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VERIFICATION OF POLYTOPES BY BRIGHTNESS FUNCTIONS

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Abstract. We show that in the class of origin centered convex bodies in Euclidean space of dimension at least three, a polytope is uniquely determined by its brigthness function in a suitably chosen, but very small set of directions.

1. Introduction and result

Aleksandrov's projection theorem (see, e.g., Gardner [1, Th. 3.3.6]) is one of the classical and central results of geometric tomography. In its simplest version, it can be formulated as follows. In d-dimensional Euclidean space $\mathbb{R}^d$ (we assume $d \geq 3$), let K be a convex body (a compact convex set with interior points, in this note). For $u \in S^{d-1}$ (the unit sphere), we denote the hyperplane through 0 orthogonal to u by $u^\perp$ and the orthogonal projection to $u^\perp$ by $\cdot|u^\perp$. The function $u \mapsto V_{d-1}(K|u^\perp)$, where V_{d-1} denotes the $(d-1)$-dimensional volume, is known as the *brightness function* of K. The body K is 0*-symmetric* (or origin centered) if $K = -K$. Aleksandrov's projection theorem says that *two* 0*-symmetric convex bodies with the same brightness function are identical.*

It is well known that in this theorem the assumption of central symmetry cannot be deleted; in the following, K and L are always 0-symmetric. It is also known that the equality of the brightness functions in all directions cannot be essentially relaxed. For a precise formulation, we define a *direction set* as a 0-symmetric closed subset of S^{d-1}. If the direction set A is a proper subset of S^{d-1}, then for any sufficiently smooth 0-symmetric convex body K there exists a 0-symmetric convex body L with

$$V_{d-1}(K|u^\perp) = V_{d-1}(L|u^\perp) \qquad \text{for all } u \in A, \tag{1.1}$$

but $L \neq K$. Examples were constructed in [5]. There have been several attempts to find additional assumptions on K and L under which smaller sets A in (1.1) still allow the conclusion that $K = L$. For example, this holds if K is a polytope and A is a neighborhood of an equator subsphere (Schneider [3]). As shown by Schneider and Weil [5], it also holds if the dimension d is odd, A is a neighborhood of an equator subsphere with pole e, and the supporting hyperplanes of K and L with normal vector e touch each of K and L at a vertex. Results of a different kind were obtained by Grinberg and Quinto [2], for example the following. Let K and L be of class C^2_+. Let A be an open connected subset of S^{d-1} such that $\mathbb{R}^d = \bigcup_{u\in A} u^\perp$.

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Assume that for some $e \in A$, the products of the principal radii of curvature of K and L agree to infinite order along the equator subsphere $e^\perp \cap S^{d-1}$. If (1.1) holds, then $K = L$.

Note that in each of the previous results, the direction set A has to be of positive (spherical Lebesgue) measure. In contrast to this, we show in the present note that 0-symmetric polytopes can be verified by their brightness function in suitable direction sets of measure zero. Here, we have adopted the terminology used by Gardner [1] (in the case of X-rays): we say that the convex body K can be *verified* by the brightness function in a direction set A, which may depend on K, if any 0-symmetric convex body L satisfying (1.1) is equal to K. We prove a result on the verification of general convex bodies. Recall that a vector $u \neq 0$ is an *extreme* normal vector of K if it cannot be written as $u = u_1 + u_2$ where u_1, u_2 are linearly independent normal vectors of K at the same boundary point. Let $E(K)$ denote the set of extreme unit normal vectors of K. For $e \in S^{d-1}$, let $S_e := e^\perp \cap S^{d-1}$; this is the equator subsphere with pole e.

Theorem 1.1. *Let $d \geq 3$. Let K and L be 0-symmetric convex bodies in $\mathbb{R}^d$. Let A be a direction set that contains S_e for each $e \in E(K)$ and, together with any d-dimensional cone spanned by finitely many vectors of $E(K)$, also a vector in the interior of the dual cone. If $V_{d-1}(K|u^\perp) = V_{d-1}(L|u^\perp)$ for all $u \in A$, then $K = L$.*

If K is a polytope, then $E(K)$ is the set of unit normal vectors of its facets, hence the set A in the theorem can be chosen as the union of finitely many great subspheres and a finite set, and thus is of spherical Lebesgue measure zero.

2. Proof of the theorem

Let the assumptions of the theorem be satisfied. Then $V_{d-1}(K|v^\perp) = V_{d-1}(L|v^\perp)$ if $v \perp u$ and $u \in E(K)$. Let $\overline{E}$ denote the closure of $E(K)$. It follows from the continuity of the brightness functions that $V_{d-1}(K|v^\perp) = V_{d-1}(L|v^\perp)$ holds also if $v \perp u$ and $u \in \overline{E}$. We shall make use of the fact that $\overline{E}$ is the support of the surface area measure $S_{d-1}(K, \cdot)$ of K (see [4, Th. 4.6.3]).

For standard notation from the theory of convex bodies, we refer to [4]. In particular, $h(K, \cdot)$ denotes the support function of K, $H(K, u)$ is the supporting hyperplane and $H^-(K, u)$ is the supporting halfspace of K, both with outer normal vector u. The scalar product of $\mathbb{R}^d$ is $\langle \cdot, \cdot \rangle$. By ΠK we denote the projection body of K. Its support function is given by

$$h(\Pi K, u) = V_{d-1}(K|u^\perp) = \frac{1}{2} \int_{S^{d-1}} |\langle u, v \rangle| \, S_{d-1}(K, \mathrm{d}v)$$

for $u \in S^{d-1}$.

Let $u \in \overline{E}$. Then

$$h(\Pi K, v) = h(\Pi L, v) \qquad \text{for all } v \in u^\perp. \tag{2.1}$$

For any convex body M we have $h(M|u^\perp, v) = h(M, v)$ if $v \in u^\perp$, hence (2.1) gives

$$h(\Pi K|u^\perp, v) = h(\Pi L|u^\perp, v) \qquad \text{for all } v \in u^\perp$$

and, therefore,

$$\Pi K|u^\perp = \Pi L|u^\perp. \tag{2.2}$$

It follows that the cylinder $C(u) := \Pi L + \mathrm{lin}\,\{u\}$ contains ΠK. The set

$$D := \bigcap_{u \in \overline{E}} C(u)$$

is a convex body, which satisfies $\Pi K \subset D$, $\Pi L \subset D$, and

$$h(\Pi L, v) = h(D, v) \qquad \text{for } v \in u^{\perp}, \text{ if } u \in \overline{E}. \tag{2.3}$$

Suppose that $\Pi K \neq D$. Then ΠK is a proper subset of D, hence the interior of D contains non-empty relatively open subset of the boundary of ΠK and hence a regular boundary point of ΠK. Therefore, there is a vector $w \in S^{d-1}$ such that the supporting hyperplane $H(\Pi K, w)$ contains a regular boundary point of ΠK and is not a supporting hyperplane of D, hence $h(\Pi K, w) < h(D, w)$. If $w \in u^{\perp}$ for some $u \in \overline{E}$, then (2.1) and (2.3) would imply $h(\Pi K, w) = h(\Pi L, w) = h(D, w)$, a contradiction. Therefore, $w \notin u^{\perp}$ for all $u \in \overline{E}$ and thus $w^{\perp} \cap \overline{E} = \emptyset$. Since $\overline{E}$ is closed, a whole neighborhood of the equator subsphere S_w does not meet $\overline{E}$. Thus, $S_y \cap \overline{E} = \emptyset$ for all y in a neighborhood of w.

We use a formula for support sets of zonoids. For a convex body M, let $F(M, y)$ be the support set of M with outer normal vector y. Then, for the zonoid ΠK we have (see [4, Lemma 3.5.5])

$$h(F(\Pi K, y), x) = \langle x, e_y \rangle + \frac{1}{2} \int_{S_y} |\langle x, v \rangle| \, S_{d-1}(K, \mathrm{d}v) \tag{2.4}$$

for $x \in \mathbb{R}^d$, with

$$e_y := \int_{S^{d-1}} \mathbf{1}\{\langle v, y \rangle > 0\} v \, S_{d-1}(K, \mathrm{d}v). \tag{2.5}$$

In our case, the integral in (2.4) vanishes for all y in a neighborhood of w, since $\overline{E}$ is the support of the measure $S_{d-1}(K, \cdot)$. This means that $F(\Pi K, y) = \{e_y\}$ for these y, and from (2.5) it follows that $e_y = e_w$ for y in a neighborhood of w. Hence, $H(\Pi K, w) \cap \Pi K = \{e_w\}$, and e_w is a singular point of ΠK; thus, the supporting hyperplane $H(\Pi K, w)$ does not contain a regular boundary point of ΠK. This contradiction shows that $\Pi K = D$.

From $\Pi K = D$ we get $\Pi L \subset \Pi K$, and by the monotonicity of mixed volumes this implies

$$V(\Pi K, \Pi L, \dots, \Pi L) \leq V(\Pi K, \Pi K, \Pi L, \dots, \Pi L) \leq \cdots \leq V(\Pi K, \dots, \Pi K), \tag{2.6}$$

where $V(\cdot, \dots, \cdot)$ is the mixed volume.

If M is a convex body, then, using a well-known representation of mixed volumes together with Fubini's theorem, we get (all integrals are over S^{d-1})

$$\begin{aligned}
V(\Pi K, M, \dots, M) &= \frac{1}{d} \int h(\Pi K, v) \, S_{d-1}(M, \mathrm{d}v) \\
&= \frac{1}{d} \int \frac{1}{2} \int |\langle u, v \rangle| \, S_{d-1}(K, \mathrm{d}u) \, S_{d-1}(M, \mathrm{d}v) \\
&= \frac{1}{d} \int \frac{1}{2} \int |\langle u, v \rangle| \, S_{d-1}(M, \mathrm{d}v) \, S_{d-1}(K, \mathrm{d}u) \\
&= \frac{1}{d} \int V_{d-1}(M|u^{\perp}) \, S_{d-1}(K, \mathrm{d}u).
\end{aligned}$$

If $u \in \overline{E}$, then $V_{d-1}(\Pi K|u^\perp) = V_{d-1}(\Pi L|u^\perp)$ by (2.2). Since this holds for all u in the support of the measure $S_{d-1}(K,\cdot)$, we get

$$V(\Pi K, \Pi L, \dots, \Pi L) = V(\Pi K, \dots, \Pi K).$$

By (2.6), this implies, in particular, that

$$V(\Pi K, \Pi L, \dots, \Pi L) = V(\Pi K, \Pi K, \Pi L, \dots, \Pi L).$$

By [4, Th. 6.6.16], this is only possible if ΠK is a 1-tangential body of ΠL. A 1-tangential body is a cap body (see [4, p. 76]); hence ΠK is the convex hull of ΠL and a (possibly empty) set X of points not in ΠL such that any segment joining two of these points meets ΠL. If $X = \emptyset$, then $\Pi K = \Pi L$. Since K and L are centrally symmetric with respect to 0, Aleksandrov's projection theorem yields $K = L$. Therefore, it remains to consider the case where $X \neq \emptyset$. (Note that a zonoid may well be a cap body of another zonoid. For example, a rhombic dodecahedron is a cap body of a cube. Therefore, we do not immediately get a contradiction. It would be interesting to classify all pairs of zonoids where one is a cap body of the other.)

Let $p \in X$. Let C_p denote the cone with apex p spanned by ΠK. Since $p \notin \Pi L$, there is a hyperplane H that strictly separates p and ΠL. It intersects the cone C_p in a $(d-1)$-dimensional convex body Q. Let x be an exposed point of Q. The halfline with endpoint p through x is an exposed ray of C_p, hence there is a supporting hyperplane of ΠK through p that intersects ΠK in a nondegenerate line segment S_x; thus $F(\Pi K, w) = S_x$ for a suitable unit vector w. Let u be a unit vector parallel to S_x. Since $F(\Pi K, w)$ is a segment of direction u, it follows from (2.4) (together with the uniqueness theorem [4, Th. 3.5.3]) that the measure $S_{d-1}(K,\cdot)$ has point masses at $\pm u$. Therefore, the support sets $F(K, \pm u)$ of K are of dimension $d-1$, which implies that $u \in E(K)$. To each exposed point x of Q there corresponds such a segment S_x. It is a summand of ΠK (by [4, Cor. 3.5.6], every support set of a zonoid is a summand of the zonoid). Since all the segments S_x have different directions and their lengths are bounded from below by a positive constant, there can only be finitely many such segments, since otherwise their sum would be unbounded. Thus, the cone $C_p - p$ is the positive hull of finitely many vectors from $E(K)$. By assumption, the interior of its dual cone contains a vector $v \in A$, and we have $F(\Pi K, v) = \{p\}$ and, therefore, $h(\Pi K, v) > h(\Pi L, v)$. On the other hand, the assumptions of the theorem give $V_{d-1}(K|v^\perp) = V_{d-1}(L|v^\perp)$ and thus $h(\Pi K, v) = h(\Pi L, v)$. This contradiction shows that the case $X \neq \emptyset$ cannot occur, which completes the proof.

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