Valuations on Convex Bodies–the Classical Basic Facts

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Abstract The purpose of this chapter is to give an elementary introduction to valuations on convex bodies. The goal is to serve the newcomer to the field, by presenting basic notions and collecting fundamental facts, which have proved of importance for the later development, either as technical tools or as models and incentives for widening and deepening the theory. We also provide hints to the literature where proofs can be found. It is not our intention to duplicate the existing longer surveys on valuations, nor to update them. We restrict ourselves to classical basic facts and geometric approaches, which also means that we do not try to describe the exciting developments of valuation theory in the last fifteen years, which involve deeper methods and will be the subject of later chapters. The sections of the present chapter treat, in varying detail, general valuations, valuations on polytopes, examples of valuations from convex geometry, continuous valuations on convex bodies, measure-valued valuations, valuations on lattice polytopes.

1 General Valuations

The natural domain for a valuation, as it is understood here, would be a lattice (in the sense of Birkhoff [4]; see p. 230, in particular). However, many important functions turning up naturally in convex geometry have a slightly weaker property, and they become valuations on a lattice only after an extension procedure. For that reason, valuations on intersectional families are the appropriate object to study here. A family $\mathcal{S}$ of sets is called *intersectional* if $A, B \in \mathcal{S}$ implies $A \cap B \in \mathcal{S}$.

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Definition 1. A function φ from an intersectional family $\mathcal{S}$ into an abelian group (with composition $+$ and zero element 0) is *additive* or a *valuation* if

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \tag{1}$$

for all $A, B \in \mathcal{S}$ with $A \cup B \in \mathcal{S}$, and if $\varphi(\emptyset) = 0$ in case $\emptyset \in \mathcal{S}$.

The abelian group in the definition may be replaced by an abelian semigroup with cancellation law, because the latter can be embedded in an abelian group. A trivial example of a valuation on $\mathcal{S}$ is given by $\varphi(A) := \mathbf{1}_A$, where $\mathbf{1}_A$ is the characteristic function of A, defined on $S := \bigcup_{A \in \mathcal{S}} A$ by

$$\mathbf{1}_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in S \setminus A. \end{cases}$$

For the abelian group appearing in Definition 1 one can take in this case, for example, the additive group of all real functions on S.

It would generally be too restrictive to assume that the intersectional family $\mathcal{S}$ is also closed under finite unions. However, we can always consider the family $U(\mathcal{S})$ consisting of all finite unions of elements from $\mathcal{S}$. Then $(U(\mathcal{S}), \cup, \cap)$ is a lattice. If φ is a valuation on $U(\mathcal{S})$ (not only on $\mathcal{S}$), then (1) is easily extended by induction to the formula

$$\varphi(A_1 \cup \cdots \cup A_m) = \sum_{\emptyset \neq J \subset \{1,\ldots,m\}} (-1)^{|J|-1} \varphi(A_J) \tag{2}$$

for $m \in \mathbb{N}$ and $A_1, \ldots, A_m \in U(\mathcal{S})$; here $A_J := \bigcap_{j \in J} A_j$ and $|J| := \operatorname{card} J$. Relation (2) is known as the *inclusion-exclusion formula*. This gives rise to another definition.

Definition 2. A function φ from the intersectional family $\mathcal{S}$ into an abelian group is called *fully additive* if (2) holds for $m \in \mathbb{N}$ and all $A_1, \ldots, A_m \in \mathcal{S}$ with $A_1 \cup \cdots \cup A_m \in \mathcal{S}$.

Thus, a valuation on $\mathcal{S}$ that has an additive extension to the lattice $U(\mathcal{S})$, is fully additive. It is a nontrivial fact that the converse is also true. We formulate a more general extension theorem. For this, we denote by $U^\bullet(\mathcal{S})$ the $\mathbb{Z}$-module spanned by the characteristic functions of the elements of $\mathcal{S}$.

Theorem 1. (Groemer's first extension theorem) *Let φ be a function from an intersectional family of sets (including $\emptyset$) into an abelian group, such that $\varphi(\emptyset) = 0$. Then the following conditions* (a) – (d) *are equivalent.*

(a) *φ is fully additive;*

(b) *If*

$$n_1 \mathbf{1}_{A_1} + \cdots + n_m \mathbf{1}_{A_m} = 0$$

with $A_i \in \mathcal{S}$ and $n_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$, then

$$n_1\varphi(A_1) + \cdots + n_m\varphi(A_m) = 0;$$

(c) *The functional* $\varphi^\bullet$ *defined by* $\varphi^\bullet(\mathbf{1}_A) := \varphi(A)$ *for* $A \in \mathcal{S}$ *has a* $\mathbb{Z}$*-linear extension to* $U^\bullet(\mathcal{S})$*;*

(d) φ *has an additive extension to the lattice* $U(\mathcal{S})$*.*

This theorem is due to Groemer [11]. His proof is reproduced in [42], Theorem 6.2.1. Actually, Groemer formulated a slightly different version. In his version, φ maps into a real vector space. The preceding theorem then remains true with $\mathbb{Z}$ replaced by $\mathbb{R}$, $U^\bullet(\mathcal{S})$ replaced by the real vector space $V(\mathcal{S})$ that is spanned by the characteristic functions of the elements of $\mathcal{S}$, and '$\mathbb{Z}$-linear' replaced by '$\mathbb{R}$-linear'.

In this case, if φ is fully additive, then Groemer defined the *φ-integral* of a function $f \in V(\mathcal{S})$ in the following way. If

$$f = a_1\mathbf{1}_{A_1} + \cdots + a_m\mathbf{1}_{A_m}, \quad a_1, \ldots, a_m \in \mathbb{R},$$

then

$$\int f \, \mathrm{d}\varphi := a_1\varphi(A_1) + \cdots + a_m\varphi(A_m).$$

This definition makes sense, since by Theorem 1 the right-hand side does not depend on the chosen representation of the function f.

Such integrals with respect to a valuation were later rediscovered by Viro [46], and they were applied by him and other authors in various ways, mainly in the case where φ is the Euler characteristic on suitable sets.

Results on general valuations, as mentioned in this section, were preceded by concrete geometric applications of valuations. We give two historic examples in subsequent sections.

2 Valuations on Polytopes

From now on, we work in n-dimensional Euclidean space $\mathbb{R}^n$, with scalar product denoted by $\cdot$ and induced norm $\|\cdot\|$. The domain of the considered valuations will be either the set $\mathcal{K}^n$ of convex bodies (nonempty, compact, convex sets) or the set $\mathcal{P}^n$ of convex polytopes in $\mathbb{R}^n$. We consider the latter case first.

Real valuations on polytopes (by which we always mean convex polytopes) are closely tied up with dissections of polytopes.

Definition 3. A *dissection* of the polytope $P \in \mathcal{P}^n$ is a set $\{P_1, \ldots, P_m\}$ of polytopes such that $P = \bigcup_{i=1}^m P_i$ and $\dim(P_i \cap P_j) < n$ for $i \neq j$.

Let G be a subgroup of the affine group of $\mathbb{R}^n$. The polytopes $P, Q \in \mathcal{P}^n$ are called *G-equidissectable* if there are a dissection $\{P_1, \ldots, P_m\}$ of P, a dissection $\{Q_1, \ldots, Q_m\}$ of Q, and elements $g_1, \ldots, g_m \in G$ such that $Q_i = g_i P_i$ for $i = 1, \ldots, m$.

The most frequently considered cases are those where G is the group T_n of translations of $\mathbb{R}^n$ or the group G_n of rigid motions of $\mathbb{R}^n$. Here a *rigid motion* is an isometry of $\mathbb{R}^n$ that preserves the orientation, thus, a mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ of the form $gx = \vartheta x + t$, $x \in \mathbb{R}^n$, with $\vartheta \in \mathrm{SO}(n)$ and $t \in \mathbb{R}^n$.

The following is a classical result of elementary geometry.

Theorem 2. (Bolyai–Gerwien, 1833/35) *In $\mathbb{R}^2$, any two polygons of the same area are G_2-equidissectable.*

The theorem remains true if the motion group G_2 is replaced by the group consisting of translations and reflections in points (Hadwiger and Glur [22]).

Hilbert's third problem from 1900 asked essentially whether a result analogous to the Bolyai–Gerwien theorem holds in three dimensions. The negative answer given by Dehn [8] is apparently the first use of valuations in convexity. We describe the essence of his answer, though in different terms and using later modifications. This gives us an opportunity to introduce some further notions and facts about valuations.

On polytopes, the valuation property follows from a seemingly weaker assumption.

Definition 4. A function φ on $\mathcal{P}^n$ with values in an abelian group is called *weakly additive* (or a *weak valuation*) if (setting $\varphi(\emptyset) := 0$) for each $P \in \mathcal{P}^n$ and each hyperplane H, bounding the two closed halfspaces H^+, H^-, the relation

$$\varphi(P) = \varphi(P \cap H^+) + \varphi(P \cap H^-) - \varphi(P \cap H) \tag{3}$$

holds.

Every valuation on $\mathcal{P}^n$ is weakly additive, but also the converse is true, even more.

Theorem 3. *Every weakly additive function on $\mathcal{P}^n$ with values in an abelian group is fully additive on $\mathcal{P}^n$.*

A proof can be found in [42], Theorem 6.2.3, and Note 1 there gives hints to the origins of this result.

Together with Groemer's first extension theorem (Theorem 1), the preceding theorem shows that every weakly additive function on $\mathcal{P}^n$ has an additive extension to the lattice $U(\mathcal{P}^n)$. The elements of $U(\mathcal{P}^n)$ are the finite unions of convex polytopes; we call them *polyhedra*.

We need two other important notions.

Definition 5. A valuation φ on a subset of $\mathcal{K}^n$ is called *simple* if $\varphi(A) = 0$ whenever $\dim A < n$.

Definition 6. Let G be a subgroup of the affine group of $\mathbb{R}^n$. A valuation φ on a subset of $\mathcal{K}^n$ (which together with A contains gA for $g \in G$) is called *G-invariant* if $\varphi(gA) = \varphi(A)$ for all A in the domain of φ and all $g \in G$.

The following is easy, but important.

Lemma 1. *Let G be a subgroup of the affine group of $\mathbb{R}^n$. If φ is a G-invariant simple valuation on $\mathcal{P}^n$ and if the polytopes $P, Q \in \mathcal{P}^n$ are G-equidissectable, then $\varphi(P) = \varphi(Q)$.*

In fact, by Theorems 3 and 1, the valuation φ has an additive extension to $U(\mathcal{P}^n)$, hence the inclusion-exclusion formula (2) can be applied to dissections $\{P_1, \dots, P_m\}$ of P and $\{Q_1, \dots, Q_m\}$ of Q, satisfying $g_i P_i = Q_i$ for $g_i \in G$. Since φ is simple, the terms in (2) with $|J| > 1$ vanish, and what remains is

$$\begin{aligned}\varphi(P) &= \varphi(P_1 \cup \dots \cup P_m) = \varphi(P_1) + \dots + \varphi(P_m) \\ &= \varphi(g_1 P_1) + \dots + \varphi(g_m P_m) = \varphi(g_1 P_1 \cup \dots \cup g_m P_m) \\ &= \varphi(Q_1 \cup \dots \cup Q_m) = \varphi(Q).\end{aligned}$$

Dehn's negative answer to Hilbert's third problem can now be obtained as follows. We have to show that there are three-dimensional polytopes of equal volume that are not G_3-equidissectable. For this, we construct a simple, G_3-invariant valuation φ on $\mathcal{P}^3$ such that $\varphi(C) = 0$ for all cubes C and $\varphi(T) \neq 0$ for all regular tetrahedra T. Denote by $\mathcal{F}_1(P)$ the set of edges of $P \in \mathcal{P}^3$, by $V_1(F)$ the length of the edge $F \in \mathcal{F}_1(P)$, and by $\gamma(P, F)$ the outer angle of P at F. Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of Cauchy's functional equation

$$f(x + y) = f(x) + f(y) \quad \text{for } x, y \in \mathbb{R} \tag{4}$$

which satisfies

$$f(\pi/2) = 0 \tag{5}$$

and

$$f(\alpha) \neq 0, \tag{6}$$

where α denotes the external angle of a regular tetrahedron T at one of its edges. That such a solution f exists, can be shown by using a Hamel basis of $\mathbb{R}$ and the fact that $\pi/2$ and α are rationally independent. Then we define

$$\varphi(P) := \sum_{F \in \mathcal{F}_1(P)} V_1(F) f(\gamma(P, F)) \quad \text{for } P \in \mathcal{P}^3.$$

Because of (4), it can be shown that φ is weakly additive and hence a valuation, and as a consequence of (5) (which implies $f(\pi) = 0$) it is simple. Clearly, it is G_3-invariant. A cube C has outer angle $\pi/2$ at its edges, hence $\varphi(C) = 0$, whereas $\varphi(T) \neq 0$, due to (6). Now it follows from Lemma 1 that C and T cannot be G_3-equidissectable (even if they have the same volume). For this approach, see Hadwiger [13], and for an elementary exposition, Boltyanskii [6].

The interrelations between the dissection theory of polytopes and valuations have been developed in great depth. For a general account, we refer to the book of Sah [37] and to the survey articles [34] (Sec. II) and [32] (Sec. 4). For a recent contribution, see Kusejko and Parapatits [27].

While Dehn's result shows that, in dimension $n \geq 3$, two polytopes of equal volume need not be G_n-equidissectable, the following result of Hadwiger [15] is rather surprising. The proof (following Hadwiger) can also be found in [42], Lemma 6.4.2. The result plays a role in the further study of valuations.

Theorem 4. *Any two parallelotopes of equal volume in $\mathbb{R}^n$ are T_n-equidissectable.*

The first main goals of a further study of valuations on polytopes will be general properties of such valuations and representation or classification results, possibly under additional assumptions, such as invariance properties or continuity.

A further extension theorem can be helpful. As we have seen, the inclusion-exclusion formula is easy to use for simple valuations, but it is a bit clumsy in the general case. We can circumvent this by decomposing a polytope into a finite disjoint union of relatively open polytopes. A *relatively open polytope*, briefly *ro-polytope*, is the relative interior of a convex polytope. We denote the set of ro-polytopes in $\mathbb{R}^n$ by $\mathcal{P}^n_{ro}$ and the set of finite unions of ro-polytopes by $U(\mathcal{P}^n_{ro})$. The elements of the latter are called *ro-polyhedra*. Every convex polytope $P \in \mathcal{P}^n$ is the disjoint union of the relative interiors of its faces (including P) and hence belongs to $U(\mathcal{P}^n_{ro})$.

Theorem 5. *Any weakly additive function on $\mathcal{P}^n$ with values in an abelian group has an additive extension to $U(\mathcal{P}^n_{ro})$.*

This can be deduced from Theorems 3 and 1; see [42], Corollary 6.2.4. The result facilitates the proof of the following theorem, which is fundamental for many of the further investigations. Here φ is called *homogeneous* of degree r if

$$\varphi_r(\lambda P) = \lambda^r \varphi(P) \quad \text{for all } P \in \mathcal{P}^n \text{ and all real } \lambda \geq 0,$$

and *rational homogeneous* of degree r if this holds for rational $\lambda \geq 0$.

Theorem 6. *Let φ be a translation invariant valuation on $\mathcal{P}^n$ with values in a rational vector space X. Then*

$$\varphi(\lambda P) = \sum_{r=0}^{n} \lambda^r \varphi_r(P) \quad \text{for } P \in \mathcal{P}^n \text{ and rational } \lambda \geq 0. \tag{7}$$

Here $\varphi_r : \mathcal{P}^n \to X$ is a translation invariant valuation which is rational homogeneous of degree r $(r = 0, \ldots, n)$.

Setting $\lambda = 1$ in (7) gives

$$\varphi = \varphi_0 + \cdots + \varphi_n, \tag{8}$$

which is known as the *McMullen decomposition*. It has the important consequence that for the investigation of translation invariant valuations on $\mathcal{P}^n$ with values in a rational vector space X one need only consider such valuations which are rational homogeneous of some degree $r \in \{0, \dots, n\}$.

Another consequence of Theorem 6 is a polynomial expansion with respect to Minkowski addition. Recall that the Minkowski sum (or vector sum) of $K, L \in \mathcal{K}^n$ is defined by

$$K + L = \{x + y : x \in K,\ y \in L\},$$

and that $K + L \in \mathcal{K}^n$. A function φ from $\mathcal{K}^n$ to some abelian group is *Minkowski additive* if

$$\varphi(K + L) = \varphi(K) + \varphi(L) \quad \text{for all } K, L \in \mathcal{K}^n.$$

By repeatedly applying (7), it is not difficult to deduce the following.

Theorem 7. *Let $\varphi : \mathcal{P}^n \to X$ (with X a rational vector space) be a translation invariant valuation which is rational homogeneous of degree $m \in \{1, \dots, n\}$. Then there is a polynomial expansion*

$$\begin{aligned}&\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k)\\ &= \sum_{r_1,\dots,r_k=0}^{m} \binom{m}{r_1 \dots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k}\, \overline{\varphi}(\underbrace{P_1, \dots, P_1}_{r_1}, \dots, \underbrace{P_k, \dots, P_k}_{r_k}),\end{aligned}$$

valid for all $P_1, \dots, P_k \in \mathcal{P}^n$ and all rational $\lambda_1, \dots, \lambda_k \geq 0$. Here $\overline{\varphi} : (\mathcal{P}^n)^m \to X$ is a symmetric mapping, which is translation invariant and Minkowski additive in each variable.

Historical note. The result of Theorem 6, even in a more general version, was stated by Hadwiger [12] (his first publication on valuations), as early as 1945, but without proof. His later work gives a proof of the decomposition (8) for simple valuations only, see [21], p. 54. The question for a result as stated in Theorem 7 was posed by Peter McMullen, at an Oberwolfach conference in 1974. He gave a proof the same year, see [28], [29]. Different proofs were provided by Meier [35] and Spiegel [45]. A variation of Spiegel's proof, using Theorem 5 instead of the inclusion-exclusion formula, is found in [42], Section 6.3. Proofs of more general versions of the polynomiality theorem were given by Pukhlikov and Khovanskii [36] and by Alesker [1].

A consequence of Theorem 7 is the fact that a valuation $\varphi : \mathcal{P}^n \to \mathbb{R}$ that is translation invariant and rational homogeneous of degree 1 is Minkowski additive. A variant of this result was first proved by Spiegel [44].

We turn to representation results for translation invariant, real valuations on $\mathcal{P}^n$. Without additional assumptions, little is known about these. Setting $\lambda = 0$ in (7), we see that any such valuation which is homogeneou of degree zero, is constant. Then we mention two classical characterizations of the volume on polytopes, which are due to Hadwiger. The volume is denoted by V_n.

Theorem 8. *Let $\varphi : \mathcal{P}^n \to \mathbb{R}$ be a translation invariant valuation which is simple and nonnegative. Then $\varphi = cV_n$ with a constant c.*

The proof can be found in Hadwiger's book [21], Sec. 2.1.3. The following result is also due to Hadwiger (see [21], p. 79; also [42], Theorem 6.4.3). The proof makes use of Theorem 4.

Theorem 9. *Let $\varphi : \mathcal{P}^n \to \mathbb{R}$ be a translation invariant valuation which is homogeneous of degree n. Then $\varphi = cV_n$ with a constant c.*

For translation invariant and *simple* valuations on polytopes, more general representations are possible. Under a weak continuity assumption, these go back to Hadwiger [18], and without that assumption to recent work of Kusejko and Parapatits [27]. We consider Hadwiger's result first, but use the terminology of [27].

For $k \in \{0, \dots, n\}$, let $\mathcal{U}^k$ denote the set of all ordered orthonormal k-tuples of vectors from the unit sphere $\mathbb{S}^{n-1}$. $\mathcal{U}^0$ contains only the empty tuple (). For $P \in \mathcal{P}^n$ and $u \in \mathbb{S}^{n-1}$, let $F(P, u)$ be the face of P with outer normal vector u. For $U = (u_1, \dots, u_k) \in \mathcal{U}^k$ and $P \in \mathcal{P}^n$ we define recursively the face P_U of P by

$$P_{()} := P, \qquad P_{(u_1,\dots,u_r)} := F(P_{(u_1,\dots,u_{r-1})}, u_k), \;\; r = 1, \dots, k.$$

The orthonormal frame $U = (u_1, \dots, u_k) \in \mathcal{U}^k$ is *P-tight* if $\dim P_{(u_1,\dots,u_r)} = n - r$ for $r = 0, \dots, k$. Let $\mathcal{U}^k_P$ denote the (evidently finite) set of all P-tight frames in $\mathcal{U}^k$. Then $V_{n-k}(P_U) > 0$ for $U \in \mathcal{U}^k_P$, where V_{n-k} denotes the $(n-k)$-dimensional volume.

A function $f : \mathcal{U}^k \to \mathbb{R}$ is called *odd* if

$$f(\varepsilon_1 u_1, \dots, \varepsilon_k u_k) = \varepsilon_1 \cdots \varepsilon_k f(u_1, \dots, u_k)$$

for $\varepsilon_i = \pm 1$.

A valuation $\varphi : \mathcal{P}^n \to \mathbb{R}$ is *weakly continuous* if it is continuous under parallel displacements of the facets of a polytope. To make this more precise, we consider the set of polytopes whose system of outer normal vectors of facets belongs to a given finite set $U = \{u_1, \dots, u_m\}$; these vectors positively span $\mathbb{R}^n$. Now a function φ on $\mathcal{P}^n$ is called weakly continuous if for any such U the function

$$(\eta_1, \dots, \eta_m) \mapsto \varphi(\{x \in \mathbb{R}^n : x \cdot u_i \le \eta_i, \; i = 1, \dots, m\})$$

is continuous on the set of all $(\eta_1, \dots, \eta_m)$ for which the argument of φ is not empty.

The following is Hadwiger's [18] result. For a version of his proof, we refer to [42], Thm. 6.4.6. The proof given in [27] appears to be simpler. We write $\mathcal{U} := \bigcup_{k=0}^{n-1} \mathcal{U}^k$.

Theorem 10. *A function $\varphi : \mathcal{P}^n \to \mathbb{R}$ is a weakly continuous, translation invariant, simple valuation if and only if for each $U \in \mathcal{U}$ there is a constant $c_U \in \mathbb{R}$ such that $U \mapsto c_U$ is odd and*

$$\varphi(P) = \sum_{k=0}^{n-1} \sum_{U \in \mathcal{U}_P^k(P)} c_U V_{n-k}(P_U) \tag{9}$$

for $P \in \mathcal{P}^n$.

For non-simple valuations, the following result holds. As usual, $\mathcal{F}_r(P)$ denotes the set of r-dimensional faces of a polytope P, and $N(P, F)$ is the cone of normal vectors of P at its face F.

Theorem 11. *A function $\varphi : \mathcal{P}^n \to \mathbb{R}$ is a weakly continuous, translation invariant valuation if and only if there are a constant c and for each $r \in \{1, \dots, n-1\}$ a simple real valuation θ_r on the system of convex polyhedral cones in $\mathbb{R}^n$ of dimension at most $n-r$ such that*

$$\varphi(P) = \varphi(\{0\}) + \sum_{r=1}^{n-1} \sum_{F \in \mathcal{F}_r(P)} \theta_r(N(P, F)) V_r(F) + c V_n(P) \tag{10}$$

for $P \in \mathcal{P}^n$.

McMullen [31] has deduced this from Hadwiger's result on simple valuations. For a different approach, see in [32] the remark after Thm. 5.19.

Satisfactory as these results are in the realm of polytopes, they seem, at present, not to lead much further in the investigation of continuous valuations on general convex bodies. Conditions on the functions θ_r, which do or do not allow a continuous extension of a valuation φ represented by (10) to general convex bodies, were investigated in [23].

Without the assumption of weak continuity, Kusejko and Parapatits [27] have obtained the following result.

Theorem 12. *A function $\varphi : \mathcal{P}^n \to \mathbb{R}$ is a translation invariant, simple valuation if and only if for each $U \in \mathcal{U}$ there exists an additive function $f_U : \mathbb{R} \to \mathbb{R}$ such that $U \mapsto f_U$ is odd and*

$$\varphi(P) = \sum_{k=0}^{n-1} \sum_{U \in \mathcal{U}_P^k(P)} f_U(V_{n-k}(P_U)) \tag{11}$$

for $P \in \mathcal{P}^n$.

The implications of this result for translative equidecomposability are explained in [27].

3 Examples of Valuations from Convex Geometry

The theory of convex bodies provides many examples of valuations that come up naturally. We explain the most important of these, before turning to classification and characterization results.

A first example is given by the identity mapping $\mathcal{K}^n \to \mathcal{K}^n$. This makes sense, since $\mathcal{K}^n$, as usual equipped with Minkowski addition, is an abelian semigroup with cancellation law. The identity mapping is a valuation, since the relation

$$(K \cup L) + (K \cap L) = K + L \tag{12}$$

holds for convex bodies $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$ (as first pointed out by Sallee [38]; the easy proof can be found in [42], Lemma 3.1.1). Consequently, also the support function defines a valuation. The support function $h(K, \cdot) = h_K$ of the convex body $K \in \mathcal{K}^n$ is defined by

$$h(K, u) := \max\{\langle u, x\rangle : x \in K\} \quad \text{for } u \in \mathbb{R}^n.$$

The function h is Minkowski additive in the first argument. The Minkowski additivity of the support function together with (12) yields

$$h(K \cup L, \cdot) + h(K \cap L, \cdot) = h(K, \cdot) + h(L, \cdot) \quad \text{if } K \cup L \text{ is convex,}$$

hence the map $K \mapsto h(K, \cdot)$, from $\mathcal{K}^n$ into (say) the vector space of real continuous functions on $\mathbb{R}^n$, is a valuation. Using the support function, the following can be shown (see, e.g., [42], Theorem 6.1.2, and, for the history, Note 2 on p. 332).

Theorem 13. *Every Minkowski additive function on $\mathcal{K}^n$ with values in an abelian group is fully additive.*

Minkowski addition plays a role in valuation theory of convex bodies in more than one way. As one example, we mention a way to construct new valuations from a given one. Let φ be a valuation on $\mathcal{K}^n$. If $C \in \mathcal{K}^n$ is a fixed convex body, then

$$\varphi_C(K) := \varphi(K + C) \quad \text{for } K \in \mathcal{K}^n$$

defines a new valuation φ_C on $\mathcal{K}^n$. If φ is translation invariant, then the same holds for φ_C.

A basic example of a valuation on $\mathcal{K}^n$ is, of course, the volume V_n. Being the restriction of a measure, the function $V_n : \mathcal{K}^n \to \mathbb{R}$ is a valuation, and since lower-dimensional convex bodies have volume zero, it is simple. Moreover, the valuation V_n is invariant under rigid motions and continuous (continuity of functions on $\mathcal{K}^n$ always refers to the Hausdorff metric). Via the construction (13) below, it gives rise to many other (non-simple) valuations. The following fact, which goes back to Minkowski at the beginning of the 20th century, was, in fact, the template for Theorem 2.5. There is a nonnegative, symmetric function $V : (\mathcal{K}^n)^n \to \mathbb{R}$, called the *mixed volume*, such that

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \dots, K_{i_n})$$

for all $m \in \mathbb{N}$, $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$. (For proofs and more information, we refer to [42], Sec. 5.1.) We write

$$V(\underbrace{K_1, \dots, K_1}_{r_1}, \dots, \underbrace{K_m, \dots, K_m}_{r_m}) =: V(K_1[r_1], \dots, K_m[r_m]).$$

For arbitrary $p \in \{1, \dots, n\}$ and fixed convex bodies $M_{p+1}, \dots, M_n \in \mathcal{K}^n$, the function φ defined by

$$\varphi(K) := V(K[p], M_{p+1}, \dots, M_n), \quad K \in \mathcal{K}^n, \tag{13}$$

is a valuation on $\mathcal{K}^n$. It is translation invariant, continuous, and homogeneous of degree p. Often in the literature, these functionals φ are also called 'mixed volumes', but we find that slightly misleading (since the mixed volume is a function of n variables) and prefer to call them *mixed volume valuations*.

Of particular importance are the special cases of the mixed volume valuations where the fixed bodies are equal to the unit ball B^n. First we recall two frequently used constants: κ_n is the volume of the unit ball in $\mathbb{R}^n$ and ω_n is its surface area; explicitly,

$$\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}, \quad \omega_n = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{14}$$

We define

$$V_j(K) := \frac{\binom{n}{j}}{\kappa_{n-j}} V(K[j], B^n[n-j]) \tag{15}$$

for $K \in \mathcal{K}^n$. The functional V_j is called the jth *intrinsic volume*. In addition to the properties that all mixed volume valuations share, it is invariant under rotations and thus under rigid motions. The normalizing factor has the effect that the intrinsic volume is independent of the dimension of the ambient space in which it is computed. In particular, if the convex body K has dimension $\dim K \leq m$, then $V_m(K)$ is the m-dimensional volume of K.

As a special case of the above approach to mixed volumes, we see that the intrinsic volumes are uniquely defined by the coefficients in the expansion

$$V_n(K+\rho B^n) = \sum_{j=0}^{n} \rho^{n-j}\kappa_{n-j}V_j(K), \quad \rho \geq 0. \tag{16}$$

Here, $K+\rho B^n$ is the outer parallel body of K at distance $\rho \geq 0$, that is, the set of all points of $\mathbb{R}^n$ that have distance at most ρ from K. Equation (16) is known as the *Steiner formula*.

The concept of the parallel body can be localized. There is a local Steiner formula, which leads to measure-valued valuations. For this, we need a few more definitions. For $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$, there is a unique point $p(K,x) \in K$ with

$$\|x - p(K,x)\| \leq \|x - y\| \quad \text{for all } y \in K.$$

The map $p(K,\cdot) : \mathbb{R}^n \to K$ is known as the *metric projection* of K. The map $K \mapsto p(K,x)$, for fixed x, is another example of a valuation, from $\mathcal{K}^n$ to $\mathbb{R}^n$. By $d(K,x) := \|x - p(K,x)\|$ the distance of x from K is defined, and, for $x \in \mathbb{R}^n \setminus K$, by

$$u(K,x) := \frac{x - p(K,x)}{d(K,x)}$$

the unit vector pointing from $p(K,x)$ to x. The pair $(p(K,x), u(K,x))$ is a support element of K. Generally, a *support element* of K is a pair (x,u), where $x \in \operatorname{bd} K$ and u is an outer unit normal vector of K at x. The set $\mathbf{nc}(K)$ of all support elements of K is called the (generalized) *normal bundle* or the *normal cycle* of K. It is a subset of the product space

$$\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1} \tag{17}$$

(which is equipped with the product topology). Now for $\eta \in \mathcal{B}(\Sigma^n)$, the σ-algebra of Borel sets of Σ^n, for $K \in \mathcal{K}^n$ and $\rho > 0$, we define the *local parallel set*

$$M_\rho(K,\eta) := \{x \in \mathbb{R}^n : 0 < d(K,x) \leq \rho \text{ and } (p(K,x), u(K,x)) \in \eta\}.$$

This is a Borel set. By $\mathcal{H}^n$ we denote n-dimensional Hausdorff measure. Again, one has a polynomial expansion, namely

$$\mathcal{H}^n(M_\rho(K,\eta)) = \sum_{j=0}^{n-1} \rho^{n-j}\kappa_{n-j}\Lambda_j(K,\eta) \quad \text{for } \rho \geq 0.$$

This defines finite Borel measures $\Lambda_0(K,\cdot), \dots, \Lambda_{n-1}(K,\cdot)$ on Σ^n. One calls $\Lambda_j(K,\cdot)$ the jth *support measure* of K. From the valuation property of the nearest point map, one can deduce that

$$\Lambda_j(K \cup L, \cdot) + \Lambda_j(K \cap L, \cdot) = \Lambda_j(K, \cdot) + \Lambda_j(L, \cdot)$$

for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$. Thus, the mapping $K \mapsto \Lambda_j(K, \cdot)$ is a valuation on $\mathcal{K}^n$, with values in the vector space of finite signed Borel measures on Σ^n.

From the support measures we get two series of marginal measures. They appear in the literature with two different normalizations. For Borel sets $\beta \subset \mathbb{R}^n$, we define

$$\frac{\binom{n}{j}}{n\kappa_{n-j}} C_j(K, \beta) = \Phi_j(K, \beta) := \Lambda_j(K, \beta \times \mathbb{S}^{n-1}).$$

The measures $C_0(K, \cdot), \dots, C_{n-1}(K, \cdot)$ are the *curvature measures* of K. They are measures on $\mathbb{R}^n$, concentrated on the boundary of K. The definition is supplemented by

$$\frac{1}{n} C_n(K, \beta) = \Phi_n(K, \beta) := \mathcal{H}^n(K \cap \beta).$$

For Borel sets $\omega \subset \mathbb{S}^{n-1}$, we define

$$\frac{\binom{n}{j}}{n\kappa_{n-j}} S_j(K, \omega) = \Psi_j(K, \omega) := \Lambda_j(K, \mathbb{R}^n \times \omega).$$

The measures $S_0(K, \cdot), \dots, S_{n-1}(K, \cdot)$ are the *area measures* of K. They are measures on the unit sphere $\mathbb{S}^{n-1}$.

4 Continuous Valuations on Convex Bodies

Among the valuations on the space $\mathcal{K}^n$ of general convex bodies in $\mathbb{R}^n$, those are of particular interest which have their values in a (here always real) topological vector space (such as $\mathbb{R}$, $\mathbb{R}^n$, tensor spaces, spaces of functions or measures) and are continuous with respect to the topology on $\mathcal{K}^n$ that is induced by the Hausdorff metric.

Before describing consequences of continuity, we wish to point out that general valuations on $\mathcal{K}^n$ can show rather irregular behaviour. For example, if we choose a non-continuous solution f of Cauchy's functional equation, $f(x+y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$, then $\varphi := f \circ V_j$ with $j \in \{1, \dots, n\}$ is a valuation on $\mathcal{K}^n$ which is not continuous, in fact not even locally bounded, since f is unbounded on every nondegenerate interval. For $j = 1$, the function φ is Minkowski additive and hence, by Theorem 13, even fully additive.

As a first consequence of continuity, we mention another extension theorem of Groemer [11]. It needs only a weaker version of continuity. A function φ from $\mathcal{K}^n$ into some topological (Hausdorff) vector space is called *σ-continuous* if for every decreasing sequence $(K_i)_{i\in\mathbb{N}}$ in $\mathcal{K}^n$ one has

$$\lim_{i\to\infty}\varphi(K_i)=\varphi\left(\bigcap_{i\in\mathbb{N}}K_i\right).$$

If φ is continuous with respect to the Hausdorff metric, then it is σ-continuous. This follows from Lemma 1.8.2 in [42].

Theorem 14. (Groemer's second extension theorem) *Let φ be a function on $\mathcal{K}^n$ with values in a topological vector space. If φ is weakly additive on $\mathcal{P}^n$ and is σ-continuous on $\mathcal{K}^n$, then φ has an additive extension to the lattice $U(\mathcal{K}^n)$.*

Groemer's proof is reproduced in [43], Theorem 14.4.2. The formulation of the theorem here is slightly more general, and we give a slightly shorter proof, based on the following lemma.

Lemma 2. *Let $K_1,\dots,K_m\in\mathcal{K}^n$ be convex bodies such that $K_1\cup\dots\cup K_m$ is convex. Let $\varepsilon>0$. Then there are polytopes $P_1,\dots,P_m\in\mathcal{P}^n$ with $K_i\subset P_i\subset K_i+\varepsilon B^n$ for $i=1,\dots,m$ such that $P_1\cup\dots\cup P_m$ is convex.*

For the proof and the subsequent argument, we refer to Weil [47], Lemma 8.1. With this lemma, Theorem 14 can be proved as follows (following a suggestion of Daniel Hug). Let φ satisfy the assumptions of Theorem 14. Let $K_1,\dots,K_m\in\mathcal{K}^n$ be convex bodies such that $K_1\cup\dots\cup K_m$ is convex. We apply Lemma 2 with K_i replaced by $K_i+2^{-k}B^n$, $k\in\mathbb{N}$, and $\varepsilon=2^{-k}$ (note that $\bigcup_{i=1}^m(K_i+2^{-k}B^n)=(\bigcup_{i=1}^m K_i)+2^{-k}B^n$ is convex). This yields polytopes $P_1^{(k)},\dots,P_m^{(k)}$ with convex union and such that $K_i+2^{-k}B^n\subset P_i^{(k)}\subset K_i+2^{1-k}B^n$. Each sequence $(P_i^{(k)})_{k\in\mathbb{N}}$ is decreasing. By Theorem 3, the function φ is fully additive on $\mathcal{P}^n$, hence

$$\varphi(P_1^{(k)}\cup\dots\cup P_m^{(k)})=\sum_{\emptyset\neq J\subset\{1,\dots,m\}}(-1)^{|J|-1}\varphi(P_J^{(k)}).$$

Since

$$\bigcap_{k\in\mathbb{N}}\left(P_1^{(k)}\cup\dots\cup P_m^{(k)}\right)=K_1\cup\dots\cup K_m$$

and

$$\bigcap_{k\in\mathbb{N}}P_J^{(k)}=K_J\ \text{ if }\ K_J\neq\emptyset,$$

the σ-continuity of φ yields

$$\varphi(K_1\cup\dots\cup K_m)=\sum_{\emptyset\neq J\subset\{1,\dots,m\}}(-1)^{|J|-1}\varphi(K_J).$$

Thus, φ is fully additive on $\mathcal{K}^n$. By Theorem 1, it has an additive extension to $U(\mathcal{K}^n)$. This proves Theorem 14.

The elements of the lattice $U(\mathcal{K}^n)$, which has been termed the *convex ring*, are finite unions of convex bodies and are known as *polyconvex sets*.

It seems to be unknown whether every valuation on $\mathcal{K}^n$ (without a continuity assumption) has an additive extension to $U(\mathcal{K}^n)$.

One consequence of Theorem 14 is the fact that the trivial valuation on $\mathcal{K}^n$, which is constantly equal to 1, has an additive extension to polyconvex sets. This extension is called the *Euler characteristic* and is denoted by χ, since it coincides, on this class of sets, with the equally named topological invariant. It should be mentioned that for the existence of the Euler characteristic on polyconvex sets, there is a very short and elegant proof due to Hadwiger [19]; it is reproduced in [42], Theorem 4.3.1.

Next, we point out that the polynomiality results from Section 2 can immediately be extended by continuity. Let φ be a translation invariant, continuous valuation on $\mathcal{K}^n$ with values in a topological vector space X. Then it follows from Theorem 6 that there are continuous, translation invariant valuations $\varphi_0, \dots, \varphi_n$ on $\mathcal{K}^n$, with values in X, such that φ_i is homogeneous of degree i $(i = 0, \dots, n)$ and

$$\varphi(\lambda K) = \sum_{i=0}^{n} \lambda^i \varphi_i(K) \quad \text{for } K \in \mathcal{K}^n \text{ and } \lambda \geq 0.$$

In particular, the McMullen decomposition $\varphi = \varphi_0 + \dots + \varphi_n$ holds, where each φ_i has the same properties as φ and is, moreover, homogeneous (not only rationally homogeneous) of degree i.

If φ is, in addition, homogeneous of degree m, then it follows from Theorem 7 that there is a continuous symmetric mapping $\overline{\varphi} : (\mathcal{K}^n)^m \to X$ which is translation invariant and Minkowski additive in each variable, such that

$$\begin{aligned} &\varphi(\lambda_1 K_1 + \dots + \lambda_k K_k) \\ &= \sum_{r_1,\dots,r_k=0}^{m} \binom{m}{r_1 \dots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \, \overline{\varphi}(\underbrace{K_1, \dots, K_1}_{r_1}, \dots, \underbrace{K_k, \dots, K_k}_{r_k}) \end{aligned}$$

holds for all $K_1, \dots, K_k \in \mathcal{K}^n$ and all real $\lambda_1, \dots, \lambda_k \geq 0$. Further, one can deduce that for $r \in \{1, \dots, m\}$ the mapping

$$K \mapsto \overline{\varphi}(\underbrace{K, \dots, K}_{r}, M_{r+1}, \dots, M_m), \tag{18}$$

with fixed convex bodies $M_{r+1}, \dots, M_m$, is a continuous, translation invariant valuation, which is homogeneous of degree r.

Now that we have the classical examples of valuations on convex bodies at our disposal, we can have a look at the second historical incentive for the development of the theory of valuations. This came from the early history of integral geometry. In his booklet on integral geometry, Blaschke [5], Sec. 43, asked a question, which we explain here in a modified form. For convex bodies $K, M \in \mathcal{K}^n$, consider the 'kinematic integral'

$$\psi(K,M) := \int_{G_n} \chi(K \cap gM)\,\mu(\mathrm{d}g).$$

Here μ denotes the (suitably normalized) Haar measure on the motion group G_n, and χ is the Euler characteristic, that is, $\chi(K) = 1$ for $K \in \mathcal{K}^n$ and $\chi(\emptyset) = 0$. In other words, $\psi(K,M)$ is the rigid motion invariant measure of the set of all rigid motions g for which gM intersects K. There are different approaches to the computation of $\psi(K,M)$, and the result is that

$$\psi(K,M) = \sum_{i,j=0}^{n} c_{ij} V_i(K) V_j(M) \tag{19}$$

with explicit constants c_{ij}. This throws new light on the importance of the intrinsic volumes. Blaschke investigated this formula in a slightly different context (three-dimensional polytopal complexes). Important was his observation that some formal properties of the involved functionals were essential for his proof of such formulas, namely the valuation property, rigid motion invariance and, in his case, the local boundedness. He claimed that these properties characterize, 'to a certain extent', the linear combinations of intrinsic volumes. He proved a result in this direction, where, however, he had to introduce an additional assumption in the course of the proof, namely the invariance under volume preserving affine transformations for the 'volume part' of his considered functional. Whether a characterization theorem for valuations on polyhedra satisfying Blaschke's original conditions is possible, seems to be unknown. Later, Hadwiger considered valuations on general convex bodies and introduced the assumption of continuity. The following is his celebrated characterization theorem.

Theorem 15. (Hadwiger's characterization theorem) *If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is a continuous and rigid motion invariant valuation, then there are constants $c_0, \dots, c_n$ such that*

$$\varphi(K) = \sum_{j=0}^{n} c_j V_j(K)$$

for all $K \in \mathcal{K}^n$.

For the three-dimensional case, Hadwiger gave a proof in [16], and for general dimensions in [17]; his proof is also found in his book [21], Sec. 6.1.10. Hadwiger expressed repeatedly ([14], p. 346, and [16], footnote 3 on p. 69) that a characterization theorem for the intrinsic volumes with the assumption of local boundedness instead of continuity would be desirable. However, the following counterexample, given in [34], p. 239, shows that this is not possible. For $K \in \mathcal{K}^n$, let

$$\varphi(K) := \sum_{u \in \mathbb{S}^{n-1}} \mathcal{H}^{n-1}(F(K,u)),$$

where $F(K,u)$ is the support set of K with outer normal vector u. This has non-zero $\mathcal{H}^{n-1}$ measure for at most countably many vectors u, hence the sum is well-defined, and its value is bounded by the surface area of K. Thus, φ is a rigid motion invariant valuation which is locally bounded, but it is not continuous and hence not a linear combination of intrinsic volumes.

Hadwiger showed in [14], [20] how his theorem immediately leads to integral-geometric results. For instance, to prove (19), one notes that for fixed K the function $\psi(K,\cdot)$ satifies the assumptions of Theorem 15 and hence is a linear combination of the intrinsic volumes of the variable convex body, with real constants that are independent of this body, thus $\psi(K,M) = \sum_{j=0}^{n} c_j(K)V_j(M)$. Then one repeats the argument with variable K and obtains that ψ must be of the form $\psi(K,M) = \sum_{i,j=0}^{n} c_{ij}V_i(K)V_j(M)$. The constants c_{ij} can then be determined by applying the obtained formula to balls of different radii. There are also different approaches to integral geometric formulas. For one result, however, called 'Hadwiger's general integral-geometric theorem' (it is reproduced in [43], Theorem 5.1.2), the proof via the characterization theorem is the only one known.

Hadwiger's proof of his characterization theorem used a fair amount of dissection theory of polytopes. A slightly simplified version of his proof was published by Chen [7]. A considerably shorter, elegant proof of Hadwiger's theorem is due to Klain [24]. This proof is reproduced in the book by Klain and Rota [26], which presents a neat introduction to integral geometry, with some emphasis on discrete aspects. Klain's proof is also reproduced in [42], Theorem 6.4.14.

An essential aspect of Hadwiger's characterization theorem is the fact that the real vector space spanned by the continuous, motion invariant real valuations on $\mathcal{K}^n$ has finite dimension. This is no longer true if the considered valuations are only translation invariant. We turn to these valuations, whose investigation is a central part of the theory. By Val we denote the real vector space of translation invariant, continuous real valuations on $\mathcal{K}^n$, and by Val_m the subspace of valuations that are homogeneous of degree m. The McMullen decomposition tells us that

$$\mathrm{Val} = \bigoplus_{m=0}^{n} \mathrm{Val}_m\,.$$

Further, a valuation φ (on $\mathcal{K}^n$ or $\mathcal{P}^n$) is called *even* (*odd*) if $\varphi(-K) = \varphi(K)$ (respectively, $\varphi(-K) = -\varphi(K)$) holds for all K in the domain of φ. We denote by Val^+ and Val^- the subspace of even, respectively odd, valuations in Val, and Val_m^+ and Val_m^- are the corresponding subspaces of m-homogeneous valuations. Since we can always write

$$\varphi(K) = \frac{1}{2}(\varphi(K)+\varphi(-K)) + \frac{1}{2}(\varphi(K)-\varphi(-K)),$$

we have

$$\mathrm{Val}_m = \mathrm{Val}_m^+ \oplus \mathrm{Val}_m^- .$$

It would be nice to have a simple explicit description of the valuations in each space Val_m. Only special cases are known. So it follows from the results on polytopes (Theorem 9, in particular), together with continuity, that the spaces Val_m are one-dimensional for $m = 0$ and $m = n$.

Corollary 1. *The space* Val_0 *is spanned by the Euler characteristic, and the space* Val_n *by the volume functional.*

An explicit description is also known for the elements of Val_{n-1}. The following result is due to McMullen [30].

Theorem 16. *Each* $\varphi \in \mathrm{Val}_{n-1}$ *has a representation*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, S_{n-1}(K, \mathrm{d}u) \quad \text{for } K \in \mathcal{K}^n,$$

with a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$*. This function is uniquely determined up to adding the restriction of a linear function.*

More complete results are known for simple valuations. The following result of Klain [24] was an essential step in his proof of Hadwiger's characterization theorem.

Theorem 17. (Klain's volume characterization) *If* $\varphi \in \mathrm{Val}^+$ *is simple, then* $\varphi(K) = cV_n(K)$ *for* $K \in \mathcal{K}^n$*, with some constant* c*.*

A counterpart for odd simple valuations was proved in [41] (the proof can also be found in [42], Theorem 6.4.13):

Theorem 18. *If* $\varphi \in \mathrm{Val}^-$ *is simple, then*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} g(u)\, S_{n-1}(K, \mathrm{d}u) \quad \text{for } K \in \mathcal{K}^n,$$

with an odd continuous function $g : \mathbb{S}^{n-1} \to \mathbb{R}$*.*

A different approach to Theorems 16 and 18 was provided by Kusejko and Parapatits [27].

Klain's volume characterization (Theorem 17) has a consequence for even valuations, which has turned out to be quite useful. By $G(n, m)$ we denote the Grassmannian of m-dimensional linear subspaces of $\mathbb{R}^n$. Now let $m \in \{1, \dots, n-1\}$, and let $\varphi \in \mathrm{Val}_m$. Let $L \in G(n, m)$. It follows from Corollary 1 that the restriction of φ to the convex bodies in L is a constant multiple of the m-dimensional volume. Thus, $\varphi(K) = c_\varphi(L)V_m(K)$ for the convex bodies $K \subset L$, where $c_\varphi(L)$ is a real constant. Since φ is continuous, this defines a continuous function c_φ on $G(n, m)$. It is called the *Klain function* of the valuation φ. This function determines even valuations uniquely, as Klain [25] has deduced from his volume characterization.

Theorem 19. *A valuation in* Val_m^+ $(m \in \{1, \dots, n-1\})$ *is uniquely determined by its Klain function.*

Klain's proofs of Theorems 17 and 19 are reproduced in [42], Theorems 6.4.10 and 6.4.11.

5 Measure-valued Valuations

We leave the translation invariant, real valuations and turn to some natural extensions of the intrinsic volumes. We have already seen the measure-valued localizations of the intrinsic volumes, the support, curvature, and area measures. Another natural extension (in the next chapter) will be that from real-valued to vector- and tensor-valued functions. In both cases, invariance (or rather, equivariance) properties with respect to the group of rigid motions play an important role.

First we recall that with each convex body $K \in \mathcal{K}^n$ we have associated its support measures

$$\Lambda_0(K, \cdot), \dots, \Lambda_{n-1}(K, \cdot)$$

and, by marginalization and renormalization, the curvature measures $C_j(K, \cdot)$ and the area measures $S_j(K, \cdot)$, $j = 0, \dots, n-1$. Each mapping $K \mapsto \Lambda_j(K, \cdot)$ is a valuation, with values in the vector space of finite signed Borel measures on $\Sigma^n = \mathbb{R}^n \times \mathbb{S}^{n-1}$, and it is weakly continuous. The latter means that $K_i \to K$ in the Hausdorff metric implies $\Lambda_j(K_i, \cdot) \xrightarrow{w} \Lambda_j(K, \cdot)$, where the weak convergence $\xrightarrow{w}$ is equivalent to

$$\lim_{i \to \infty} \int_{\Sigma^n} f \, \mathrm{d}\Lambda_j(K_i, \cdot) = \int_{\Sigma^n} f \, \mathrm{d}\Lambda_j(K, \cdot)$$

for every continuous function $f : \Sigma^n \to \mathbb{R}$. The measure $\Lambda_j(K, \cdot)$ is concentrated on the normal bundle $\mathbf{nc}(K)$ of K. Valuation property and weak continuity carry over to the mappings C_j and S_j. The measure $C_j(K, \cdot)$ is a Borel measure on $\mathbb{R}^n$, concentrated on $\mathrm{bd}\, K$ for $j \leq n-1$. The area measure $S_j(K, \cdot)$ is a Borel measure on the unit sphere $\mathbb{S}^{n-1}$.

The behaviour of these measures under the motion group is as follows. First, if $g \in G_n$, we denote the rotation part of g by g_0 (that is, $gx = g_0 x + t$ for all $x \in \mathbb{R}^n$, with a fixed translation vector t). Then we define

$$\begin{aligned} g\eta &:= \{(gx, g_0 u) : (x, u) \in \eta\} \quad \text{for } \eta \subset \Sigma^n, \\ g\beta &:= \{gx : x \in \beta\} \quad \text{for } \beta \subset \mathbb{R}^n, \\ g\omega &:= \{g_0 u : u \in \omega\} \quad \text{for } \omega \subset \mathbb{S}^{n-1}. \end{aligned}$$

For $K \in \mathcal{K}^n$, $g \in G_n$ and Borel sets $\eta \subset \Sigma^n$, $\beta \subset \mathbb{R}^n$ and $\omega \subset \mathbb{S}^{n-1}$ we then have

$$\Lambda_j(gK, g\eta) = \Lambda_j(K, \eta), \quad C_j(gK, g\beta) = C_j(K, \beta), \quad S_j(gK, g\omega) = S_j(K, \omega).$$

In each case, we talk of this behaviour as *rigid motion equivariance*.

One may ask whether, for these measure-valued extensions of the intrinsic volumes, there are classification results similar to Hadwiger's characterization theorem. It turns out that in addition to the valuation, equivariance, and continuity properties we need, because we are dealing with measures, some assumption of local determination, saying roughly that the value of the considered measure of K at a Borel set α depends only on a local part of K determined by α. With an appropriate assumption of this kind, the following characterization theorems have been obtained. If $\varphi(K)$ is a measure, we write here $\varphi(K)(\alpha) =: \varphi(K, \alpha)$.

Theorem 20. *Let φ be a map from $\mathcal{K}^n$ into the set of finite Borel measures on $\mathbb{R}^n$, satisfying the following conditions.*

(a) *φ is a valuation;*

(b) *φ is rigid motion equivariant;*

(c) *φ is weakly continuous;*

(d) *φ is locally determined, in the following sense: if $\beta \subset \mathbb{R}^n$ is open and $K \cap \beta = L \cap \beta$, then $\varphi(K, \beta') = \varphi(L, \beta')$ for every Borel set $\beta' \subset \beta$.*

Then there are real constants $c_0, \dots, c_n \geq 0$ such that

$$\varphi(K, \beta) = \sum_{i=0}^{n} c_i C_i(K, \beta)$$

for $K \in \mathcal{K}^n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$.

In the following theorem, $\tau(K, \omega)$ denotes the inverse spherical image of K at ω, that is, the set of all boundary points of the convex body K at which there is an outer normal vector belonging to the given set $\omega \subset \mathbb{S}^{n-1}$.

Theorem 21. *Let φ be a map from $\mathcal{K}^n$ into the set of finite signed Borel measures on $\mathbb{S}^{n-1}$, satisfying the following conditions.*

(a) *φ is a valuation;*

(b) *φ is rigid motion equivariant;*

(c) *φ is weakly continuous;*

(d) *φ is locally determined, in the following sense: if $\omega \subset \mathbb{S}^{n-1}$ is a Borel set and if $\tau(K, \omega) = \tau(L, \omega)$, then $\varphi(K, \omega) = \varphi(L, \omega)$.*

Then there are real constants $c_0, \dots, c_{n-1}$ such that

$$\varphi(K, \omega) = \sum_{i=0}^{n-1} c_i S_i(K, \omega)$$

for $K \in \mathcal{K}^n$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$.

Theorem 20 was proved in [40] and Theorem 21 in [39]. The following result is due to Glasauer [10].

Theorem 22. *Let φ be a map from $\mathcal{P}^n$ into the set of finite signed Borel measures on Σ^n, satisfying the following conditions.*

(a) *φ is rigid motion equivariant;*

(b) *φ is locally determined, in the following sense: if $\eta \in \mathcal{B}(\Sigma^n)$ and $K, L \in \mathcal{K}^n$ satisfy $\eta \cap \mathbf{nc}(K) = \eta \cap \mathbf{nc}(L)$, then $\varphi(K, \eta) = \varphi(L, \eta)$.*

Then there are real constants $c_0, \dots, c_{n-1}$ such that

$$\varphi(K, \eta) = \sum_{j=0}^{n-1} c_j \Lambda_j(K, \eta)$$

for $K \in \mathcal{K}^n$ and $\eta \in \mathcal{B}(\Sigma^n)$.

Here the valuation property has not been forgotten! Indeed, the last theorem has a character different from the two previous ones: the assumption that $\varphi(K, \cdot)$ is a locally determined measure on Σ^n, is strong enough to allow a simpler proof, without assuming the valuation property. The latter point will be important in the treatment of local tensor valuations (in Chap. 2).

6 Valuations on Lattice Polytopes

We denote by $\mathcal{P}(\mathbb{Z}^n)$ the set of all polytopes with vertices in $\mathbb{Z}^n$. In contrast to $\mathcal{P}^n$ and $\mathcal{K}^n$ considered so far, $\mathcal{P}(\mathbb{Z}^n)$ is not an intersectional family. For that reason, we modify the definition of a valuation in this case and say that a mapping φ from $\mathcal{P}(\mathbb{Z}^n)$ into some abelian group is a valuation if

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q) \tag{20}$$

holds whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\mathbb{Z}^n)$; moreover, we define that $\emptyset \in \mathcal{P}(\mathbb{Z}^n)$ and assume that $\varphi(\emptyset) = 0$. In a similar vein, we say that φ satisfies the *inclusion-exclusion principle* if

$$\varphi(A_1 \cup \dots \cup A_m) = \sum_{\emptyset \neq J \subset \{1,\dots,m\}} (-1)^{|J|-1} \varphi(A_J)$$

holds whenever $m \in \mathbb{N}$, $A_1 \cup \dots \cup A_m \in \mathcal{P}(\mathbb{Z}^n)$ and $A_J \in \mathcal{P}(\mathbb{Z}^n)$ for all nonempty $J \subset \{1, \dots, m\}$. Further, a valuation φ on $\mathcal{P}(\mathbb{Z}^n)$ is said to have the *extension property* if there is a function $\widetilde{\varphi}$ on the family of finite unions of polytopes in $\mathcal{P}(\mathbb{Z}^n)$ such that

$$\widetilde{\varphi}(A_1 \cup \dots \cup A_m) = \sum_{\emptyset \neq J \subset \{1,\dots,m\}} (-1)^{J|-1} \varphi(A_J)$$

whenever $A_J \in \mathcal{P}(\mathbb{Z}^n)$ for all nonempty $J \subset \{1, \ldots, m\}$. The following theorem was proved by McMullen [33].

Theorem 23. *A valuation on $\mathcal{P}(\mathbb{Z}^n)$ satisfies the inclusion-exclusion principle and has the extension property.*

For polytopes in $\mathcal{P}(\mathbb{Z}^n)$, the natural counterpart to the volume functional is the *lattice point enumerator* G. It is defined by

$$G(P) := \mathrm{card}(P \cap \mathbb{Z}^n) \quad \text{for} P \in \mathcal{P}(\mathbb{Z}^n).$$

It was first proved by Ehrhart [9] that there is a polynomial expansion

$$G(kP) = \sum_{i=0}^{n} k^i G_i(P), \quad P \in \mathcal{P}(\mathbb{Z}^n),\ k \in \mathbb{N}. \tag{21}$$

We refer to the surveys [34], [32] for information about how this fact embeds into the general polynomiality theorems proved later.

The expansion (21) defines valuations $G_0, \ldots, G_n (= G)$ on $\mathcal{P}(\mathbb{Z}^n)$, which are invariant under unimodular transformations, that is, volume preserving affine maps of $\mathbb{R}^n$ into itself that leave $\mathbb{Z}^n$ invariant. A result of Betke [2] (see also Betke and Kneser [3]), together with Theorem 23, gives the following characterization theorem.

Theorem 24. *If φ is a real valuation on $\mathcal{P}(\mathbb{Z}^n)$ which is invariant under unimodular transformations, then*

$$\varphi(P) = \sum_{i=0}^{n} c_i G_i(P) \quad \textit{for } P \in \mathcal{P}(\mathbb{Z}^n),$$

with real constants $c_0, \ldots, c_n$.

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