely many coordinates, there exists a σ_{n-1} almost everywhere defined measurable mapping $f : \mathbb{S}^{n-1} \to \mathbb{T}^r$ with

$$F(B) = \int_B f \, \mathrm{d}\sigma_{n-1}, \qquad B \in \mathcal{B}(\mathbb{S}^{n-1}).$$

We fix a vector $u \in \mathbb{S}^{n-1}$ and denote by $B_{u,\rho}$ the spherical cap with centre u and spherical radius $\rho \in (0, \pi/2)$. Let $T := F(B_{u,\rho})$. Then $\vartheta T = T$ for all rotations $\vartheta \in \mathrm{O}(n)$ fixing u. Therefore, we can apply Lemma 3 with $L = u^\perp$, where we choose the orthonormal basis $(e_1, \dots, e_n)$ such that $e_n = u$. Since every tensor in $\mathbb{T}^{r-2j}(\mathrm{lin}\{u\})$ is of the form $b_j e_n^{r-2j}$ with a real constant b_j, we obtain that

$$T = \sum_{j=0}^{\lfloor r/2 \rfloor} b_j(u, \rho)(Q - e_n^2)^j e_n^{r-2j}, \tag{14}$$

where we have indicated by the notation that the numbers b_j depend on u and ρ. Noting that $(Q - e_n^2)(e_i, e_j) = 0$ if $i = n$ or $j = n$ and $(Q - e_n^2)(e_i, e_j) = \langle e_i, e_j \rangle$ if $i, j \le n - 1$, we get, for $m \in \{0, \dots, \lfloor r/2 \rfloor\}$,

$$\binom{r}{2m} T(\underbrace{e_1, \dots, e_1}_{2m}, \underbrace{e_n, \dots, e_n}_{r-2m}) = b_m(u, \rho). \tag{15}$$

By the definition of T and the choice of the basis, we see from (14) that

$$F(B_{u,\rho}) = \sum_{j=0}^{\lfloor r/2 \rfloor} b_j(u, \rho)(Q - u^2)^j u^{r-2j}. \tag{16}$$

This is a relation between tensors and does not depend on the choice of a basis; it holds for all $u \in \mathbb{S}^{n-1}$. Since F intertwines rotations, we have $\vartheta F(B_{u,\rho}) = F(\vartheta B_{u,\rho}) = F(B_{\vartheta u,\rho})$ for any $\vartheta \in \mathrm{O}(n)$, and hence

$$\begin{aligned} \vartheta \sum_{j=0}^{\lfloor r/2 \rfloor} b_j(u, \rho)(Q - u^2)^j u^{r-2j} &= \sum_{j=0}^{\lfloor r/2 \rfloor} b_j(\vartheta u, \rho)(Q - (\vartheta u)^2)^j (\vartheta u)^{r-2j} \\ &= \vartheta \sum_{j=0}^{\lfloor r/2 \rfloor} b_j(\vartheta u, \rho)(Q - u^2)^j u^{r-2j}. \end{aligned}$$

The tensors $(Q - u^2)^j u^{r-2j}$, $j = 0, \dots, \lfloor r/2 \rfloor$, are linearly independent, and it follows that $b_j(\vartheta u, \rho) = b_j(u, \rho)$. Thus, $b_j(u, \rho)$ does not depend on u, and we denote it by $b_j(\rho)$.

Since $F(B) = \int_B f \, \mathrm{d}\sigma_{n-1}$, Lebesgue's differentiation theorem (applied coordinate-wise) yields

$$\lim_{\rho \to 0} \frac{F(B_{u,\rho})}{\sigma_{n-1}(B_{u,\rho})} = f(u) \qquad \text{for } \sigma_{n-1} \text{ almost all } u \in \mathbb{S}^{n-1}.$$

At a point u where this limit exists, we can choose $e_n = u$ as above and then deduce from (15) that also the limit

$$\lim_{\rho\to 0} \frac{\binom{r}{2j}}{\sigma_{n-1}(B_{u,\rho})} F(B_{u,\rho})(\underbrace{e_1,\dots,e_1}_{2j},\underbrace{e_n,\dots,e_n}_{r-2j}) = \lim_{\rho\to 0} \frac{b_j(\rho)}{\sigma_{n-1}(B_{u,\rho})} =: b_j$$

exists, for $j = 0,\dots,\lfloor r/2\rfloor$. We conclude that

$$f(u) = \sum_{j=0}^{\lfloor r/2\rfloor} b_j (Q-u^2)^j u^{r-2j} \qquad \text{a.e.}$$

and hence

$$F(B) = \sum_{j=0}^{\lfloor r/2\rfloor} b_j \int_B (Q-u^2)^j u^{r-2j}\,\mathcal{H}^{n-1}(\mathrm{d}u) \tag{17}$$

for $B \in \mathcal{B}(\mathbb{S}^{n-1})$. Expanding $(Q-u^2)^j$ by the binomial theorem, we obtain

$$F(B) = \sum_{k=0}^{\lfloor r/2\rfloor} a_k Q^k \int_B u^{r-2k}\,\mathcal{H}^{n-1}(\mathrm{d}u), \qquad B \in \mathcal{B}(\mathbb{S}^{n-1}),$$

with

$$a_k = \sum_{j=0}^{\lfloor r/2\rfloor} \binom{j}{k} b_j.$$

This completes the proof of Lemma 4. □

4 Proof of Theorem 1

We assume that $p \in \mathbb{N}_0$ and that a mapping $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^p$ with the properties of Theorem 1 is given. Thus, (a) for each $P \in \mathcal{P}^n$, $\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure, concentrated on $\mathrm{Nor}\,P$, (b) Γ is isometry covariant, and (c) Γ is locally defined. For fixed $P \in \mathcal{P}^n$ and $\eta \in \mathcal{B}(\Sigma)$, the assumption (b) implies, in particular, that $\Gamma(P+t,\eta+t)$ is a p-tensor polynomial in t, hence there are tensors $\Gamma^{(p-i)}(P,\eta) \in \mathbb{T}^{p-i}$ such that

$$\Gamma(P+t,\eta+t) = \sum_{i=0}^{p} \Gamma^{(p-i)}(P,\eta)t^i \tag{18}$$

for $t \in \mathbb{R}^n$.

Let $k \in \{0,\dots,n-1\}$, let $L \subset \mathbb{R}^n$ be a k-dimensional linear subspace, and let $\mathbb{S}_{L^\perp} := \mathbb{S}^{n-1} \cap L^\perp$ be the unit sphere in the orthogonal complement of L. We fix a set $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$. Let E be a translate of L and let $A \in \mathcal{B}_b(E)$. We choose a polytope $P \subset E$ with $A \subset \mathrm{relint}\,P$ (where relint denotes the relative interior). Then $A \times B \subset \mathrm{Nor}\,P$, and if we define

$$\varphi(A) := \Gamma(P, A \times B) \tag{19}$$

for $A \in \mathcal{B}_b(E)$, then this does not depend on the choice of P, since Γ is locally defined. For any $t \in \mathbb{R}^n$, (18) gives

$$\varphi(A+t) = \Gamma(P+t, (A+t) \times B) = \sum_{i=0}^{p} \Gamma^{(p-i)}(P, A \times B) t^i.$$

Since the left side does not depend on P and a tensor polynomial determines its coefficient tensors uniquely, the tensor $\Gamma^{(p-i)}(P, A \times B)$ does not depend on P, and we have

$$\varphi(A+t) = \sum_{i=0}^{p} \varphi^{(p-i)}(A \times B) t^i \tag{20}$$

with tensors $\varphi^{(p-i)}(A \times B) \in \mathbb{T}^{p-i}$.

Let φ_L denote the restriction of φ to $\mathcal{B}_b(L)$. Then φ_L is a local tensor measure on L, and $\varphi_L(A+t)$ is a p-tensor polynomial in $t \in L$, for each $A \in B_b(L)$. From Lemma 2 we deduce that

$$\varphi(A) = \sum_{j=0}^{p} a^{(j)} \int_A x^{p-j} \, \mathcal{H}^k(\mathrm{d}x) \qquad \text{for } A \in \mathcal{B}_b(L), \tag{21}$$

with tensors $a^{(j)} \in \mathbb{T}^j$ (depending on L and B).

For $t \in L$, (21) gives

$$\varphi(A+t) = \sum_{m=0}^{p} \left(\sum_{j=0}^{p-m} \binom{p-j}{m} a^{(j)} \int_A x^{p-j-m} \, \mathcal{H}^k(\mathrm{d}x) \right) t^m. \tag{22}$$

If $k \geq 1$, we choose the orthonormal basis $(e_1, \dots, e_n)$ in such a way that $e_n \in L$. Then Lemma 1 can be applied to (20) and (22) and yields that

$$\varphi^{(p-m)}(A \times B) = \sum_{j=0}^{p-m} \binom{p-j}{m} a^{(j)} \int_A x^{p-j-m} \, \mathcal{H}^k(\mathrm{d}x). \tag{23}$$

Now let E be a translate of L, and let $A \in \mathcal{B}_b(E)$. Then $A = A' + t$ with suitable $A' \in \mathcal{B}_b(L)$ and $t \in \mathbb{R}^n$. From (20) and (23) we get

$$\begin{aligned}
\varphi(A'+t) &= \sum_{m=0}^{p} \varphi^{(p-m)}(A \times B) t^m \\
&= \sum_{m=0}^{p} \sum_{j=0}^{p-m} \binom{p-j}{m} a^{(j)} \int_{A'} x^{p-j-m} \, \mathcal{H}^k(\mathrm{d}x) t^m \\
&= \sum_{j=0}^{p} a^{(j)} \int_{A'+t} x^{p-j} \, \mathcal{H}^k(\mathrm{d}x).
\end{aligned}$$

Thus, we have

$$\varphi(A) = \sum_{j=0}^{p} a^{(j)}(L,B) \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) \qquad \text{for } A \in \mathcal{B}_b(E), \tag{24}$$

which extends (21) to $A \in \mathcal{B}_b(E)$ for all translates E of L. The notation shows that $a^{(j)}$ depends on L and B.

If $k = 0$, a representation of the form (24) follows immediately from (21), with $a^{(j)}(\{0\}, B) = \varphi^{(j)}(\{0\} \times B)$.

Our next goal is to investigate how the tensor $a^{(j)}(L,B)$ depends on L and $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$. Let $A \in \mathcal{B}_b(L)$ and choose a polytope $P \subset L$ with $A \subset \operatorname{relint} P$. By (19) and (24),

$$\Gamma(P, A \times B) = \sum_{j=0}^{p} a^{(j)}(L,B) \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x). \tag{25}$$

Here we replace A by λA with $\lambda > 0$ and use the fact that $\int_{\lambda A} x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) = \lambda^{p-j+k} \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x)$. Arguing as in the proof of Lemma 1, we obtain representations

$$a^{(j)}(L,B) \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) = \sum_{m=1}^{p+1} b_{jm} \Gamma(P, mA \times B), \qquad j = 0, \dots, p, \tag{26}$$

with real constants b_{jm} depending only on p, j, m.

Since Γ is isometry covariant, for $\vartheta \in \mathrm{O}(n)$ we have $\Gamma(\vartheta P, \vartheta(A \times B)) = \vartheta \Gamma(P, A \times B)$. From (26) we obtain

$$\begin{aligned} a^{(j)}(\vartheta L, \vartheta B) \int_{\vartheta A} x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) &= \vartheta \left(a^{(j)}(L,B) \int_A x^{p-j}\, \mathcal{H}(\mathrm{d}x) \right) \\ &= \vartheta a^{(j)}(L,B) \int_{\vartheta A} x^{p-j}\, \mathcal{H}(\mathrm{d}x) \end{aligned}$$

for $j = 0, \dots, p$. Since the symmetric tensor algebra has no zero divisors (and the right-hand factor is not zero for suitable A), we deduce that

$$a^{(j)}(\vartheta L, \vartheta B) = \vartheta a^{(j)}(L,B) \qquad \text{for } \vartheta \in \mathrm{O}(n). \tag{27}$$

In a similar way, we obtain from (26) and the properties of Γ that $a^{(j)}(L, \cdot)$ is a $\mathbb{T}^j$-valued measure on $\mathcal{B}(\mathbb{S}_{L^\perp})$.

Let $\mathrm{O}(L) \subset \mathrm{O}(n)$ be the subgroup of rotations that fix $L^\perp$ pointwise (and hence map L into itself). For $\vartheta \in \mathrm{O}(L)$ it follows from (27) that

$$\vartheta a^{(j)}(L,B) = a^{(j)}(\vartheta L, \vartheta B) = a^{(j)}(L,B).$$

Now we infer from Lemma 3 that

$$a^{(j)}(L,B) = \sum_{l=0}^{\lfloor j/2 \rfloor} (\pi_L^* Q_L^l) \pi_{L^\perp}^* b^{(j-2l)}(L,B) \tag{28}$$

with tensors $b^{(j-2l)}(L,B) \in \mathbb{T}^{j-2l}(L^\perp)$.

With $y \in L \cap \mathbb{S}^{n-1}$ and $x_1, \dots, x_j \in L^\perp$, we apply both sides of (28) to $(y, \dots, y, x_1, \dots, x_{j-2q})$, for $q = 0, \dots, \lfloor j/2 \rfloor$ and obtain

$$a^{(j)}(L,B)(\underbrace{y, \dots, y}_{2q}, x_1, \dots, x_{j-2q}) = b^{(j-2q)}(L,B)(x_1, \dots, x_{j-2q}), \tag{29}$$

thus expressing $b^{(j-2q)}(L,B)$ in terms of $a^{(j)}(L,B)$.

For $\vartheta \in \mathrm{O}(L^\perp)$ we get from (27) and (29) that

$$b^{(j-2l)}(L, \vartheta B) = \vartheta b^{(j-2l)}(L,B), \qquad l = 0, \dots, \lfloor j/2 \rfloor. \tag{30}$$

From (29) and the fact that $a^{(j)}(L,\cdot)$ is a $\mathbb{T}^j$-valued measure we deduce that $b^{(j-2l)}(L,\cdot)$ is a $\mathbb{T}^{j-2l}(L^\perp)$-valued measure on $\mathbb{S}_{L^\perp}$. By (30), it intertwines rotations of $L^\perp$. Now it follows from Lemma 4 (applied in $L^\perp$) that

$$b^{(j-2l)}(L,B) = \sum_{i=0}^{\lfloor j/2 \rfloor - l} \alpha_{i,j,l}(L) Q^i_{L^\perp} \int_B u^{j-2l-2i} \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \tag{31}$$

for $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$, with real constants $\alpha_{i,j,l}(L)$.

For arbitrary $\vartheta \in \mathrm{O}(n)$, we use (27) and an argument similar to (29) to obtain

$$b^{(j-2l)}(\vartheta L, \vartheta B) = \vartheta b^{(j-2l)}(L,B) \tag{32}$$

(which are tensors in $\mathbb{T}^{j-2l}(\vartheta L^\perp)$).

Now we fix a k-dimensional linear subspace L_0 and put $\alpha_{i,j,l}(L_0) =: \alpha_{i,j,l}$. For a given k-dimensional subspace L, there is a rotation $\vartheta \in \mathrm{O}(n)$ with $L = \vartheta L_0$. From (32) and (31) we obtain, for $B \in \mathcal{B}(\mathbb{S}_{L^\perp})$ and $B_0 = \vartheta^{-1} B$,

$$\begin{aligned} b^{(j-2l)}(L,B) &= b^{(j-2l)}(\vartheta L_0, \vartheta B_0) = \vartheta b^{(j-2l)}(L_0, B_0) \\ &= \vartheta \sum_{i=0}^{\lfloor j/2 \rfloor - l} \alpha_{i,j,l}(L_0) Q^i_{L_0^\perp} \int_{B_0} u^{j-2l-2i} \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \\ &= \sum_{i=0}^{\lfloor j/2 \rfloor - l} \alpha_{i,j,l} Q^i_{\vartheta L_0^\perp} \int_{\vartheta B_0} u^{j-2l-2i} \, \mathcal{H}^{n-k-1}(\mathrm{d}u), \end{aligned}$$

thus

$$b^{(j-2l)}(L,B) = \sum_{i=0}^{\lfloor j/2 \rfloor - l} \alpha_{i,j,l} Q^i_{L^\perp} \int_B u^{j-2l-2i} \, \mathcal{H}^{n-k-1}(\mathrm{d}u). \tag{33}$$

Relations (25), (28), (33) together yield

$$\begin{aligned}
&\Gamma(P, A \times B) \\
&= \sum_{j=0}^{p} a^{(j)}(L,B) \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) \\
&= \sum_{j=0}^{p} \sum_{l=0}^{\lfloor j/2 \rfloor} (\pi_L^* Q_L^l) \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) \pi_{L^\perp}^* \\
&\quad \times \sum_{i=0}^{\lfloor j/2 \rfloor - l} \alpha_{i,j,l} Q_{L^\perp}^i \int_B u^{j-2l-2i}\, \mathcal{H}^{n-k-1}(\mathrm{d}u) \\
&= \sum_{j=0}^{p} \sum_{l=0}^{\lfloor j/2 \rfloor} \sum_{i=0}^{\lfloor j/2 \rfloor - l} \alpha_{i,j,l} (\pi_L^* Q_L^l)(\pi_{L^\perp}^* Q_{L^\perp}^i) \\
&\quad \times \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) \int_B u^{j-2l-2i}\, \mathcal{H}^{n-k-1}(\mathrm{d}u) \\
&= \sum_{j=0}^{p} \sum_{m=0}^{\lfloor j/2 \rfloor} \left(\sum_{l=0}^{m} c_{j,m,l} Q_{m,l}(L) \right) \int_A x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) \int_B u^{j-2m}\, \mathcal{H}^{n-k-1}(\mathrm{d}u)
\end{aligned}$$

with $c_{j,m,l} := \alpha_{m-l,j,l}$ and

$$Q_{m,l}(L) = (\pi_L^* Q_L^l)(\pi_{L^\perp}^* Q_{L^\perp}^{m-l}).$$

Now let $P \in \mathcal{P}^n$ be an arbitrary polytope, $A \in \mathcal{B}(\mathbb{R}^n)$, $B \in \mathcal{B}(\Sigma)$. Since $\Gamma(P, \cdot)$ is concentrated on $\operatorname{Nor} P$ and

$$(A \times B) \cap \operatorname{Nor} P = \bigcup_{k=0}^{n-1} \bigcup_{F \in \mathcal{F}_k(P)} (A \cap \operatorname{relint} F) \times (B \cap \nu(P,F)),$$

where $\nu(P,F) := N(P,F) \cap \mathbb{S}^{n-1}$ and $N(P,F)$ is the normal cone of P at F, we obtain

$$\begin{aligned}
&\Gamma(P, A \times B) \\
&= \Gamma(P, (A \times B) \cap \operatorname{Nor} P) \\
&= \sum_{k=0}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \Gamma(P, (A \cap \operatorname{relint} F) \times (B \cap \nu(P,F))) \\
&= \sum_{k=0}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \sum_{j=0}^{p} \sum_{m=0}^{\lfloor j/2 \rfloor} \sum_{l=0}^{m} c_{j,m,l} Q_{m,l}(D(F)) \\
&\quad \times \int_{A \cap F} x^{p-j}\, \mathcal{H}^k(\mathrm{d}x) \int_{B \cap \nu(P,F)} u^{j-2m}\, \mathcal{H}^{n-k-1}(\mathrm{d}u)
\end{aligned}$$

$$= \sum_{j=0}^{p} \sum_{m=0}^{\lfloor j/2 \rfloor} \sum_{l=0}^{m} c_{j,m,l} \sum_{k=0}^{n-1} \sum_{F \in \mathcal{F}_k(P)}$$
$$\times Q_{m,l}(D(F)) \int_F \int_{\nu(P,F)} \mathbf{1}_{A \times B}(x,u) x^{p-j} u^{j-2m} \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \, \mathcal{H}^k(\mathrm{d}x).$$

Since both sides are measures (evaluated at the product set $A \times B$), we can replace $A \times B$ in the result by an arbitrary set $\eta \in \mathcal{B}(\Sigma)$. This completes the proof of the theorem. □

References

1. Alesker, S.: Continuous valuations on convex sets. Geom. Funct. Anal. 8, 402–409 (1998)
2. Alesker, S.: Continuous rotation invariant valuations on convex sets. Ann. of Math. 149, 977–1005 (1999)
3. Alesker, S.: Description of continuous isometry covariant valuations on convex sets. Geom. Dedicata 74, 241–248 (1999)
4. Beisbart, C., Barbosa, M. S., Wagner, H., da F. Costa, L.: Extended morphometric analysis of neuronal cells with Minkowski valuations. Eur. Phys. J. B Condens. Matter Phys. 52, 531–546 (2006)
5. Beisbart, C., Dahlke, R., Mecke, K., Wagner, H.: Vector- and tensor-valued descriptors for spatial patterns. In: Mecke, K., Stoyan, D. (eds.) Morphology of Condensed Matter, pp. 238–260. Lecture Notes in Physics, vol. 600, Springer, Berlin (2002)
6. Glasauer, S.: A generalization of intersection formulae of integral geometry. Geom. Dedicata 68, 101–121 (1997)
7. Greub, W. H.: Multilinear Algebra. Springer, Berlin (1967)
8. Hug, D., Schneider, R., Schuster, R.: The space of isometry covariant tensor valuations. Algebra i Analiz 19, 194–224 (2007) = St. Petersburg Math. J. 19, 137–158 (2008)
9. Hug, D., Schneider, R., Schuster, R.: Integral geometry of tensor valuations. Adv. Appl. Math. 41, 482–509 (2008)
10. Klain, D. A.: A short proof of Hadwiger's characterization theorem. Mathematika 42, 329–339 (1995)
11. Klain, D. A., Rota, G.–C.: Introduction to Geometric Probability. Cambridge University Press, Cambridge (1997)
12. McMullen, P.: Isometry covariant valuations on convex bodies. Rend. Circ. Mat. Palermo (2) Suppl. 50, 259–271 (1997)
13. Schneider, R.: Kinematische Berührmaße für konvexe Körper. Abh. Math. Sem. Univ. Hamburg 44, 12–23 (1975)
14. Schneider, R.: Curvature measures of convex bodies. Ann. Mat. Pura Appl. 116, 101–134 (1978)
15. Schneider, R.: Convex Bodies: the Brunn-Minkowski Theory. Cambridge University Press, Cambridge (1993)
16. Schneider, R., Weil, W.: Stochastic and Integral Geometry. Springer, Berlin (2008)
17. Schröder–Turk, G. E., Kapfer, S., Breidenbach, B., Beisbart, C., Mecke, K.: Tensorial Minkowski functionals and anisotropy measures for planar patterns. J. Microscopy 238, 57–74 (2010)
18. Schröder–Turk, G. E., Mickel, W., Kapfer, S. C., Klatt, M. A., Schaller, F. M., Hoffmann, M. J. F., Kleppmann, N., Armstrong, P., Inayat, A., Hug, D., Reichelsdorfer, M., Peukert, W., Schwieger, W., Mecke K.: Minkowski tensor shape analysis of cellular, granular and porous structures. Advanced Materials, Special Issue: Hierarchical Structures Towards Functionality 23, 2535–2553 (2011)
19. Schröder–Turk, G. E., Mickel, W., Kapfer, S. C., Schaller, F. M., Breidenbach, B., Hug, D., Mecke, K.: Minkowski tensors of anisotropic spatial structure. arXiv:1009.2340v1

20. Sporer, S., Goll, C., Mecke, K.: Motion by stopping: rectifying Brownian motion of nonspherical particles. Phys. Rev. E 78, 011917 (2008)
21. Zähle, M. Approximation and characterization of generalized Lipschitz–Killing curvatures. Ann. Global Anal. Geom. 8, 249–260 (1990)