On a formula for the volume of polytopes

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Abstract

We carry out an elementary proof of a formula for the volume of polytopes, due to A. Esterov, from which it follows that the mixed volume of polytopes depends only on the product of their support functions.

1 Introduction

Esterov [1] has proved the surprising fact that the mixed volume of n convex polytopes in $\mathbb{R}^n$ depends only on the product of their support functions. That an extension of this result to general convex bodies is not true, was pointed out by Kazarnovskiĭ [2, Remark 2]. Esterov deduced his result from a new formula for the volume of a polytope in terms of the nth power of its support function. It is the purpose of this note to carry out an elementary proof of this formula. A motivation will be given after we have stated this formula, in the next section.

2 Formulation of the result

First we fix some terminology. We work in n-dimensional Euclidean space $\mathbb{R}^n$ with scalar product $\langle \cdot\,, \cdot \rangle$. Its unit sphere is denoted by $\mathbb{S}^{n-1}$. Polytopes are always nonempty, compact, and convex. The volume of a polytope P is denoted by $V_n(P)$. A polyhedral cone is the intersection of a finite family of closed halfspaces with the origin o in their boundaries; equivalently, it is the positive hull of a finite set of vectors. The positive hull of a basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ is a *simple cone* and is denoted by $<v_1, \dots, v_n>$; this cone is said to be *generated* by $v_1, \dots, v_n$. A *fan* in $\mathbb{R}^n$ is a family $\mathcal{F}$ of polyhedral cones with the following properties: every face of a cone in $\mathcal{F}$ is a cone in $\mathcal{F}$, and the intersection of two cones in $\mathcal{F}$ is a face of both. A fan is called *simple* if all its n-dimensional cones are simple. A fan $\mathcal{F}'$ is a *refinement* of the fan $\mathcal{F}$ if every cone of $\mathcal{F}'$ is contained in a cone of $\mathcal{F}$. For a polytope P and a (nonempty) face F of P, we denote by $N(P,F)$ the normal cone of P at F. The family of all normal cones of P at its faces is a fan, called the *normal fan* of P.

If $(v_1, \dots, v_n)$ is an ordered basis of $\mathbb{R}^n$, we denote by $(v_1^\perp, \dots, v_n^\perp)$ its Gram–Schmidt orthonormalization. This means that $(v_1^\perp, \dots, v_n^\perp)$ is orthonormal, the set $\{v_1^\perp, \dots, v_k^\perp\}$ spans the same subspace as $\{v_1, \dots, v_k\}$, and $\langle v_k, v_k^\perp \rangle > 0$, for $k = 1, \dots, n$.

The following is a special case of a more general result of Esterov [1] (with a corrected factor).

Theorem 1. (Esterov) *Let $P \subset \mathbb{R}^n$ be a polytope, and let the fan Γ be a simple refinement of the normal fan of P. Let $\mathcal{B}(\Gamma)$ be the set of all ordered n-tuples of unit vectors generating cones of Γ. For each n-dimensional cone $C \in \Gamma$, let f_C be the restriction of the nth power of the support function of P to the interior of C. Then*

$$\frac{1}{(n!)^2} \sum_{(v_1,\dots,v_n)\in\mathcal{B}(\Gamma)} \frac{\partial^n f_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} = V_n(P). \tag{1}$$

Esterov writes about his result: "We ... represent it as a specialization of the isomorphism between two well known combinatorial models of the cohomology of toric varieties." He also gives a brief sketch of an elementary proof, however, for the polytope A under consideration, "assuming for simplicity that the orthogonal complement to the affine span of every (relatively open) face $B \subset A$ intersects B". This is too much of a simplification, since the construction becomes non-trivial if this assumption is not satisfied. Moreover, the statement about the subdivision into "simplices that are in one to one correspondence with the terms of the sum" (the sum in (1) is meant) is not correct, since the simplices depend only on the polytope P, whereas the sum gets more terms if the fan Γ is refined. That these extra terms (which are in general not zero) add up to zero, requires an additional argument. The author's statement, "Independence of subdivisions of Γ and linearity follow by definition", seems unjustified.

Since Esterov's surprising result has never been observed in the development of the classical theory of mixed volumes, it might be desirable to have a complete proof along classical lines. Therefore, in the following we carry out Esterov's brief sketch with the necessary details.

3 Proof of Theorem 1

First we recall the notion of an *orthoscheme* Δ in $\mathbb{R}^n$. Given a base point z (which will later be the origin, but is needed in greater generality for an induction argument), an ordered orthonormal basis $(u_1, \dots, u_n)$ of $\mathbb{R}^n$, and a sequence $(a_1, \dots, a_n)$ of real numbers, it is defined by

$$\Delta := \mathrm{conv}\{z, z_1, \dots, z_n\}, \quad z_k := z + \sum_{i=1}^{k} a_i u_i \text{ for } k = 1, \dots, n.$$

This is a simplex, but it may be degenerate, since $a_i = 0$ is allowed.

Let $P \subset \mathbb{R}^n$ be a polytope with interior points. Under the special assumption in Esterov's proof sketch quoted above, P can be decomposed into orthoschemes. Without this assumption, one has to decompose the indicator function of P into the indicator functions of signed orthoschemes.

To prepare this, we denote by $H(Q, u)$ the supporting hyperplane of a polytope $Q \subset \mathbb{R}^n$ with outer normal vector $u \in \mathbb{R}^n \setminus \{o\}$ (for notions from convex geometry that are not explained here, we refer to [3]). By $H^-(Q, u), H^+(Q, u)$ we denote the two closed halfspaces bounded by $H(Q, u)$, where $H^-(Q, u)$ contains Q. In the following, for a point $p \in \mathbb{R}^n$ and a hyperplane $H \subset \mathbb{R}^n$, we denote by $p|H$ the image of p under orthogonal projection to H.

Let $(v_1, \dots, v_n)$ be a basis of $\mathbb{R}^n$ such that $v_1, \dots, v_n \in N(P, \{y\})$ for some vertex y of P. Let $(v_1^\perp, \dots, v_n^\perp)$ be the Gram–Schmidt orthonormalization of $(v_1, \dots, v_n)$. We define

$$S_1 := P \cap H(P, v_1^\perp), \quad S_2 := S_1 \cap H(S_1, v_2^\perp), \quad \dots \quad S_n := S_{n-1} \cap H(S_{n-1}, v_n^\perp).$$

Then $\dim S_k \le n-k$, so that $S_n = \{y\}$ for the vertex y of P, and $S_k \supseteq S_{k+1}$ for $k = 1, \dots, n-1$ (equality may hold). We say that $(S_1, \dots, S_n)$ is *generated by* $(v_1, \dots, v_n)$. A sequence $(S_1, \dots, S_n)$ of faces of P is called a *complete tower* of P if $S_1 \supset S_2 \supset \cdots \supset S_n$ and $\dim S_k = n - k$ for $k = 1, \dots, n$.

Now we define the required orthoschemes. Let $z \in \mathbb{R}^n$. Given $(v_1, \dots, v_n)$ as above and its generated sequence $(S_1, \dots, S_n)$ (so that $S_n = \{y\}$), we define a sequence of points by

$$z_1 := z|H(P, v_1^\perp), \quad z_2 := z_1|H(S_1, v_2^\perp), \quad \dots \quad z_n := z_{n-1}|H(S_{n-1}, v_n^\perp).$$

We also define a sequence $(a_1, \dots, a_n)$ of numbers by

$$z_1 = z + a_1 v_1^\perp, \quad z_2 := z_1 + a_2 v_2^\perp, \quad \dots \quad z_n = z_{n-1} + a_n v_n^\perp.$$

Here $z_n = y$ and hence $y = z + a_1 v_1^\perp + \dots + a_n v_n^\perp$. Therefore,

$$a_i = \langle y - z, v_i^\perp \rangle. \tag{2}$$

We define the orthoscheme

$$\Delta := \mathrm{conv}\{z, z_1, \dots, z_n\}.$$

Its volume is given by

$$V_n(\Delta) = \frac{1}{n!}|a_1 \cdots a_n|. \tag{3}$$

We denote by $\delta \in \{-1, 0, 1\}$ the sign of $a_1 \cdots a_n$ and call the pair (Δ, δ) a *signed orthoscheme*. It is said to be *induced by* $(v_1, \dots, v_n)$ (if P and z are given).

We apply this construction from two different starting points. First, we start from the polytope P and associate a signed orthoscheme with each of its complete towers. Let $(S_1, \dots, S_n)$ be a complete tower of P. Then there is a unique ordered orthonormal basis $(u_1, \dots, u_n)$ of $\mathbb{R}^n$ such that

$$S_1 := P \cap H(P, u_1), \quad S_2 := S_1 \cap H(S_1, u_2), \quad \dots \quad S_n := S_{n-1} \cap H(S_{n-1}, u_n). \tag{4}$$

We call $(u_1, \dots, u_n)$ the orthonormal basis *associated with* the complete tower $(S_1, \dots, S_n)$ of P. Let (Δ, δ) be the signed orthoscheme induced by $(u_1, \dots, u_n)$. It is also said to be the signed orthoscheme *induced by the complete tower* $(S_1, \dots, S_n)$.

Definition. For given P and z, we denote by $\mathcal{O}(P, z)$ the set of all signed orthoschemes induced by complete towers of P.

Let U be the union of the affine hulls of the facets of all orthoschemes Δ, for $(\Delta, \delta) \in \mathcal{O}(P, z)$. Denoting the indicator function of a set $A \subset \mathbb{R}^n$ by $\mathbb{1}_A$, we state the following

Proposition 1.

$$\sum_{(\Delta,\delta)\in\mathcal{O}(P,z)} \delta \mathbb{1}_\Delta(x) = \mathbb{1}_P(x) \quad \textit{for all } x \in \mathbb{R}^n \setminus U. \tag{5}$$

We set

$$g_n(P, z, x) := \sum_{(\Delta,\delta)\in\mathcal{O}(P,z)} \delta \mathbb{1}_\Delta(x)$$

and prove (5) by induction with respect to the dimension. The case $n = 1$ is clear. We assume that $n \geq 2$ and that the assertion has been proved in smaller dimensions, for all polytopes and base points. Let P be an n-polytope and z a point in $\mathbb{R}^n$. Let $(\Delta, \delta) \in \mathcal{O}(P, z)$. Then (Δ, δ) is induced by some complete tower $(S_1, \dots, S_n)$ of P. If $F_1, \dots, F_m$ are the facets of P, then $S_1 = F_i$ for some $i \in \{1, \dots, m\}$, and $(S_2, \dots, S_n)$ is a complete tower of F_1. We have $\Delta = \mathrm{conv}(\Delta' \cup \{z\})$ with some $(\Delta', \delta') \in \mathcal{O}(F_i, z|\mathrm{aff}\, F_i)$ and $\delta = \delta' \sigma(F_i, z)$, where we define, for a facet F of P with outer unit normal vector u,

$$\sigma(F, z) := \begin{cases} 1 & \text{if } z \in \mathrm{int}\, H^-(P, u), \\ 0 & \text{if } z \in H(P, u), \\ -1 & \text{if } z \in \mathrm{int}\, H^+(P, u). \end{cases}$$

For $x \in \mathbb{R}^n \setminus \{z\}$ we define the ray

$$R(z,x) := \{x + \lambda(x-z) : \lambda \geq 0\}.$$

Let $x \in \mathbb{R}^n \setminus U$. Then $x \neq z$. If $i \in \{1,\dots,k\}$ and if the ray $R(z,x)$ meets $\operatorname{aff} F_i$, we denote the intersection point by $q(x,F_i)$. Clearly,

$$x \in \Delta \Leftrightarrow R(z,x) \text{ meets aff } F_i \text{ and } q(x,F_i) \in \Delta',$$

and if this holds, then $\delta = \delta'\sigma(F_i,z)$. Thus $\mathbb{1}_\Delta(x) = \mathbb{1}_{\Delta'}(q(x,F_i))$. This gives

$$\begin{aligned} g_n(P,z,x) &= \sum_{(\Delta,\delta)\in\mathcal{O}(P,z)} \delta \mathbb{1}_\Delta(x) \\ &= \sum_{i=1}^m \sigma(F_i,z) \sum_{(\Delta',\delta')\in\mathcal{O}(F_i,z|\mathrm{affF_i})} \delta' \mathbb{1}_{\Delta'}(q(x,F_i)) \\ &= \sum_{i=1}^m \sigma(F_i,z) g_{n-1}(F_i, z|\operatorname{aff} F_i, q(x,F_i)), \end{aligned}$$

where we put $g_{n-1}(F_i, z|\operatorname{aff} F_i, q(x,F_i)) := 0$ if the ray $R(z,x)$ does not meet $\operatorname{aff} F_i$. This yields

$$\begin{aligned} & g_n(P,z,x) \\ &= \sum_{i=1}^m \sigma(F_i,z)[\mathbb{1}\{q(x,F_i)\in F_i\} g_{n-1}(F_i,z|\operatorname{aff} F_i, q(x,F_i)) \\ &\quad + \mathbb{1}\{q(x,F_i)\notin F_i\} g_{n-1}(F_i,z|\operatorname{aff} F_i, q(x,F_i))] \\ &= \sum_{i=1}^m \sigma(F_i,z) \mathbb{1}\{q(x,F_i)\in F_i\}, \end{aligned}$$

where the induction hypothesis was applied to g_{n-1}. This is possible, since the point $q(x,F_i)$ is not contained in the union of the affine hulls of the $(n-2)$-faces of the orthoschemes Δ', $(\Delta',\delta') \in \mathcal{O}(F_i, z|\operatorname{aff} F_i)$, $i = 1,\dots,m$. If $x \in P$, there is exactly one index $i \in \{1,\dots,k\}$ with $q(x,F_i) \in F_i$, and we have $\sigma(F_i,z) = 1$. Therefore, $g_n(P,z,x) = 1$. If $x \notin P$ and some point $q(F_i,x) \in F_i$ exists, then precisely one other index j exists with $q(F_j,x) \in F_j$, and we have $\sigma(F_i,z) = -\sigma(F_j,z)$. This gives $g_n(P,z,x) = 0$. We have proved (5).

Integrating (5) with respect to Lebesgue measure, we obtain

$$V_n(P) = \sum_{(\Delta,\delta)\in\mathcal{O}(P,z)} \delta V_n(\Delta). \tag{6}$$

Our second starting point for constructing signed orthoschemes is the fan Γ. Let (Δ,δ) be the signed orthoscheme induced by $(v_1,\dots,v_n) \in \mathcal{B}(\Gamma)$, where $<v_1,\dots,v_n> \subseteq N(P,\{y\})$ for a vertex y of P. To express the volume of Δ in a suitable way, we note that the restriction of the support function of P to $<v_1,\dots,v_n>$ is given by $u \mapsto \langle y,u\rangle$, hence

$$f_{<v_1,\dots,v_n>}(u) = \langle y,u\rangle^n \quad \text{for } u \in \operatorname{int}<v_1,\dots,v_n>.$$

Writing

$$u = \alpha_1 v_1^\perp + \cdots + \alpha_n v_n^\perp,$$

we have

$$f_{<v_1,\ldots,v_n>}(u) = \Big(\alpha_1\langle v_1^\perp, y\rangle + \cdots + \alpha_n\langle v_n^\perp, y\rangle\Big)^n$$

and therefore

$$\frac{\partial f_{<v_1,\ldots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} = n!\langle v_1^\perp, y\rangle \cdots \langle v_n^\perp, y\rangle = n! a_1 \cdots a_n = (n!)^2 \delta V_n(\Delta) \tag{7}$$

by (3), where the numbers $a_1, \ldots, a_n$ are those defined by (2) with $z = o$.

To utilize this, we need to know which $(v_1, \ldots, v_n) \in \mathcal{B}(\Gamma)$ induce signed orthoschemes from $\mathcal{O}(P, o)$. We introduce the following definition.

Definition. Let $(v_1, \ldots, v_n)$ be an ordered basis of $\mathbb{R}^n$ such that $v_1, \ldots, v_n$ are contained in a polyhedral cone C. Then $(v_1, \ldots, v_n)$ is called *adapted to* C if there is a sequence $T_1 \subset T_2 \subset \cdots \subset T_n$ where T_k is a k-face of C and $v_1 \in T_1$, $v_k \in T_k \setminus T_{k-1}$ for $k = 2, \ldots, n$.

An ordered basis $(v_1, \ldots, v_n) \in \mathcal{B}(\Gamma)$ is called *tidy* if it is adapted to the normal cone $N(P, \{y\})$ containing $v_1, \ldots, v_n$. The set of all tidy ordered bases $(v_1, \ldots, v_n) \in \mathcal{B}(\Gamma)$ is denoted by $\mathcal{T}$, and we set $\mathcal{B}(\Gamma) \setminus \mathcal{T} =: \mathcal{U}$.

From now on, the point $z \in \mathbb{R}^n$ chosen earlier is the origin o.

Proposition 2. *The signed orthoscheme induced by $(v_1, \ldots, v_n) \in \mathcal{B}(\Gamma)$ belongs to $\mathcal{O}(P, o)$ if and only if $(v_1, \ldots, v_n)$ is tidy. Every signed orthoscheme from $\mathcal{O}(P, o)$ is induced by a unique element of $\mathcal{B}(\Gamma)$.*

For the proof, we assume first that $(v_1, \ldots, v_n) \in \mathcal{B}(\Gamma)$ induces the signed orthoscheme that is induced by the complete tower $(S_1, \ldots, S_n)$ of P, with $S_n = \{y\}$. Then the Gram–Schmidt orthonormalization of $(v_1, \ldots, v_n)$ is equal to the ordered basis $(u_1, \ldots, u_n)$ associated with the complete tower $(S_1, \ldots, S_n)$ (that is, defined by (4)). By the definition of the Gram–Schmidt orthonormalization, this implies that $v_1 \in N(P, S_1)$ and $v_k \in N(P, S_k) \setminus N(P, S_{k-1})$ for $k = 2, \ldots, n$. The normal cone $N(P, S_k)$ is an $(n-k)$-face of the normal cone $N(P, \{y\})$. Thus, $(v_1, \ldots, v_n)$ is adapted to $N(P, \{y\})$, and hence $(v_1, \ldots, v_n)$ is tidy. Conversely, suppose that $(v_1, \ldots, v_n) \in \mathcal{B}(\Gamma)$ is tidy, say that $v_1 \in T_1$ and $v_k \in T_k \setminus T_{k-1}$ for $k = 2, \ldots, n$, where $T_1 \subset T_2 \subset \cdots \subset T_n$ are faces of $N(P, \{y\})$ (for some vertex y of P) with $\dim T_k = k$. Then there is a complete tower $(S_1, \ldots, S_n)$ of P with $N(P, S_k) = T_k$ for $k = 1, \ldots, n$. The signed orthoscheme induced by this tower is induced by $(v_1, \ldots, v_n)$.

A given signed orthoscheme $(\Delta, \delta) \in \mathcal{O}(P, o)$ is induced by a unique complete tower $(S_1, \ldots, S_n)$ of P, say with $S_n = \{y\}$. Then $T_k := N(P, S_k)$ is a k-face of $N(P, \{y\})$ and $T_1 \subset T_2 \subset \cdots \subset T_n$. Since $N(P, \{y\})$ is the union of simple cones from Γ, there is a unique cone $C = <v_1, \ldots, v_n> \in \Gamma$ which has k-faces F_k, $k = 1, \ldots, n$, satisfying $F_k \subseteq T_k$. With a suitable (unique) ordering, we then have $v_1 \in F_1$ and $v_k \in F_k \setminus F_{k-1}$ for $k = 2, \ldots, n$. This element $(v_1, \ldots, v_n) \in \mathcal{B}(\Gamma)$ induces (Δ, δ), and it is the only one with this property. This completes the proof of Proposition 2.

We now see from (6) and (7) that

$$\frac{1}{(n!)^2} \sum_{(v_1,\ldots,v_n)\in\mathcal{T}} \frac{\partial^n f_{<v_1,\ldots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} = V_n(P).$$

To complete the proof of (1), it remains to show that

$$\sum_{(v_1,\dots,v_n)\in\mathcal{U}} \frac{\partial^n f_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} = 0. \tag{8}$$

To prove (8), we state a more general version, which can be proved by induction.

Proposition 3. *Let $C \subset \mathbb{R}^n$ be an n-dimensional polyhedral cone, and let Γ_C be a simple fan such that C is the union of its cones. Let $\mathcal{B}(\Gamma_C)$ be the set of all ordered n-tuples of unit vectors generating cones of Γ_C, and let $\mathcal{U}_C$ be the subset of ordered n-tuples that are not adapted to C. Let $y \in \mathbb{R}^n$ and $f(u) := \langle y, u\rangle^n$ for $u \in C$. Then*

$$\sum_{(v_1,\dots,v_n)\in\mathcal{U}_C} \frac{\partial^n f_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} = 0. \tag{9}$$

We abbreviate

$$Df(v_1,\dots,v_n) := \frac{1}{n!}\frac{\partial^n f_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} \quad \text{for } (v_1,\dots,v_n) \in \mathcal{B}(\Gamma_C),$$

then

$$Df(v_1,\dots,v_n) = \langle v_1^\perp, y\rangle \cdots \langle v_n^\perp, y\rangle, \tag{10}$$

by (7).

We prove Proposition 3 by induction with respect to the dimension, starting with $n = 2$. Let the data be as in the proposition, with $n = 2$. Let $(v_1, v_2) \in \mathcal{U}_C$. Then $v_1 \in \operatorname{int} C$, since otherwise (v_1, v_2) would be adapted to C. Therefore, there exists precisely one cone $<v_1, w_2> \in \Gamma_C$ with w_2 independent from v_2. The Gram–Schmidt orthonormalizations of (v_1, v_2), (v_1, w_2) are, respectively, $(v_1^\perp, v_2^\perp)$ and $(v_1^\perp, -v_2^\perp)$. It follows from (10) that

$$Df(v_1, v_2) + Df(v_1, w_2) = 0.$$

Thus, the elements (v_1, v_2) of $\mathcal{U}_C$ can be grouped into pairs for which the expressions $Df(v_1, v_2)$ sum to 0. Therefore, (9) holds for $n = 2$.

Now we assume that $n \geq 3$ and that Proposition 3 has been proved in smaller dimensions. Let the data be as in the proposition. We consider two classes of elements $(v_1,\dots,v_n) \in \mathcal{U}_C$.

Class 1 contains the tuples $(v_1,\dots,v_n) \in \mathcal{U}_C$ with $\operatorname{pos}\{v_1,\dots,v_{n-1}\} \cap \operatorname{int} C \neq \emptyset$. For $(v_1,\dots,v_n)$ in this class, the positive hull $\operatorname{pos}\{v_1,\dots,v_{n-1}\}$ is an $(n-1)$-dimensional face of the cone $< v_1,\dots,v_n > \in \Gamma_C$ that meets $\operatorname{int} C$ and hence is a face of precisely one other cone $< v_1,\dots,v_{n-1}, w_n > \in \Gamma_C$. Let $(v_1^\perp,\dots,v_n^\perp)$ be the Gram–Schmidt orthonormalization of $(v_1,\dots,v_n)$, and let $(v_1^\perp,\dots,v_{n-1}^\perp, w_n^\perp)$ be the Gram–Schmidt orthonormalization of $(v_1,\dots,v_{n-1}, w_n)$. Since v_n and w_n lie in different halfspaces bounded by $\operatorname{lin}\{v_1,\dots,v_{n-1}\} = \operatorname{lin}\{v_1^\perp,\dots,v_{n-1}^\perp\}$, we have $v_n^\perp = -w_n^\perp$. It follows from (10) that

$$Df(v_1,\dots,v_n) + Df(v_1,\dots,v_{n-1}, w_n) = 0.$$

Thus, the elements $(v_1,\dots,v_n)$ of $\mathcal{U}_C$ in class 1 can be grouped into pairs for which the expressions $Df(v_1,\dots,v_n)$ sum to 0.

Class 2 contains the tuples $(v_1,\dots,v_n) \in \mathcal{U}_C$ with $\operatorname{pos}\{v_1,\dots,v_{n-1}\} \cap \operatorname{int} C = \emptyset$. Let $(v_1,\dots,v_n)$ be in this class. Then the cone $\operatorname{pos}\{v_1,\dots,v_{n-1}\}$ is contained in an $(n-1)$-dimensional face F_i of the cone C. We have $v_n \in C \setminus \operatorname{lin} F_i$, since $v_1,\dots,v_n$ are linearly independent. If the $(n-1)$-tuple $(v_1, \dots, v_{n-1})$ were adapted to the cone F_i, then the n-tuple $(v_1, \dots, v_n)$ were adapted to the cone C, a contradiction. Thus, $(v_1, \dots, v_{n-1})$ is not adapted to F_i. We can now apply the inductional hypothesis to the cone F_i and the simple fan induced in F_i by Γ_C. This yields

$$\sum_{(v_1,\dots,v_n)\in \text{class } 2_i} \langle v_1^\perp, y\rangle \cdots \langle v_{n-1}^\perp, y\rangle = 0, \tag{11}$$

where class 2_i contains the tuples $(v_1, \dots, v_n) \in \mathcal{U}_C$ with $\text{pos}\{v_1, \dots, v_{n-1}\} \subseteq F_i$. For each $(v_1, \dots, v_n) \in \text{class } 2_i$, we have

$$\text{lin}\{v_1^\perp, \dots, v_{n-1}^\perp\} = \text{lin}\{v_1, \dots, v_{n-1}\} = \text{lin}\, F_i,$$

and v_n is contained in the open halfspace bounded by $\text{lin}\, F_i$ whose closure contains C. Therefore, $v_n^\perp$ is the same vector for all $(v_1, \dots, v_n) \in \text{class } 2_i$. Multiplying (11) by $\langle v_n^\perp, y\rangle$, we obtain

$$\sum_{(v_1,\dots,v_n)\in \text{class } 2_i} Df(v_1, \dots, v_n) = 0.$$

Summing this over $i = 1, \dots, m$, where $F_1, \dots, F_m$ are the facets of C, we complete the induction and thus the proof of Proposition 3.

To prove (8), we now apply Proposition 3 to the normal cone of each vertex of P and sum over the vertices. This completes the proof of (1).

Formula (1) is useful if one has to consider a common simple refinement of several normal fans, as in the next section. In a volume formula for a single polytope, the superfluous terms may well be omitted. We state an appropriate reformulation of the above result. Let $(S_1, \dots, S_n)$ be a complete tower of the n-polytope P. We say that *it ends at* the vertex y if $S_n = \{y\}$. The orthonormal basis $(u_1, \dots, u_n)$ associated with the complete tower $(S_1, \dots, S_n)$ (defined by (4)) is also the unique orthonormal basis defined by

$$u_1 \in N(P, S_1), \quad u_2 \in \text{lin}\{u_1\} + N(P, S_2), \quad \dots \quad u_n \in \text{lin}\{u_1, \dots, u_{n-1}\} + N(P, S_n).$$

For each vertex y of P, we denote by $\mathcal{B}(y)$ the set of all orthonormal bases associated with all complete towers of P ending at y. With these notations, the volume formula obtained from (6) and (7) can also be written in the form

$$V_n(P) = \frac{1}{n!} \sum_{y\in \text{vert}\, P} \sum_{(u_1,\dots,u_n)\in\mathcal{B}(y)} \langle u_1, y\rangle \cdots \langle u_n, y\rangle. \tag{12}$$

Note that $\langle u_1, y\rangle \cdots \langle u_n, y\rangle$ is the product of the coordinates of the vertex y with respect to the orthonormal basis $(u_1, \dots, u_n)$.

4 Mixed volumes

For the extension of Theorem 1 to mixed volumes $V(\cdot, \dots, \cdot)$, it is crucial that the normal fans of finitely many polytopes have a common simple refinement.

Theorem 2. *Let $P_1, \dots, P_n \subset \mathbb{R}^n$ be polytopes, and let the fan Γ be a simple refinement of the normal fans of $P_1, \dots, P_n$. Let $\mathcal{B}(\Gamma)$ be the set of all ordered n-tuples of unit vectors generating cones of Γ. For each n-dimensional cone $C \in \Gamma$, let g_C be the restriction of the product of the support functions of $P_1, \dots, P_n$ to the interior of C. Then*

$$\frac{1}{(n!)^2} \sum_{(v_1,\dots,v_n)\in\mathcal{B}(\Gamma)} \frac{\partial^n g_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} = V(P_1, \dots, P_n). \tag{13}$$

Proof. Let $C \in \Gamma$. To each $i \in \{1,\dots,n\}$, there is a vertex y_i of P_i such that the support function of P_i on C is given by $\langle \cdot\,, y_i\rangle$. Let $\lambda_1,\dots,\lambda_n \geq 0$ and $P = \lambda_1 P_1 + \dots + \lambda_n P_n$ with polytopes $P_1,\dots,P_n \subset \mathbb{R}^n$. The support function of P on C is given by $\langle \cdot\,, \lambda_1 y_1 + \dots + \lambda_n y_n\rangle$. With f_C defined as in Theorem 1 for the polytope P, we have

$$\sum_{i_1,\dots,i_n=1}^{n} \lambda_{i_1}\cdots\lambda_{i_n} V(P_{i_1},\dots,P_{i_n}) = V_n(P)$$

$$= \frac{1}{(n!)^2} \sum_{(v_1,\dots,v_n)\in\mathcal{B}(\Gamma)} \frac{\partial^n f_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp}.$$

Here

$$f_{<v_1,\dots,v_n>}(u) = \langle u, \lambda_1 y_1 + \dots + \lambda_n y_n\rangle^n \quad \text{for } u \in \mathrm{pos}\{v_1,\dots,v_n\},$$

hence

$$\begin{aligned} \frac{\partial^n f_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp} &= n!\langle v_1^\perp, \lambda_1 y_1 + \dots + \lambda_n y_n\rangle \cdots \langle v_n^\perp, \lambda_1 y_1 + \dots + \lambda_n y_n\rangle \\ &= n! \sum_{i_1,\dots,i_n=1}^{n} \lambda_{i_1}\cdots\lambda_{i_n} \langle v_1^\perp, y_{i_1}\rangle \cdots \langle v_n^\perp, y_{i_n}\rangle. \end{aligned}$$

By comparison we get, in particular,

$$\begin{aligned} V(P_1,\dots,P_n) &= \frac{1}{n!} \sum_{(v_1,\dots,v_n)\in\mathcal{B}(\Gamma)} \langle v_1^\perp, y_1\rangle \cdots \langle v_n^\perp, y_n\rangle \\ &= \frac{1}{(n!)^2} \sum_{(v_1,\dots,v_n)\in\mathcal{B}(\Gamma)} \frac{\partial^n g_{<v_1,\dots,v_n>}}{\partial v_1^\perp \cdots \partial v_n^\perp}, \end{aligned}$$

which completes the proof. □

References

[1] Esterov, A., Tropical varieties with polynomial weights and corner loci of piecewise polynomials. *Mosc. Math. J.* **12** (2012), 55–76.

[2] Kazarnovskiĭ, B.Ya., On the action of the complex Monge–Ampère operator on piecewise linear functions. (Russian) *Funktsional. Anal. i Prilozhen.* **48** (2014), 19–29. English translation in *Funct. Anal. Appl.* **48** (2014), 15–23.

[3] Schneider, R., *Convex Bodies: The Brunn–Minkowski Theory.* 2nd edn., Encyclopedia of Mathematics and Its Applications, vol. **151**, Cambridge University Press, Cambridge, 2014.

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