Convexity and geometric probabilities

Rolf Schneider

In the development of the subject of Geometric Probabilities, there was always a close relationship to Convex Geometry. In these lectures, I want to demonstrate this relationship with a number of examples. These examples are of different types, since I want to cover various aspects. I start with hitting probabilities for convex bodies, which can be treated by means of integral geometry. Then I want to explain how several classical results on convex bodies can be applied to solve some extremal and uniqueness questions for various parameters connected with random systems of convex sets. The third topic will be more elementary in view of the questions to be asked, but not so as far as some of the answers are concerned: I will consider convex hulls of finitely many random points, under various different aspects.

1 Hitting probabilities for convex bodies

The purpose of this first section is twofold. First, I want to explain how results of classical integral geometry, in the style of Blaschke and Santaló, have been interpreted to give results about hitting probabilities for convex bodies. I will sketch how the basic integral geometric result can be obtained, using more recent developments. Second, I take this opportunity to introduce, in a natural way, a basic model of contemporary stochastic geometry, the Poisson process of convex particles.

To do this, I start with a heuristic intuitive question, which was posed and treated in an old paper by Hadwiger and Giger (1967). Let K and L be two given convex bodies in $\mathbb{R}^n$. We use K to generate a random field of congruent copies of K. That means, countably many congruent copies of K are laid out randomly and independently in space. The bodies may overlap. It is assumed that the random system has a well defined number density, that is, an expected mean number of particles per unit volume. The body L is used as a fixed test body. For a given number $j \in \mathbb{N}_0$, we ask for the probability p_j of the event that the test body L is hit by exactly j bodies of the random field.

So far, of course, this is only an imprecise heuristic question. It will require several steps to make the question precise. In a first step, we choose a large ball B_r, of radius r, that contains L, and we consider only one randomly moving congruent copy of K under the condition that it hits B_r. What is the probability that it hits also L? To make this a meaningful question, we have to specify the probability distribution of the randomly moving body. The geometrically most natural assumption is that this distribution is induced from the motion invariant measure μ on the motion group G_n of the Euclidean space $\mathbb{R}^n$. This means that we represent the congruent copies of K in the form gK, where $g \in G_n$ is a rigid motion. We define a probability distribution on the space of congruent copies of K by

$$\mathbb{P}(gK \in \mathcal{A}) = \frac{\mu(\{g \in G_n : gK \cap B_r \neq \emptyset \wedge gK \in \mathcal{A}\})}{\mu(\{g \in G_n : gK \cap B_r \neq \emptyset\})}$$

for Borel sets $\mathcal{A}$ of convex bodies.

Now it makes sense to ask for the probability p of the event that gK meets the body $L \subset B_r$, and this probability is given by

$$p = \frac{\mu(K, L)}{\mu(K, B_r)},$$

where we have put

$$\mu(K, M) := \mu(\{g \in G_n : gK \cap M \neq \emptyset\})$$

for convex bodies K, M.

How can we compute $\mu(K, M)$? The answer is given by integral geometry. Let us first suppose that K is a ball of radius ρ. Then the measure of all motions g that bring K in a hitting position with M is (under suitable normalization) equal to the measure of all translations which bring the centre of K into the parallel body $M + B_\rho$, and hence to the volume of this body. The volume of the parallel body is, by Steiner's formula, given by

$$V(M + B_\rho) = \sum_{i=0}^{n} \rho^{n-i} \kappa_{n-i} V_i(M)$$

(κ_j = volume of j-dimensional unit ball). This polynomial expansion defines important functionals from the theory of convex bodies, the *intrinsic volumes* $V_0, \dots, V_n$. With a different normalization, they are also known as *quermassintegrals* or *Minkowski functionals*. Special cases are: V_n, the volume; $2V_{n-1}$, the surface area; V_1, a multiple of the mean width, and V_0, the Euler characteristic.

We see already from this special case, $K = B_\rho$, that in the computation of the measure $\mu(K, M)$ the intrinsic volumes must play an essential role. It is a remarkable fact that no further functionals are needed for the general case. The *principal kinematic formula* of integral geometry, in its specialization to convex bodies, says that

$$\mu(K, M) = \sum_{i=0}^{n} \alpha_{ni} V_i(K) V_{n-i}(M),$$

with certain explicit constants α_{ni}. For the moment, we take this formula for granted. Later, I will say more about its proof.

Recall that the probability p, of the event that a randomly moving copy of K hitting B_r also hits L, is given by

$$p = \frac{\mu(K, L)}{\mu(K, B_r)}.$$

Hence, we have now found that

$$p = \frac{\sum_{i=0}^{n} \alpha_{ni} V_i(K) V_{n-i}(L)}{\sum_{i=0}^{n} \alpha_{ni} V_i(K) r^{n-i} \kappa_n},$$

which depends only on the intrinsic volumes of K and L.

In the second step, we consider m randomly chosen congruent copies of K, given in the form $g_1 K, \dots, g_m K$ with random motions $g_1, \dots, g_m$. We assume that these random motions are stochastically independent and that they all have the same distribution, as described above. For $j \in \{0, 1, \dots, m\}$, let p_j denote the probability of the event that the test body L is hit by exactly j of the random congruent copies of K. It is well known that the assumed independence leads to a binomial distribution, thus

$$p_j = \binom{m}{j} p^j (1-p)^{m-j}$$

with p as before, i.e.,

$$p = \frac{\mu(K, L)}{\mu(K, B_r)}.$$

In the third step, we let the ball radius r and the number m of particles tend to infinity, but in such a way that

$$\lim_{m \to \infty, r \to \infty} \frac{m}{V(B_r)} = \gamma$$

with a positive constant γ. From

$$mp = \frac{m}{V(B_r)} \frac{V(B_r)}{\mu(K, B_r)} \mu(K, L)$$

and

$$\lim_{r \to \infty} \frac{\mu(K, B_r)}{V(B_r)} = 1$$

we get $mp \to \gamma \mu(K, L) =: \lambda$ and hence

$$\binom{m}{j} p^j (1-p)^{m-j} \sim \frac{m!}{j!(m-j)!} \left(\frac{\lambda}{m}\right)^j \left(1 - \frac{\lambda}{m}\right)^{m-j} \sim \frac{\lambda^j}{j!} e^{-\lambda}.$$

Thus the result is

$$\lim_{r \to \infty} p_j = \frac{\lambda^j}{j!} e^{-\lambda}$$

with

$$\lambda = \gamma \mu(K, L) = \gamma \sum_{i=0}^{n} \alpha_{ni} V_i(K) V_{n-i}(L).$$

We have found, not surprisingly, a Poisson distribution. Its parameter, λ, is expressed explicitly in terms of the intrinsic volumes of K and L and involves the constant γ, which can be interpreted as the asymptotic density of our random system of convex bodies.

This is the answer given by Hadwiger and Giger to their initial question. The answer is explicit and elegant, but it is still not the final answer. What the authors have computed is a limit of probabilities, and this turned out to be a Poisson distribution. However, this Poisson distribution is not yet interpreted as the distribution of a well-defined random variable. What we need is a model that allows us to consider from the beginning countably infinite systems of randomly placed convex bodies, with suitable independence properties. This requirement leads us, inevitably and in a natural way, to the notion of a Poisson process of convex particles. Since this is a basic notion of contemporary stochastic geometry, it seems worthwhile to explain it briefly.

First we consider an arbitrary locally compact space E with a countable base. A subset $F \subset E$ is called *locally finite* if $E \cap C$ is finite for every compact subset C of E. Let $\mathcal{F}_{lf}$ be the system of all locally finite subsets of E. One equips $\mathcal{F}_{lf}$ with the smallest σ-algebra for which all counting functions

$$F \mapsto |F \cap A|, \qquad A \subset E \text{ Borel set},$$

are measurable. A (simple) *point process* in E is a random variable X on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathcal{F}_{lf}$. The expectation

$$\Theta(A) := \mathbb{E}|X \cap A|, \qquad A \subset E \text{ Borel set},$$

defines the *intensity measure* of the point process X. The point process X with intensity measure Θ is called a *Poisson process* if Θ is finite on compact sets and if, for every Borel set $A \subset E$ with $\Theta(A) < \infty$ and all $j \in \mathbb{N}_0$ one has

$$\mathbb{P}(|X \cap A| = j) = \frac{\Theta(A)^j}{j!} e^{-\Theta(A)}.$$

Such a Poisson process has strong independence properties: For pairwise disjoint Borel sets $A_1, A_2, \ldots \subset E$, the point processes $X \cap A_1, X \cap A_2, \ldots$ are stochastically independent. Moreover, the following holds. Under the condition, that in the Borel set A with $0 < \Theta(A) < \infty$ there are precisely k points of the process, the process $X \cap A$ is stochastically equivalent to the point process defined by k independent, identically distributed random points with the distribution

$$\frac{\Theta \llcorner A}{\Theta(A)}.$$

There are sufficiently many Poisson processes. To any atom free measure Θ on the Borel sets of E which is finite on compact sets, there is a Poisson process on Θ with intensity measure Θ. It is unique up to stochastic equivalence.

All this can now be applied to $E = \mathcal{K}^n$, the space of convex bodies in $\mathbb{R}^n$ (topologized by the Hausdorff metric). We consider only point processes in $\mathcal{K}^n$ whose intensity measure Θ satisfies

$$\Theta(\mathcal{K}_L) < \infty \qquad \text{for all } L \in \mathcal{K}^n,$$

where

$$\mathcal{K}_L := \{K \in \mathcal{K}^n : K \cap L \neq \emptyset\}.$$

A point process X in $\mathcal{K}^n$ satisfying this assumption will be called a *particle process*. If X is, in particular, a Poisson process, we call X a *Poisson process of convex particles.* It is called *stationary* if its intensity measure Θ (which is a Borel measure on $\mathcal{K}^n$) is translation invariant, and X is called *isotropic* if Θ is invariant under rotations.

Let us now see how the original question of Hadwiger and Giger can be treated with this general model. Now we need not restrict ourselves to a field of congruent convex bodies, but can consider more generally random fields of convex bodies where also the shapes of the particles are random. We assume that X is a stationary and isotropic Poisson process of convex particles in $\mathbb{R}^n$. Let L be a fixed convex body; we call it the *test body.* For $j \in \mathbb{N}_0$, we ask for the probability p_j that the test body is hit be exactly j bodies of the particle process. Since our particle process is a Poisson process, this probability is given by

$$p_j = \mathbb{P}(|X \cap \mathcal{K}_L| = j) = \frac{\Theta(\mathcal{K}_L)^j}{j!} e^{-\Theta(\mathcal{K}_L)}.$$

Hence, we have to determine the value $\Theta(\mathcal{K}_L)$ of the intensity measure. Due to the assumption of stationarity, the intensity measure can be decomposed in the following way. For $K \in \mathcal{K}^n$, let $c(K)$ denote the centre of the circumball of K, and put

$$\mathcal{K}_0 := \{K \in \mathcal{K}^n : c(K) = 0\}.$$

Thus $\mathcal{K}_0$ represents the translation classes of convex bodies. Then there is a unique probability measure $\mathbb{P}_0$ on $\mathcal{K}_0$ and a constant $\gamma > 0$ such that

$$\int_{\mathcal{K}^n} f \, d\Theta = \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^n} f(K + x) \, d\lambda(x) \, d\mathbb{P}_0(K) \tag{1}$$

holds for every Θ-integrable function f. Here λ is Lebesgue measure on $\mathbb{R}^n$. One calls $\mathbb{P}_0$ the *shape distribution* and γ the *intensity* of the particle process X. This intensity satisfies

$$\gamma = \frac{1}{\lambda(B)} \mathbb{E}|\{K \in X : c(K) \in B\}|$$

for every Borel set B with $0 < \lambda(B) < \infty$ and can hence be interpreted as the mean number of particles per unit volume.

Since our particle process is assumed as isotropic, the shape distribution $\mathbb{P}_0$ is invariant under rotations. Hence, when we apply (1) to the indicator function of the set $\mathcal{K}_L$, we may introduce an additional integration over all rotations of K, with the invariant measure on the rotation group, and this yields

$$\Theta(\mathcal{K}_L) = \gamma \int_{\mathcal{K}_0} \int_{G_n} \mathbf{1}_{\mathcal{K}_L}(gK) \, d\mu(g) \, d\mathbb{P}_0(K).$$

Here

$$\int_{G_n} \mathbf{1}_{\mathcal{K}_L}(gK) \, d\mu(g) = \mu(K, L) = \sum_{i=0}^{n} \alpha_{ni} V_i(K) V_{n-i}(L)$$

by the kinematic formula, hence we end up with

$$\Theta(\mathcal{K}_L) = \sum_{i=0}^{n} \alpha_{ni} V_{n-i}(L) \overline{V}_i(X)$$

with

$$\overline{V}_i(X) = \gamma \int_{\mathcal{K}_0} V_i(K)\, d\mathbb{P}_0(K).$$

This number is called the *density* of the ith intrinsic volume of our particle process X. It can be interpreted as

$$\overline{V}_i(X) = \frac{1}{\lambda(B)} \mathbb{E} \sum_{K \in X,\, c(K) \in B} V_i(K),$$

with B as above, or also as

$$\overline{V}_i(X) = \lim_{r \to \infty} \frac{1}{V_n(B_r)} \sum_{K \in X, X \subset B_r} V_i(K).$$

We have finally arrived at an explicit expression for the hitting probabilities p_j.

The purpose of the foregoing was twofold: *to develop a basic model of stochastic geometry, the Poisson particle process, and to demonstrate the role of integral geometry for the computation of geometric hitting probabilities in the presence of invariance assumptions under the motion group.*

2 The employed integral geometry

I will now say more about the proof of the employed principal kinematic formula. It can be written in the form

$$\int_{G_n} \chi(gK \cap M)\, d\mu(g) = \sum_{i=0}^{n} \alpha_{ni} V_i(K) V_{n-i}(M), \tag{2}$$

where χ denotes the Euler characteristic ($\chi(\emptyset) = 0$, and $\chi(K) = 1$ for a nonempty compact convex set). There is a very elegant way to prove (5), due to Hadwiger. It makes use of an axiomatic characterization of the intrinsic volumes, also due to Hadwiger. Whereas Hadwiger's original proof was quite long, one has since 1995 a shorter proof due to Daniel Klain. I will present this proof here, except that a certain extension theorem for additive functionals and a certain analytical result have to be taken for granted.

Recall that the intrinsic volumes can be defined via the Steiner formula

$$V_n(K + \rho B^n) = \sum_{i=0}^{n} \rho^{n-i} \kappa_{n-i} V_i(K), \qquad \rho \geq 0,$$

where now B^n denotes the unit ball. From this definition it is not difficult to deduce that the intrinsic volume V_i, as a real function on the space of convex bodies, inherits certain properties from the volume functional. The most important properties are the following: V_i is additive, continuous, and rigid motion invariant. Generally, a function $\varphi : \mathcal{K}^n \to \mathbb{R}$ is *additive* if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

for all convex bodies K, L satisfying $K \cup L \in \mathcal{K}^n$. One extends the definition of φ by putting $\varphi(\emptyset) = 0$. Continuity of a function on $\mathcal{K}^n$ refers, of course, to the Hausdorff metric, and φ is rigid motion invariant if $\varphi(gK) = \varphi(K)$ for all $K \in \mathcal{K}^n$ and every rigid motion $g \in G_n$. With these definitions, Hadwiger's celebrated characterization theorem says that every additive, continuous, rigid motion invariant real function on $\mathcal{K}^n$ is a linear combination of the intrinsic volumes with constant coefficients.

The crucial step for a proof of Hadwiger's characterization theorem is the following result.

1.1 Theorem. *Suppose that $\psi : \mathcal{K}^n \to \mathbb{R}$ is an additive, continuous, motion invariant function satisfying $\psi(K) = 0$ whenever either $\dim K < n$ or K is a unit cube. Then $\psi = 0$.*

Proof. The proof proceeds by induction with respect to the dimension. For $n = 0$, there is nothing to prove. If $n = 1$, ψ vanishes on (closed) segments of unit length, hence on segments of length $1/k$ for $k \in \mathbb{N}$ and therefore on segments of rational length. By continuity, ψ vanishes on all segments and thus on $\mathcal{K}^1$.

Now let $n > 1$ and suppose that the assertion has been proved in dimensions less than n. Let $H \subset \mathbb{R}^n$ be a hyperplane and I a closed segment of length 1, orthogonal to H. For convex bodies $K \subset H$ define $\varphi(K) := \psi(K + I)$. Clearly φ has, relative to H, the properties of ψ in the Theorem, hence the induction hypothesis yields $\varphi = 0$. For fixed $K \subset H$, we thus have $\psi(K + I) = 0$, and a similar argument as used above for $n = 1$ shows that $\psi(K + S) = 0$ for any closed segment S orthogonal to H. Thus ψ vanishes on right convex cylinders.

Let $K \subset H$ again be a convex body and let $S = \operatorname{conv}\{0, s\}$ be a segment not parallel to H. If $m \in \mathbb{N}$ is sufficiently large, the cylinder $Z := K + mS$ can be cut by a hyperplane H' orthogonal to S so that the two closed halfspaces H^-, H^+ bounded by H' satisfy $K \subset H^-$ and $K + ms \subset H^+$. Then $\overline{Z} := [(Z \cap H^-) + ms] \cup (Z \cap H^+)$ is a right cylinder, and we deduce that $m\mu(K + s) = \mu(Z) = \mu(\overline{Z}) = 0$. Thus ψ vanishes on arbitrary convex cylinders.

It can be shown that the continuous additive function ψ on $\mathcal{K}^n$ has an additive extension to the convex ring, which is the system of all finite unions of convex bodies, or polyconvex sets. The proof is elementary, but will not be given here. We denote this extension also by ψ. Then

$$\psi\left(\bigcup_{i=1}^{k} K_i\right) = \sum_{i=1}^{k} \psi(K_i)$$

whenever $K_1, \dots, K_k$ are convex bodies such that $\dim(K_i \cap K_j) < n$ for $i \neq j$. This follows from the fact that the additive extension satisfies the inclusion-exclusion principle and that ψ has been assumed to vanish on convex bodies of dimension less than n.

Let P be a polytope and S a segment. The sum $P + S$ has a decomposition

$$P + S = \bigcup_{i=1}^{k} P_i,$$

where $P_1 = P$, the polytope P_i is a convex cylinder for $i > 1$, and $\dim(P_i \cap P_j) < n$ for $i \neq j$. It follows that $\psi(P + S) = \psi(P)$. By induction, we obtain $\psi(P + Z) = \psi(P)$ if Z is a finite sum of segments. Such a body Z is called a *zonotope*, and a convex body which can be approximated by zonotopes is called a zonoid. Since the function ψ is continuous, it follows that $\psi(K + Z) = \psi(K)$ for arbitrary convex bodies K and zonoids Z.

Now we have to use an analytic result for which we do not give a proof. Let K be a centrally symmetric convex body which is sufficiently smooth (say, its support function is of class C^∞). Then there exist zonoids Z_1, Z_2 so that $K + Z_1 = Z_2$ (this can be seen from Section 3.5 in my book on Convex Bodies, expecially Theorem 3.5.3). We conclude that $\psi(K) = \psi(K + Z_1) = \psi(Z_2) = 0$. Since every centrally symmetric convex body K can be approximated by bodies which are centrally symmetric and sufficiently smooth, it follows from the continuity of ψ that $\psi(K) = 0$ for all centrally symmetric convex bodies.

Now let Δ be a simplex, say $\Delta = \operatorname{conv}\{0, v_1, \dots, v_n\}$, without loss of generality. Let $v := v_1 + \dots + v_n$ and $\Delta' := \operatorname{conv}\{v, v - v_1, \dots, v - v_n\}$, then $\Delta' = -\Delta + v$. The vectors $v_1, \dots, v_n$ span a parallelotope P. It is the union of Δ, Δ' and the part of P lying between the hyperplanes spanned by $v_1, \dots, v_n$ and $v - v_1, \dots, v - v_n$, respectively. The latter, say Q, is a centrally symmetric polytope, and $\Delta \cap Q$, $\Delta' \cap Q$ are of dimension $n - 1$. We deduce that $0 = \psi(P) = \psi(\Delta) + \psi(Q) + \psi(\Delta')$, thus $\psi(-\Delta) = -\psi(\Delta)$. If the dimension n is even, then $-\Delta$ is obtained from Δ by a proper rigid motion, and the motion invariance of ψ yields $\psi(\Delta) = 0$. If the dimension $n > 1$ is odd, we decompose Δ as follows. Let z be the centre of the inscribed ball of Δ, and let p_i be the point where this ball touches the facet F_i of Δ $(i = 1, \dots, n + 1)$. For $i \neq j$, let Q_{ij} be the convex hull of the face $F_i \cap F_j$ and the points z, p_i, p_j. The polytope Q_{ij} is invariant under reflection in the hyperplane spanned by $F_i \cap F_j$ and z. If $Q_1, \dots, Q_m$ are the polytopes Q_{ij} for $1 \leq i < j \leq n + 1$ in any order, then $P = \bigcup_{r=1}^{m} Q_r$ and $\dim(Q_r \cap Q_s) < n$ for $r \neq s$. Since $-Q_r$ is the image of Q_r under a proper rigid motion, we have $\psi(-\Delta) = \sum \psi(-Q_r) = \sum \psi(Q_r) = \psi(\Delta)$. Thus $\psi(\Delta) = 0$ for every simplex Δ.

Decomposing a polytope P into simplices, we obtain $\psi(P) = 0$. The continuity of ψ now implies $\psi(K) = 0$ for all convex bodies K. This finishes the induction and hence the proof of Theorem 1.1. ■

Hadwiger's characterization theorem is now an easy consequence.

1.2 Theorem. *Suppose that* $\psi : \mathcal{K}^n \to \mathbb{R}$ *is an additive, continuous, motion invariant function. Then there are constants* $c_0, \dots, c_n$ *so that*

$$\psi(K) = \sum_{i=0}^{n} c_i V_i(K)$$

for all $K \in \mathcal{K}^n$.

Proof. We use induction on the dimension. For $n = 0$, the assertion is trivial. Suppose that $n > 0$ and the assertion has been proved in dimensions less than n. Let $H \subset \mathbb{R}^n$ be a hyperplane. The restriction of ψ to the convex bodies lying in H is additive, continuous and invariant under motions of H into itself. By the induction hypothesis, there are constants $c_0, \dots, c_{n-1}$ so that $\psi(K) = \sum_{i=0}^{n-1} c_i V_i(K)$ holds for convex bodies $K \subset H$ (note that the intrinsic volumes do not depend on the dimension of the surrounding space, as can be deduced, with a little computation, from the Steiner formula). By the motion invariance of ψ and V_i, this holds for all $K \in \mathcal{K}^n$ of dimension less than n. It follows that the function ψ' defined by

$$\psi'(K) := \psi(K) - \sum_{i=0}^{n} c_i V_i(K)$$

for $K \in \mathcal{K}^n$, where c_n is chosen so that ψ' vanishes at a fixed unit cube, satisfies the assumptions of Theorem 1.1. Hence $\psi' = 0$, which completes the proof of Theorem 1.2. ■

We are now in a position to prove the principal kinematic formula, that is, to evaluate the integral

$$\int_{G_n} \chi(K \cap gM)\, d\mu(g)$$

for arbitrary convex bodies K and M. For this, we first fix the convex body M and define a functional ψ by

$$\psi(K) := \int_{G_n} \chi(K \cap gM)\, d\mu(g).$$

It is not difficult to see that the integral is well defined, that is, that the function $g \mapsto \chi(K \cap gM)$ is μ-integrable. This follows from the fact that this function is continuous on the set $G_n \setminus G_n(K, M)$, where

$$G_n(K, M) := \{g \in G_n : K \text{ and } gM \text{ touch}\},$$

and that $G_n(K, M)$ has μ-measure zero. The latter is a consequence of the fact that the boundary of a convex body has Lebesgue measure zero.

Thus the function ψ is defined on $\mathcal{K}^n$, and one can show by similar arguments and using the bounded convergence theorem that it is continuous. From the invariance of the measure μ it follows easily that ψ is rigid motion invariant. Since the Euler characteristic

χ is additive, one sees immediately that ψ is additive. Thus the functional ψ has all the properties required for Hadwiger's characterization theorem, and we deduce that

$$\int_{G_n} \chi(K \cap gM)\, d\mu(g) = \sum_{i=0}^{n} c_i(M) V_i(K) \tag{3}$$

for $K \in \mathcal{K}^n$, where the constants $c_i(M)$ depend, of course, on the fixed body M. But we can interchange the roles of K and M and deduce that (3) must also be of the form

$$\sum_{j=0}^{n} c'_j(K) V_j(M) \qquad \text{for all } M \in \mathcal{K}^n.$$

Both results together give an expression of the form

$$\int_{G_n} \chi(K \cap gM)\, d\mu(g) = \sum_{i,j=1}^{n} \alpha_{nij} V_i(K) V_j(M)$$

for $K, M \in \mathcal{K}^n$, with constants α_{nij} depending only on n, i, j. To determine these constants, one can choose for K and M balls of different radii. The left-hand side is then easily computed, and by comparing equal powers of the radii, one finds the coefficients. In this way, one obtains

$$\int_{G_n} \chi(K \cap gM)\, d\mu(g) = \sum_{i=0}^{n} \alpha_{ni} V_i(K) V_{n-i}(M)$$

with

$$\alpha_{ni} = \frac{\kappa_i \kappa_{n-i}}{\binom{n}{i} \kappa_n}.$$

The late Gian-Carlo Rota was so enthusiastic about Hadwiger's characterization theorem and the type of applications sketched here that he called it, in a Colloquium Lecture at the Annual Meeting of the AMS in 1997, the 'Main Theorem of Geometric Probability'. This seems a bit exaggerating, since the theory of geometric probabilities has developed well beyond this type of applications of integral geometry.

3 Particle processes and intrinsic volumes, I

In my first lecture, I have considered elementary hitting probabilities for random systems of convex bodies. I did this with two goals in mind: first, to introduce a basic model of stochastic geometry that is built from convex bodies and, second, to show the role that integral geometry plays in treating probabilities and expectations related to such systems. In the second lecture, I sketched a quick approach to the required kinematic formulae of integral geometry. I would now like to present an advanced synthesis of both topics, by proving a basic result about the so-called Boolean model of stochastic geometry.

Let me start with explaining a problem that has its origin in practice. Assume that we observe a realisation of a random system of convex sets, for example in the plane a microscopic image of blood cells or, as in material sciences, the polished surface of some material that contains particles of some other material. Assume further that we need to know some quantitative aspects. These could be the mean number of particles per unit area, or the mean perimeter, or the mean area. In general, however, we will not be able to observe individual particles, but only their union set. We assume that for the union set we can measure, for a given realisation inside an observation window, the area, the perimeter, the Euler characteristic. We will show that, under suitable assumptions, this allows us to obtain estimators for the corresponding parameters of the underlying particle process. Of course, such a correspondence can only be expected if the particle process satisfies strong independence assumptions. We shall see that a Poisson particle process is a perfect model to permit such conclusions.

Let me first recall the definition of a Poisson process of convex particles in $\mathbb{E}^n$. This is a measurable map X from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the set of all locally finite systems of convex bodies in $\mathbb{E}^n$ with the property that the corresponding counting variables have Poisson distributions: For every Borel set $\mathcal{B} \subset \mathcal{K}^n$,

$$\mathbb{P}(|X \cap B| = j) = e^{-\Theta(B)} \frac{\Theta(B)^j}{j!},$$

if $\Theta(B) := \mathbb{E}|X \cap \mathcal{B}| < \infty$. Here Θ is a Borel measure on the space $\mathcal{K}^n$ of convex bodies, the *intensity measure* of the process X. We make two additional assumptions. Writing

$$\mathcal{K}_L := \{K \in \mathcal{K}^n : K \cap L \neq \emptyset\} \qquad \text{for } L \in \mathcal{K}^n,$$

we assume that

$$\Theta(\mathcal{K}_L) < \infty \qquad \text{for all } L \in \mathcal{K}^n.$$

This means that a given convex test body L is almost surely hit by only finitely many particles of the process X.

The second assumption is that the measure Θ be invariant under translations. We express this by saying that the particle process is *stationary*. As a consequence, the intensity measure has a decomposition, as already noted. We define

$$\mathcal{K}_0 := \{K \in \mathcal{K}^n : c(K) = 0\},$$

where $c(K)$ is the centre of the circumball of K. The space $\mathcal{K}_0$ contains exactly one representative of each translation class of convex bodies. One can show that there is a probability measure $\mathbb{P}_0$ on the space $\mathcal{K}_0$ and a constant γ so that

$$\int_{\mathcal{K}^n} f \, d\Theta = \gamma \int_{\mathcal{K}_0} \int_{E^n} f(K + x) \, d\lambda_n(x) \, d\mathbb{P}_0(K)$$

for every nonnegative measurable function f on $\mathcal{K}^n$.

Now we introduce parameters for a quantitative description of the particle process X. We might, for example, be interested in the mean expected volume of the particles per unit volume, or the mean expected area, or the mean number of particles per unit volume. All these descriptions are included if we define densities of the intrinsic volumes $V_0, \dots, V_n$. For this, we choose any Borel set $B \subset \mathbb{E}^n$ with $0 < \lambda(B) < \infty$ and compute the expected value

$$\mathbb{E} \sum_{K \in X,\, c(K) \in B} V_i(K).$$

Here we use Campbell's theorem about point processes, which says in our case that

$$\mathbb{E} \sum_{K \in X} f(K) = \int_{\mathcal{K}^n} f(K)\, d\Theta(K).$$

Using further the decomposition of the intensity measure, we obtain

$$\begin{aligned}
&\mathbb{E} \sum_{K \in X, c(K) \in B} V_i(K) \\
&= \mathbb{E} \sum_{K \in X} \mathbf{1}\{c(K) \in B\} V_i(K) \\
&= \int_{\mathcal{K}^n} \mathbf{1}\{c(K) \in B\} V_i(K)\, d\Theta(K) \\
&= \gamma \int_{\mathcal{K}_0} \int_{\mathbb{E}^n} \mathbf{1}\{c(K+x) \in B\} V_i(K+x)\, d\lambda_n(x)\, d\mathbb{P}_0(K) \\
&= \lambda_n(B) \gamma \int_{\mathcal{K}_0} V_i(K)\, d\mathbb{P}_0(K).
\end{aligned}$$

Hence, the quotient

$$\overline{V}_i(X) = \frac{1}{\lambda_n(B)} \mathbb{E} \sum_{K \in X,\, c(K) \in B} V_i(K)$$

does not depend on B, and it has the integral representation

$$\overline{V}_i(X) = \gamma \int_{\mathcal{K}_0} V_i(K)\, d\mathbb{P}_0(K).$$

We call $\overline{V}_i(X)$ the *density* of the ith intrinsic volume for the particle process X.

Since $V_0(K) = 1$ for every nonempty convex body K, it is clear from the last integral that

$$\overline{V}_0(X) = \gamma.$$

So γ is the expected number of particles K in the process X with $c(K)$ in a given set of unit volume. Therefore, the number γ is called the *intensity* or *particle intensity* of the process X.

As explained initially, we are looking for a possibility to estimate the parameters $\overline{V}_i(X)$ of the particle process if only the union set of the process can be observed and can be an object for measurements. Therefore, we now consider the union set

$$Z_X := \bigcup_{K \in X} K.$$

This is the most prominent and often used model for a random closed set. Also for Z_X, we can define densities of the intrinsic volumes. This is obvious for the volume density, therefore we consider this case first. It can be introduced as

$$p := \mathbb{P}(O \in Z_X),$$

the probability that O is covered by the random set Z_X. Since X is stationary, the random sets Z_X and $Z_X + t$ (for any fixed $t \in \mathbb{E}^n$) have the same distribution. Hence,

$$p = \mathbb{P}(y \in Z_X) = \mathbb{E}\mathbf{1}_{Z_X}(y)$$

holds for every $y \in \mathbb{E}^n$. Let $B \subset \mathbb{E}^n$ be a Borel set with $0 < \lambda_n(B) < \infty$. By Fubini's theorem,

$$\begin{aligned} p\lambda_n(B) &= \int_B \mathbb{E}\mathbf{1}_{Z_X}(y)\, d\lambda_n(y) \\ &= \mathbb{E} \int_B \mathbf{1}_{Z_X}(y)\, d\lambda_n(y) \\ &= \mathbb{E}\lambda_n(Z_X \cap B). \end{aligned}$$

Thus

$$p = \frac{\mathbb{E}\lambda_n(Z_X \cap B)}{\lambda_n(B)} =: \Delta_n(Z_X)$$

is independent of the set B, and we call this number the *volume density* of Z_X.

We can now find a connection with the volume density $\overline{V}_n(X)$ of the underlying particle process. In fact, we have

$$\begin{aligned} \Delta_n(Z_X) &= \mathbb{P}(O \in Z_X) = 1 - \mathbb{P}(O \notin Z_X) \\ &= 1 - \mathbb{P}(|X \cap \mathcal{K}_{\{0\}}| = 0) \\ &= 1 - e^{-\Theta(\mathcal{K}_{\{0\}})} \end{aligned}$$

and

$$\begin{aligned} \Theta(\mathcal{K}_{\{0\}}) &= \gamma \int_{\mathcal{K}_0} \int_{\mathbb{E}^n} \mathbf{1}_{\mathcal{K}_{\{0\}}}(K + x)\, d\lambda_n(x)\, d\mathbb{P}_0(K) \\ &= \gamma \int_{\mathcal{K}_0} V_n(K)\, d\mathbb{P}_0(K) \\ &= \overline{V}_n(X). \end{aligned}$$

Thus we have found

$$\Delta_n(Z_X) = 1 - e^{-\overline{V}_n(X)}$$

This equality should have come as a surprise: it says that the volume density $\overline{V}_n(X)$ of the particle process X is determined by the volume density $\Delta_n(Z_X)$ of the union set.

This is surprising, since in a given realization of Z_X one cannot identify the generating particles, since they overlap, and some particles may even be covered totally by others. The reason for the existence of the exact relation above lies in the strong independence properties of Poisson processes.

Densities of the other intrinsic volumes for the union set can also be defined, but this is less straightforward. We have to use the additive extensions of the intrinsic volumes. Let us denote by $\mathcal{R}^n$ the so-called convex ring, the set of polyconvex sets (finite unions of convex sets) in $\mathbb{E}^n$. As I have already mentioned (and used) in my second lecture, the intrinsic volumes have additive extensions to $\mathcal{R}^n$; these extensions will be denoted by the same symbols.

If now L is a convex body, then

$$Z_X \cap L = \bigcup_{K \in X} (K \cap L)$$

is a union of convex bodies, and almost surely of only finitely many of them. Hence, the intrinsic volume $V_i(Z_X \cap L)$ is well defined. Since the additive extension of the intrinsic volumes are measurable, $V_i(Z_X \cap L)$ is a random variable. Hence, the expectation $\mathbb{E}V_i(Z_X \cap L)$ is defined. In order to derive from this a density, we need a limit process. It can be shown that

$$\Delta_i(Z_X) := \lim_{r \to \infty} \frac{\mathbb{E}V_i(Z_X \cap rL)}{V_n(rL)}$$

always exists and is independent of the choice of the convex body L (with $V_n(L) > 0$). The number $\Delta_i(Z_X)$ is the density of the ith intrinsic volume of the random set Z_X. In particular, $\Delta_n(Z_X)$ is the volume density, $\Delta_{n-1}(Z_X)$ is the density of the surface area, and $\Delta_0(Z_X)$ is the density of the Euler characteristic. Thus, in dimension two, all densities Δ_i have a simple intuitive geometric meaning.

We recall that it is our aim to establish relations between the densities $\Delta_i(Z_X)$ of the union set Z_X and densities $\overline{V}_i(X)$ of the underlying particle process. One such relation, for the volume densities, has already been established, namely

$$\Delta_n(Z_X) = 1 - e^{-\overline{V}_n(X)}. \tag{4}$$

We will establish a similar relation for the surface area densities, namely

$$\Delta_{n-1}(Z_X) = \overline{V}_{n-1}(X) e^{-\overline{V}_n(X)}. \tag{5}$$

To derive (5), the stationarity assumption as for (7) is sufficient. For the other densities, however, we must also assume that the process X is isotropic. Together with stationarity this means that its distribution is invariant under rigid motions. I will not derive these further relations, since the pattern of proof will already be clear after (5), and the other cases tend to become complicated. I mention, however, the result for $n = 2$, together with (7) and (5) for this case:

$$\begin{aligned}
\Delta_2(Z_X) &= 1 - e^{-\overline{V}_2(X)}, \\
\Delta_1(Z_X) &= e^{-\overline{V}_2(X)} \overline{V}_1(X) \\
\Delta_0(Z_X) &= e^{-\overline{V}_2(X)} \left(\gamma - \frac{1}{4\pi} \overline{V}_1(X)^2 \right).
\end{aligned}$$

Observe that on the left-hand side we have the density of the area, the boundary length, and the Euler characteristic of the union sets Z_X. These parameters can be measured (estimated) from observations of the realisation of Z_X in an observation window. Then from the right-hand sides of the equations one gets, successively, estimators for the area density $\overline{V}_2(X)$, the perimeter density $\overline{V}_1(X)$ and, finally, the number density or intensity $\gamma = \overline{V}_0(X)$ of the particle process. Thus one can determine, at least in principle, the mean particle number γ by measuring, at the union set in a convex window, areas, boundary lengths and Euler characteristics.

4 Particle processes and intrinsic volumes, II

I want now to explain how the relation (5) is obtained. The first part of the proof is quite general, so we can consider any V_i. The density of V_i for the union set Z_X was defined by

$$\Delta_i(Z_X) = \lim_{r\to\infty} \frac{\mathbb{E}V_i(Z_X \cap rL)}{V_n(rL)}$$

for any convex body L with interior points. Therefore, we have to consider the expected value

$$\mathbb{E}V_i(Z_X \cap K)$$

for a convex body K. We use the hitting number

$$\nu := |X \cap \mathcal{K}_K|,$$

the random variable giving the number of particles of the process X that hit K.

For a given realization $X(\omega)$ $(\omega \in \Omega)$, let

$$M_1(\omega), \dots, M_{\nu(\omega)}(\omega)$$

be the particles that hit K. Since the functional V_i is additive on the convex ring, we can use the inclusion-exclusion principle and obtain

$$V_i(Z_{X(\omega)} \cap K) = V_i\left(\bigcup_{M\in X(\omega)} (M \cap K)\right)$$

$$= \sum_{k=1}^{\nu(\omega)} (-1)^{k-1} \sum_{1\le i_1<\dots<i_k\le\nu(\omega)} V_i(K \cap M_{i_1}(\omega) \cap \dots \cap M_{i_k}(\omega))$$

$$= \sum_{k=1}^{\nu(\omega)} \frac{(-1)^{k-1}}{k!} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}(\omega)} V_i(K \cap K_1 \cap \dots \cap K_k).$$

Here $X^k_{\neq} := \{(K_1,\dots,K_k) \in (\mathcal{K}^n)^k : K_i \text{ pairwise distinct}\}$ is a point process in the space $(\mathcal{K}^n)^k$. Let $\Lambda^{(k)}$ denote its intensity measure. It is a property of Poisson processes that

$$\Lambda^{(k)} = \Theta^k \quad (= \Theta \otimes \dots \otimes \Theta,\ k \text{ factors}).$$

To compute the expectation, we have to interchange integration and summation, and for this we use the bounded convergence theorem. To justify its application, the following estimate is required. Since the intrinsic volume V_i is monotone on convex bodies, we have $V_i(M) \leq V_i(K)$ for all convex bodies $M \subset K$. This gives

$$\left| \sum_{k=1}^{\nu(\omega)} \frac{(-1)^{k-1}}{k!} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}(\omega)} V_i(K \cap K_1 \cap \ldots \cap K_k) \right|$$

$$\leq \sum_{k=1}^{\nu(\omega)} \binom{\nu(\omega)}{k} V_i(K) \leq V_i(K) 2^{\nu(\omega)}.$$

Now ν has a Poisson distribution, hence

$$\begin{aligned} \mathbb{E}2^{\nu} &= \sum_{k=0}^{\infty} 2^k \mathbb{P}(|X \cap \mathcal{K}_K| = k) \\ &= e^{-\Theta(\mathcal{K}_K)} \sum_{k=0}^{\infty} 2^k \frac{\Theta(\mathcal{K}_K)^k}{k!} \\ &= e^{-\Theta(\mathcal{K}_K)} e^{2\Theta(\mathcal{K}_K)} = e^{\Theta(\mathcal{K}_K)} < \infty, \end{aligned}$$

since we have assumed that $\Theta(\mathcal{K}_K) < \infty$ for every convex body K. Thus, we can apply the bounded convergence theorem and obtain (using Campbell's theorem for the process $X^k_{\neq}$, together with $\Lambda^{(m)} = \Theta^m$)

$$\begin{aligned} &\mathbb{E}V_i(Z_X \cap K) \\ = &\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \mathbb{E} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}} V_i(K \cap K_1 \cap \ldots \cap K_k) \\ = &\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}^n} \cdots \int_{\mathcal{K}^n} V_i(K \cap K_1 \cap \ldots \cap K_k) \, d\Theta(K_1) \cdots d\Theta(K_k) \end{aligned}$$

Now integral geometry enters the scene again. Using the decomposition of the intensity measure Θ, for the last iterated integral, which we denote by I_K, we get

$$\begin{aligned} I_k = \gamma^k \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{\mathbb{E}^n} \cdots \int_{\mathbb{E}^n} & V_i(K \cap (K_1 + x_1) \cap \ldots \cap (K_n + x_n)) \\ & d\lambda_n(x_1) \cdots d\lambda_n(x_k) \, d\mathbb{P}_0(K_1) \cdots d\mathbb{P}_0(K_n). \end{aligned}$$

At this point, we specialize to the case $i = n-1$. In this case, we have a translative integral geometric formula, namely

$$\int_{\mathbb{E}^n} V_{n-1}(K \cap (K_1 + x_1)) \, d\lambda_n(x_1) = V_n(K)V_{n-1}(K_1) + V_{n-1}(K)V_n(K_1).$$

A similar translative formula holds for the volume, namely

$$\int_{\mathbb{E}^n} V_n(K \cap (K_1 + x))\, d\lambda(x) = V_n(K)V_n(K_1).$$

However, for the other intrinsic volumes, the kinematic formula requires integration over the full rotation group, if one wants the result to depend only on intrinsic volumes of K and K_1.

Since both, V_n and V_{n-1}, satisfy translational formulas, the above formula for V_{n-1} can be iterated, replacing K by $K \cap (K_2 + x_2)$, and so on. The result is

$$\begin{aligned} &\int_{\mathbb{E}^n} \cdots \int_{\mathbb{E}^n} V_{n-1}(K_0 \cap (K_1 + x_1) \cap \ldots \cap (K_k + x_k))\, d\lambda_n(x_1) \cdots d\lambda_n(x_k) \\ = &\sum_{i=0}^{k} V_n(K_0) \cdots V_n(K_{i-1})V_{n-1}(K_i)V_n(K_{i+1}) \cdots V_n(K_k). \end{aligned}$$

Using this in the computation of I_k and observing that

$$\int_{\mathcal{K}_0} V_i(M)\, d\mathbb{P}_0(M) = \overline{V}_i(X),$$

we finally get

$$\begin{aligned} &\mathbb{E}V_{n-1}(Z_X \cap K) \\ = &\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \left[V_{n-1}(K)\overline{V}_n(X)^k + kV_n(K)\overline{V}_{n-1}(X)\overline{V}_n(X)^{k-1}\right] \\ = &V_{n-1}(K)\left(1 - e^{-\overline{V}_n(X)}\right) + V_n(K)\overline{V}_{n-1}(X) \sum_{k=1}^{\infty} \frac{\left(-\overline{V}_n(X)\right)^{k-1}}{(k-1)!} \\ = &V_{n-1}(K)\left(1 - e^{-\overline{V}_n(X)}\right) + V_n(K)\overline{V}_{n-1}(X)e^{-\overline{V}_n(X)}. \end{aligned}$$

Here we replace K by rL (with $r > 0$), divide by $V_n(rL)$ and let r tend to infinity. We conclude that

$$\Delta_{n-1}(Z_X) = \lim_{r\to\infty} \frac{\mathbb{E}(V_{n-1}(Z_X \cap rL)}{V_n(rL)} = e^{-\overline{V}_n(X)}\overline{V}_{n-1}(X).$$

We have seen so far how, in principle, the densities $\overline{V}_i(X)$ can be determined from the densities $\Delta_i(Z_X)$ of the union set. The latter densities were defined by

$$\Delta_i(Z_X) = \lim_{r\to\infty} \frac{\mathbb{E}V_i(Z_X \cap rK)}{V_n(rK)}.$$

In practice, one cannot consider arbitrarily large windows, but has to work with a fixed observation window. The question arises, therefore, whether it is reasonable to estimate $\Delta_i(Z_X)$ by $V_i(Z \cap K)/V_n(K)$. The first question would be whether this is an unbiased estimator, that is, whether its expectation is equal to $\Delta_i(Z_X)$. Here again integral geometry provides the answer.

We must, however, assume that our underlying Poisson process X is stationary and isotropic.

By B^n we denote the unit ball. We choose a rotation $\vartheta \in SO_n$, a vector $t \in \mathbb{E}^n$ and a number $r > 0$ and consider the random variable

$$V_i(Z_X \cap K \cap (\vartheta r B^n j + t)).$$

For its expectation we get, using the motion invariance of V_i and the stationarity and isotropy of the random set Z_X,

$$\begin{aligned}
&\mathbb{E}V_i(Z_X \cap K \cap (\vartheta r B^n + t)) \\
&\quad = \mathbb{E}V_i(\vartheta^{-1}(Z_X - t) \cap \vartheta^{-1}(K - t) \cap rB^n) \\
&\quad = \mathbb{E}V_i(Z_X \cap \vartheta^{-1}(K - t) \cap rB^n).
\end{aligned}$$

One can show that the function

$$\begin{array}{ccc}
\mathbb{E}^n \times SO_n \times \Omega & \to & \mathbb{R} \\
(t, \vartheta, \omega) & \mapsto & V_i(Z_{X(\omega)} \cap K \cap (\vartheta B^n + t))
\end{array}$$

is integrable with respect to the product measure $\lambda_n \otimes \nu \otimes \mathbb{P}$ (where ν is the invariant probability measure on the group SO_n). Therefore, Fubini's theorem can be applied, and we get

$$\begin{aligned}
&\mathbb{E} \int_{SO_n} \int_{\mathbb{E}^n} V_i(Z_X \cap K \cap (\vartheta r B^n + t))\, d\lambda_n(t)\, d\nu(\vartheta) \\
&\quad = \mathbb{E} \int_{SO_n} \int_{\mathbb{E}^n} V_i(Z_X \cap (\vartheta K + t) \cap rB^n)\, d\lambda_n(t)\, d\nu(\vartheta).
\end{aligned}$$

Here the principal kinematic formula can be applied on both sides, and we obtain

$$\sum_{k=i}^{n} \alpha_{nik} \mathbb{E}V_k(Z_X \cap K) V_{n+i-k}(rB^n) = \sum_{k=i}^{n} \alpha_{nik} V_k(K) \mathbb{E}V_{n+i-k}(Z_X \cap rB^n).$$

We divide both sides by $V_n(rB^n)$ and let $r \to \infty$. Since $V_m(rB^n) = r^m V_m(B^n)$ and $\alpha_{nii} = 1$, we obtain

$$\mathbb{E}V_i(Z_X \cap K) = \sum_{k=i}^{n} \alpha_{nik} V_k(K) \overline{\Delta}_{n+i-k}(Z_X). \tag{6}$$

This shows that

$$\frac{\mathbb{E}V_i(Z_X \cap K)}{V_n(K)} = \overline{\Delta}_i(Z_X) + \frac{1}{V_n(K)} \sum_{k=i}^{n-1} \alpha_{nik} V_k(K) \overline{\Delta}_{n+i-k}(Z_X).$$

If K expands to the whole space, the last term on the right side tends to zero, thus the estimator

$$\frac{V_i(Z_{X(\omega)} \cap K)}{V_n(K)}$$

is asymptotically unbiased. But one can also obtain unbiased estimators, by solving the system (1), which is easy, since it is triangular.

For example, in dimension 2, the following relations are obtained:

$$\begin{aligned}
\mathbb{E}V_2(Z \cap K) &= V_2(K)\Delta_2(Z) \\
\mathbb{E}V_1(Z \cap K) &= V_2(K)\Delta_1(Z) + V_1(K)\Delta_2(Z) \\
\mathbb{E}V_0(Z \cap K) &= V_2(K)\Delta_0(Z) + \frac{2}{\pi}V_1(K)\Delta_1(Z) + V_0(K)\Delta_2(Z)
\end{aligned}$$

with the solutions:

$$\begin{aligned}
\Delta_2(Z) &= \mathbb{E}\frac{1}{V_2(K)}V_2(Z \cap K) \\
\Delta_1(Z) &= \mathbb{E}\left[\frac{1}{V_2(K)}V_1(Z \cap K) - \frac{V_1(K)}{V_2(K)^2}V_2(Z \cap K)\right] \\
\Delta_0(Z) &= \mathbb{E}\left[\frac{1}{V_2(K)}V_0(Z \cap K) - \frac{2}{\pi}\frac{V_1(K)}{V_2(K)^2}V_1(Z \cap K)\right. \\
&\qquad \left. + \left(\frac{2}{\pi}\frac{V_1(K)^2}{V_2(K)^3} - \frac{1}{V_2(K)^2}\right)V_2(Z \cap K)\right]
\end{aligned}$$

5 Random planes

In this lecture, I want to give an example for a different type of relation between convexity and geometric probabilities. I will consider an extremal problem for intersection densities of certain systems of random hyperplanes. The solution of this problem will be achieved by applying a classical inequality from the geometry of convex bodies, the Aleksandrov-Fenchel inequality. The convex body to which this inequality is applied is an auxiliary body which is constructed from the data of the considered random hyperplane system. This method of auxiliary bodies has several applications in stochastic geometry, but I give only one example. The bodies that will be used are of a special type, namely zonoids, and I will first give a brief introduction to zonoids. I have already mentioned (and used) zonoids in the sketch of Klain's proof for Hadwiger's characterization theorem. These special convex bodies appear in several different contexts. They play a role in geometry, measure theory, functional analysis, and some other fields.

If we have a non-atomic $\mathbb{R}^n$-valued measure μ, then Liapounoff's theorem tells us that the range of μ is a compact convex set. Not every convex body can appear as such a range. If Z is obtained in this way and is suitably translated, one can show that its support function h_Z, defined by

$$h_Z(u) := \max\{\langle u, x\rangle : x \in K\}, \qquad u \in \mathbb{R}^n,$$

has a representation

$$h_Z(u) = \int_{S^{n-1}} |\langle u, v\rangle| \, d\varphi(v), \qquad u \in \mathbb{R}^n, \tag{7}$$

with a finite positive measure φ on the unit sphere $S^{n-1} := \{u \in \mathbb{R}^n : \|u\| = 1\}$. This measure can be assumed as even (i.e., satisfying $\varphi(-A) = \varphi(A)$ for every Borel set $A \subset S^{n-1}$), and then it is uniquely determined.

The representation (7) can be interpreted geometrically. The function $u \mapsto |\langle u, v\rangle|$ is the support function of the line segment conv $\{-v, v\}$. If the measure φ is concentrated on finitely many points, then h_Z is of the form

$$h_Z(u) = \sum_{i=1}^{k} \alpha_i |\langle u, v_i\rangle|$$

with $\alpha_1, \ldots, \alpha_k > 0$. This means that Z is a vector sum of finitely many segments, a so-called zonotope. This is a polytope with the property that all of its faces are centrally symmetric. A general zonoid can be approximated, in the Hausdorff metric, by zonotopes, and conversely every limit of zonotopes is a zonoid. In the space of convex bodies, the zonoids form a closed, nowhere dense subset.

We will need a formula for the intrinsic volumes of a zonoid. Suppose, first, that Z is a zonotope, say $Z = S_1 + \ldots + S_k$ with line segments $S_1, \ldots, S_k$. Then it is easy to see that

$$V_n(S_1 + \ldots + S_k) = \sum_{1 \le i_1 < \ldots < i_n \le k} V_n(S_{i_1} + \ldots + S_{i_n}).$$

For vectors $u_1, \ldots, u_j$, we denote by $[u_1, \ldots, u_j]$ the j-dimensional volume of the parallelepiped spanned by these vectors. Then the above formula for the volume of a zonotope extends to a formula for the volume of the zonoid Z represented by (7), namely

$$V_n(Z) = \frac{2^n}{n!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \ldots, u_n] \, d\varphi(u_1) \cdots d\varphi(u_n).$$

This formula, in turn, can be generalized to a formula for the intrinsic volumes, namely

$$V_j(Z) = \frac{2^j}{j!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \ldots, u_j] \, d\varphi(u_1) \cdots d\varphi(u_j).$$

After these preliminaries on zonoids, I will now explain the first problem of random geometry where they have proved useful. In my first lecture, I introduced the notion of a Poisson point process in a general locally compact (and second countable) space. This space was then taken to be the space of convex bodies in $\mathbb{E}^n$. Now we consider the space of k-dimensional planes in $\mathbb{E}^n$, which we denote by $A(n, k)$ (the affine Grassmannian). With its standard topology, it satisfies the assumptions. Hence, we may investigate Poisson processes in this space.

Let me start with an old example for a possible application of line-processes in the plane. I quote from a paper by R. Davidson [1974]:

“One can consider them [i.e., the line processes] as models for the arrangement of fibres in a sheet of paper. It being clear that the strength of a piece of paper depends largely on the number of crossings of its fibres, it becomes of interest to find that process which has the largest number of intersections per unit area relative (...) to its density.”

In order to answer this question, we first need a mathematical model for the system of fibres. To simplify the situation, we assume that the fibres are modelled by lines, which are spread out over the plane by some random mechanism. The question for the intersection point density only makes sense if the lines are sufficiently independent from each other. The appropriate model, therefore, is that of a Poisson process in the space of lines. We will, however, immediately study a generalization of the question in higher dimensions. Instead of lines in the plane, we consider hyperplanes in $\mathbb{E}^n$, together with the processes of lower dimensional flats they induce by intersections.

Recall that in the first lecture I have introduced Poisson point processes on an arbitrary locally compact space E with a countable base. For this space E we choose now the space $A(n,k)$, the affine Grassmannian of k-flats in $\mathbb{R}^n$, with the usual topology.

We first assume that X is a stationary process of k-flats in $\mathbb{E}^n$. Stationarity again means that the distribution of the process is invariant under translations. As usual, the *intensity measure* Θ of the process X is defined by

$$\Theta(B) = \mathbb{E}|X \cap B|$$

for Borel set $B \subset A(n,k)$. Due to the stationarity assumption, it again has a decomposition, this time of the form

$$\int_{A(n,k)} f \, d\Theta = \gamma \int_{G(n,k)} \int_{L^\perp} f(L+x) \, d\lambda_{L^\perp}(x) \, d\mathbb{P}_0(L).$$

Here $G(n,k)$ denotes the Grassmannian of linear subspaces of $\mathbb{E}^n$, $\mathbb{P}_0$ is a probability measure on $G(n,k)$, the *direction distribution* of X, and γ is a constant. This can be interpreted intuitively in the following ways. Writing

$$\mathcal{F}_K := \{A \subset \mathbb{E}^n : A \cap K \neq \emptyset\},$$

we have

$$\begin{aligned}
\mathbb{E}|X \cap \mathcal{F}_{B^n}| &= \mathbb{E} \sum_{E \in X} \mathbf{1}_{\mathcal{F}_{B^n}}(E) \\
&= \int_{A(n,k)} \mathbf{1}_{\mathcal{F}_{B^n}}(E) \, d\Theta(E) \\
&= \gamma \int_{G(n,k)} \int_{L^\perp} \mathbf{1}_{\mathcal{F}_{B^n}}(L+x) \, d\lambda_{L^\perp}(x) \, d\mathbb{P}_0(L) = \gamma \kappa_{n-k},
\end{aligned}$$

thus

$$\gamma = \frac{1}{\kappa_{n-k}} \mathbb{E}|X \cap \mathcal{F}_{B^n}|.$$

The number γ is called the intensity of the process X. For another interpretation, we denote by λ_E, for $E \in A(n,k)$, the k-dimensional Lebesgue measure on E, and we choose a Borel set $B \subset \mathbb{E}^n$. Then

$$\begin{aligned}\mathbb{E} \sum_{E \in X} \lambda_E(B) &= \int_{A(n,k)} \lambda_E(B)\, d\Theta(E) \\ &= \gamma \int_{G(n,k)} \int_{L^\perp} \lambda_{L+x}(B)\, d\lambda_{L^\perp}(x)\, d\mathbb{P}_0(L) \\ &= \gamma \lambda_n(B).\end{aligned}$$

Now let X be a stationary Poisson hyperplane process. If $n = 2$, this is a line process, and our initial question asked for the maximal intensity of the process of intersection points, given the intensity of the line process. For a hyperplane process, there are more such questions. Let $k \in \{2, \dots, n\}$. In any realization of X, we take all intersections of any k hyperplanes of the process which are in general position. This yields a stationary process X_k of $(n-k)$-flats. Its intensity γ_k we call the k-th *intersection density* of X. Now the question arises: if the intensity γ $(= \gamma_1)$ of X is given, which processes yield the maximal k-th intersection density? This question can be answered by using a classical result from the theory of convex bodies. The application is based on the following observations.

It is convenient to parametrize hyperplanes in the form

$$H_{u,\tau} = \{x \in \mathbb{E}^n : \langle x, u \rangle = \tau\}$$

with $u \in S^{n-1}$ and $\tau \in \mathbb{R}$. Thus u is one of the two unit normal vectors of the hyperplane, and $H_{u,\tau} = H_{-u,-\tau}$.

The direction distribution $\mathbb{P}_0$ of the hyperplane process X is a measure on $G(n, n-1)$, and it induces in an obvious way an even measure $\tilde{\mathbb{P}}$ on the sphere S^{n-1}. The decomposition of the intensity measure of X can so be written in the form

$$\int_{A(n,n-1)} f\, d\Theta = \gamma \int_{S^{n-1}} \int_{\mathbb{R}} f(H_{u,\tau})\, d\tau\, d\tilde{\mathbb{P}}(u).$$

We have now to compute the intensity γ_k of the k-th intersection process. For this, we use the representation

$$\gamma_k = \frac{1}{\kappa_k} \mathbb{E}|X_k \cap \mathcal{F}_{B^n}|.$$

For hyperplanes $H_1, \dots, H_k$, let

$$f(H_1, \dots, H_k) := \begin{cases} 1, & \text{if } H_1 \cap \ldots \cap H_k \cap B_n \neq \emptyset, \dim(H_1 \cap \ldots \cap H_k) = n-k \\ 0 & \text{otherwise.} \end{cases}$$

In the following we use again the fact that for the Poisson process X the intensity measure of the process $X^k_{\neq}$ is equal to the product measure Θ^k. We get

$$\mathbb{E}|X_k \cap \mathcal{F}_{B^n}| = \frac{1}{k!}\mathbb{E} \sum_{(H_1,\dots,H_k)\in X^k_{\neq}} f(H_1,\dots,H_k)$$

$$= \frac{1}{k!} \int\limits_{A(n,n-1)^k} f\, d\Lambda^{(k)}$$

$$= \frac{1}{k!} \int\limits_{A(n,n-1)} \cdots \int\limits_{A(n,n-1)} f(H_1,\dots,H_k)\, d\Theta(H_1)\cdots d\Theta(H_k)$$

$$= \frac{\gamma^k}{k!} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} \mathbf{1}\{u_1,\dots,u_k \text{ lin. indep.}\}$$

$$\underbrace{\int\limits_{\mathbb{R}} \cdots \int\limits_{\mathbb{R}} \mathbf{1}_{\mathcal{F}_{B^n}}(H_{u_1,\tau_1},\dots,H_{u_k,\tau_k})\, d\tau_1\cdots d\tau_k}_{I_k}\, d\tilde{\mathbb{P}}(u_1)\cdots d\tilde{\mathbb{P}}(u_2)$$

Let $u_1,\dots,u_k$ be linearly independent. Then in order to compute the integral I_k, we can transform the integral by introducing the intersection point of the hyperplanes $H_{u_1,\tau_1},\dots,H_{u_k,\tau_k}$ with the subspace spanned by $u_1,\dots,u_k$ as integration variable. This gives

$$I_k = \kappa_k[u_1,\dots,u_k].$$

The result then is

$$\gamma_k = \frac{1}{k!}\gamma^k \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} [u_1,\dots,u_k]\, d\tilde{\mathbb{P}}(u_1)\cdots d\tilde{\mathbb{P}}(u_k).$$

This should remind us of the formula for the k-th intrinsic volume of a zonoid. We now use the orientation distribution $\tilde{\mathbb{P}}$ as the generating measure of a zonoid Z. The function

$$h_Z(u) := \frac{1}{2}\gamma \int\limits_{S^{n-1}} |\langle u,v\rangle|\, d\tilde{\mathbb{P}}(v), \qquad u\in S^{n-1},$$

is the support function of a zonoid Z. A comparison with the earlier formula reveals the astonishing fact that the k-th intersection density of our Poisson hyperplane process X is given by

$$\gamma_k = V_k(Z).$$

We can now use the fact that the intrinsic volumes of a convex body satisfy various inequalities. In particular, the Aleksandrov-Fenchel inequality gives the following result.

Theorem. *The k-th intersection density γ_k ($k\in\{2,\dots,n\}$) of a stationary Poisson hyperplane process of intensity $\gamma>0$ in $\mathbb{E}^n$ satisfies the inequality*

$$\gamma_k \le c_{nk}\gamma^k \qquad \left(c_{nk} = \frac{\binom{n}{k}\kappa_{n-1}^k}{n^k\kappa_{n-k}\kappa_n^{k-1}}\right).$$

Equality holds if and only if the process is isotropic.

We indicate briefly how the equality condition comes out. The Brunn-Minkowski theory tells us the in the employed special case of the Aleksandrov-Fenchel inequalities equality holds only if the convex body Z is a ball. There is an analytical uniqueness theorem for the integral equation connecting h_Z and $\tilde{\mathbb{P}}$, and this tells us that in this case $\tilde{\mathbb{P}}$ must be rotation invariant. This implies that the intensity measure of the process X is motion invariant, and since X is a Poisson process, this implies that X itself has a motion invariant distribution.

The interest of the result lies in the fact that a most symmetric situation is characterized by an extremal property.

The method of "associated zonoids", which goes back to G. Matheron, has several more applications.

For example, the Poisson hyperplane process generates, in the obvious, the *zero-cell* or *Poisson polytope* C_0. For the expected number of its vertices one finds

$$\mathbb{E}|\mathrm{vert}\, C_0| = \frac{n!}{2^n} V_n(Z) V_n(Z^0).$$

Hence, known inequalities for the volume product of a zonoid (due, respectively, to Blaschke-Santaló and Reisner) give the sharp inequalities

$$2^n \leq \mathbb{E}|\mathrm{vert}\, C_0| \leq \frac{n!}{2^n} \kappa_n^2.$$

Equality on the right holds if and only if a suitable affine image of the hyperplane process is isotropic. Equality on the left holds if and only if the hyperplanes of X are almost surely parallel to n fixed hyperplanes.

6 Convex hulls of random points, I

In the remaining part of these lectures, I will consider a more elementary and basic situation in geometric probability theory, which has often been studied: finitely many random points, which are independently and identically distributed. One can ask many simple questions, but the anwers will in general not be easy to obtain. I concentrate here on questions concerning the convex hull of the random points. First I give a brief survey over some of the questions that have been treated in the past.

I start with an historical example from the 19th century, Sylvester's four point problem. Given four i.i.d. random points in the plane, what is the probability that they form a convex quadrilateral, that is, are the vertices of their convex hull? Of course, this question does not make sense as long as we do not specify the distribution of the random points. Suppose that the distribution is the uniform distribution in a convex body K in the plane. Let $p(K)$ denote the probability that four random points, chosen independently according to the uniform distribution in K, form a convex quadrilateral, that is, are the vertices of their convex hull. Denoting the area in the plane $\mathbb{R}^2$ by A, it is easy to see that

$$\begin{aligned} p(K) &= 1 - \frac{4}{A(K)^4} \int_K \int_K \int_K A(\operatorname{conv}\{x_1, x_2, x_3\})\, dx_1\, dx_2\, dx_3 \\ &= 1 - \frac{4}{A(K)} \mathbb{E} A(K, 4). \end{aligned}$$

The expectation occurring here is of the following type. Let φ be a (measurable) real function defined on polytopes in $\mathbb{R}^d$. Let $K \in \mathcal{K}^d$ be a convex body. If $X_1, \ldots, X_n$ (where $n \in \mathbb{N}$) are independent uniform random points in K, then

$$\varphi(K, n) := \varphi(\operatorname{conv}\{X_1, \ldots, X_n\})$$

is a random variable, and we denote its expectation by $\mathbb{E}\,\varphi(K, n)$. Such expectations have been the subject of many investigations, for functions φ like volume, surface area, mean width, number of vertices, number of facets, Hausdorff distance from K, and others. In the following, we will mainly consider the volume in $\mathbb{R}^d$, denoted by V. For $d = 2$, we continue to use A for the area.

What can we say about the random variable $V(K, n)$, for a given convex body K? It is not surprising that explicit computations are only possible for very special convex bodies, like simplex T^d, parallelepiped P^d, ball B^d. We suppose these convex bodies to be of volume 1; the same assumption is made for general convex bodies K appearing later.

We give a small list of cases where explicit computations have been possible:

$\underline{d = 2}$

Reed 1974:	moments of $A(T^2, 3)$, $A(P^2, 3)$
Alagar 1977:	distribution fct. of $A(T^2, 3)$
Henze 1983:	distribution fct. of $A(P^2, 3)$
Buchta 1984:	$\mathbb{E}A(Q, n)$, Q convex m-gon

$\underline{d \geq 2}$

Pederzoli 1985 – 87:	density function of $V(B^d, d+1)$
Affentranger 1988:	$\mathbb{E}V(B^d, n)$

Explicit computations may be difficult even in seemingly simple cases. A famous example is the problem to compute

$$\mathbb{E}V(T^3, 4),$$

the expected volume of a tetrahedron whose vertices are chosen at random from a tetrahedron of unit volume. Victor Klee posed this as a research problem in the Amer.

Math. Monthly in 1969. Although this seems to be just an elementary exercise in integration, it is so complicated that it took 30 years until someone succeeded with the calculation. In the meantime, Monte Carlo experiments have been done at several places. In this way, Do & Solomon (1986) found the interval [0.01686, 0.01756] as a 95% confidence interval. Recently Buchta & Reitzner (announced 1993) proved (on more than 100 pages) that

$$\mathbb{E}V(T^3, 4) = \frac{13}{720} - \frac{\pi^2}{15015} = 0,0173982\ldots$$

Making heavy use of computer algebra, Mannion (1994) obtained the same result; Buchta and reitzner, However, have more general results, also for $\mathbb{E}V(T^3, n)$.

For general convex bodies K (remember: $V(K) = 1$) of course, one will have to be satisfied with estimates. It is known that

$$\mathbb{E}V(K, n) \geq \mathbb{E}V(B^d, n)$$

for $n \geq d + 1$, with equality if and only if K is an ellipsoid. This result is due to Blaschke (1917) for $d = 2$ and $n = 3$ and to Groemer (1974) in the general case. The method of proof is the well known Steiner symmetrization. A sharp estimate in the other direction is harder to obtain. Blaschke (1917) used the so-called process of "Schüttelung" (shakedown) to prove that

$$\mathbb{E}A(K, 3) \leq \mathbb{E}A(T^2, 3) \qquad \text{for } d = 2,$$

with equality only if K is a triangle. This was extended by Dalla & Larman (1991) to the inequality

$$\mathbb{E}A(K, n) \leq \mathbb{E}A(T^2, n) \qquad \text{for } n \geq 3,$$

and Giannopoulos (1992) showed that equality holds only for triangles. An extension to higher dimensions is unknown; in particular, it is one of the most intriguing open questions in this area to decide whether

$$\mathbb{E}V(K, d + 1) \leq \mathbb{E}V(T^d, d + 1).$$

In the plane, the known results give

$$\frac{35}{48\pi^2} \leq \mathbb{E}A(K, 3) \leq \frac{1}{12}.$$

Henze (1983) has observed that the proofs yield similar estimates for the distribution function of $A(K, 3)$.

Coming back to Sylvester's problem, we now see that

$$\frac{2}{3} \leq p(K) \leq 1 - \frac{35}{12\pi^2} = 0,7048\ldots$$

We now turn to asymptotic results for the random variable $V(K, n)$, that is, results referring to the limit procedure $n \to \infty$. In other words, how does the volume of conv $\{X_1, \ldots, X_n\}$ behave for large n? In the plane, such questions have first been treated in three influential papers by Rényi and Sulanke in the Sixties. These papers have led to many subsequent investigations. I mention only a few of the more recent results. It turns out that the behaviour of $V(K,n)$ for large n is very sensitive against the boundary structure of the convex body K. Let us first consider the case of a d-polytope P. A rather deep result, due to Affentranger and Wieacker (1991) for simple polytopes and to Bárány & Buchta (1993+) in general, states that

$$\mathbb{E}V(P,n) = 1 - \frac{T(P)}{(d+1)^{d-1}(d-1)!}\frac{\log^{d-1} n}{n} + O\left(\frac{\log^{d-2} n \log\log n}{n}\right).$$

Here $T(P)$ is a combinatorial invariant of P, the number of complete towers $F_0 \subset F_1 \subset \ldots \subset F_{d-1}$ of i-dimensional faces of P.

A very different asymptotic behaviour is observed for sufficiently smooth convex bodies. Let $K \in \mathcal{K}^d$ be a convex body with a boundary of class C^3 and with positive Gauss-Kronecker curvature κ. Then Bárány (1992) proved that

$$\mathbb{E}V(K,n) = 1 - c(d) \int_{\partial K} \kappa^{\frac{1}{d+1}} \, dF \left(\frac{n}{V}\right)^{-\frac{2}{d+1}} + O\left(n^{-\frac{3}{d+1}} \log^2 n\right),$$

where $c(d)$ is a constant. Thus the approximations in the case of polytopes and in the case of smooth bodies are of different orders of magnitude.

These results and the methods involved in their proofs are particularly interesting from a geometric point of view, but from the stochastic view point they may be a bit disappointing, since they concern only convergence of expectations. In the plane one has a few results of a more stochastic nature; exhibiting convergence of random variables. For example, Cabo & Groeneboom (1991) were able to show that for a k-gon in $\mathbb{R}^2$ one has

$$\frac{1 - A(P,n) - \frac{2}{3}k\frac{\log n}{n}}{\sqrt{\frac{100}{189}k\frac{\log n}{n}}} \xrightarrow{\mathcal{D}} N(0,1),$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $N(0,1)$ is the standard normal distribution.

In all the examples considered so far, the random points were chosen in K. Now we choose random points on the boundary of K. We assume that $X_1, X_2, \ldots$ is a sequence of independent, identically distributed random points in ∂K, and we put

$$P_n := \text{conv}\{X_1, \ldots, X_n\} \qquad \text{for } n \in \mathbb{N}.$$

Thus we get a sequence of random polytopes inscribed to K. Under suitable assumptions on the distribution μ of X_i, this sequence will almost surely converge to K (in the Hausdorff metric), and we may ask, for example, how fast the random variables $V(P_n)$ converge to 1. First we consider the case $d = 2$. Let ∂K be of class C^2 and of positive curvature κ. About the distribution μ we assume that it has a positive continuous density h with respect to the arc length measure. Under these assumptions it was proved by Schneider (1988) that

$$\lim_{n\to\infty} n^2[1 - A(P_n)] = \frac{1}{2}\int_{\partial K} h^{-2}\kappa \, dS \qquad \text{almost surely.}$$

It follows from Hölder's inequality that the best approximation (the smallest right-hand side) is obtained if h is proportional to $\kappa^{\frac{1}{3}}$.

We can interpret the above as a result on random approximation, measured in terms of the symmetric-difference metric. A similar result holds for the more familiar Hausdorff metric δ:

$$\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^2 \delta(K,P_n) = \frac{1}{8}\max_{s\in\partial K}\frac{\kappa(s)}{h(s)^2} \qquad \text{a.s.}$$

In this case, the optimal approximation is obtained if h is proportional to $\sqrt{\kappa}$.

It would be interesting to have similar results on random approximation in higher dimensions. For the Hausdorff metric, the following was proved recently (Glasauer & Schneider). Let $K \in \mathcal{K}^d$, $d \geq 3$, be of class C^3 with positive Gauss-Kronecker curvature. Let μ be a probability distribution on ∂K which has a positive density h of class C^1 with respect to the surface area measure. Now let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random points on K with distribution μ. Then

$$P - \lim_{n\to\infty}\left(\frac{n}{\log n}\right)^{\frac{2}{d-1}} \delta(K,P_n) = \frac{1}{2}\left(\frac{1}{b_{d-1}}\max_{x\in\partial K}\frac{\sqrt{\kappa(x)}}{h(x)}\right)^{\frac{2}{d-1}},$$

where b_{d-1} is the volume of the $(d-1)$-dimensional unit ball and $P - \lim$ denotes a stochastic limit. Thus for $d \geq 3$, we have to make stronger differentiability assumptions, and we have only stochastic convergence.

Finally, I would like to consider an extremal problem for a geometrical probability of a different kind. First, let K be a convex body in the plane. Three independently and uniformly distributed random points in K determine, with probability 1, a unique circle on which they lie. What is the probability that this circle is contained in K? Affentranger has shown that this probability is $\leq \frac{2}{5}$ and that equality holds if and only if K is a circular disk. Next, we ask, not for the circle through the random points, but for the smallest circular disk containing the points. The probability that this smallest circular disk is contained in K turns out to be $\leq \frac{3}{5}$, again with equality if and only if K is itself a circular disk. I would like to say a few words how the extension of this question to higher dimensions and arbitrary finite numbers of points can be treated.

For a finite set $\{x_1,\dots,x_m\}$ in $\mathbb{R}^d$, let $B(x_1,\dots,x_m)$ be the circumball (the smallest closed ball containing the point set), and denote its boundary by $\partial B(x_1,\dots,x_m)$ and its radius by $r(x_1,\dots,x_m)$.

We are interested in the probability

$$p_m(K) := \mathbb{P}(B(X_1,\dots,X_m) \subset K)$$

for convex bodies K in $\mathbb{R}^d$ and $m \geq 2$ independent uniform random points $X_1,\dots,X_m$ in K. We first derive a formula for $p_m(K)$.

With probability 1, at most $d+1$ (and at least two) of the points $X_1,\dots,X_m$ lie on the boundary of $B(X_1,\dots,X_m)$, and $B(X_1,\dots,X_m)$ is the circumball of these points.

Writing

$$f_q(x_1,\dots,x_m) := \begin{cases} 1, & \text{if } B(x_1,\dots,x_m)\subset K,\ x_1,\dots,x_{q+1}\in\partial B(x_1,\dots,x_m), \\ & \quad x_{q+2},\dots,x_m\in \operatorname{int} B(x_1,\dots,x_m), \\ 0 & \text{otherwise,} \end{cases}$$

we therefore get

$$\begin{aligned} p_m(K) &= \frac{1}{V(K)^m}\sum_{q=1}^{d}\binom{m}{q+1}\int\cdots\int f_q(x_1,\dots,x_m)\,dx_1\cdots dx_m \\ &= \frac{1}{V(K)^m}\sum_{q=1}^{d}\binom{m}{q+1}\kappa_d^{m-q-1}A_q \end{aligned}$$

with

$$A_q := \int\cdots\int f_q(x_1,\dots,x_{q+1})r(x_1,\dots,x_{q+1})^{d(m-q-1)}\,dx_1\cdots dx_{q+1}.$$

Using some integral-geometric transformations, the latter integral can be expressed in the form

$$A_q = \frac{1}{d(m-1)}c_{dq}(q!)^{d-q+1}M(q,d-q+1)\int_K \operatorname{dist}(z,dx)^{d(m-1)}\,dz$$

with

$$c_{dq} = \frac{\omega_{q-q+1}\cdots\omega_d}{\omega_1\cdots\omega_q}, \qquad \omega_j = j\kappa_j,$$

$$\begin{aligned} M(d,k) := &\int_{S^{d-1}}\cdots\int_{S^{d-1}} \mathbf{1}\{u_1,\dots,u_{d+1} \text{ not in an open hemisphere}\} \\ & V(\operatorname{conv}\{u_1,\dots,u_{d+1}\})^k\ d\sigma(u_1)\cdots d\sigma(u_{d+1}). \end{aligned}$$

Writing

$$I_k(K) := \int_K \operatorname{dist}(z,\partial K)^k\,dz,$$

the resulting formula for the probability $p_m(K)$ is

$$p_m(K) = \frac{1}{d(m-1)}\frac{I_{d(m-1)}(K)}{V(K)^m}\sum_{q=1}^{m}\binom{m}{q+1}\kappa_d^{m-q-1}c_{dq}(q!)^{d-q+1}M(q,d-q+1).$$

An inequality of Bol says that

$$I_k(K) \le \binom{d+m}{d}^{-1} V(K)^{\frac{d+m}{m}}\kappa_d^{-\frac{m}{d}},$$

with equality if and only if K is a ball. Hence,

$$p_m(K) \le p_m(B^d),$$

with equality if and only if K is a ball.

For $d=2$, one gets

$$p_m(B^2) = \frac{m}{2m-1}.$$

7 Convex hulls of random points, II

In the introduction of my last lecture, I have mentioned the historical Sylvester four point problem: what is the probability that four independent, identically distributed random points in the plane are the vertices of a convex quadrilateral? As I said, a common interpretation of this question assumes that the points are uniformly distributed in a given convex body. One may ask whether there are other natural ways of specifying a probability measure, not depending on an additional choice like that of a convex body. Now Sylvester's question is of an affine nature rather than Euclidean. It would, therefore, be rather more natural to specify a probability distribution on the set of affine equivalence classes of n-tuples of points. The following approach leads to a natural distribution. Every configuration of $n+1$ numbered points in general position in $\mathbb{R}^d$ is affinely equivalent to the orthogonal projection of the set of numbered vertices of a fixed regular simplex $T^n \subset \mathbb{R}^n$ onto a unique d-dimensional linear subspace of $\mathbb{R}^n$. This establishes a one-to-one correspondence between the (orientation-preserving) affine equivalence classes of such configurations and an open dense subset of the Grassmannian $G(n, d)$ of oriented d-spaces in $\mathbb{R}^n$. The unique rotation invariant probability measure on $G(n, d)$ thus leads to a probability distribution on the set of affine equivalence classes of $(n + 1)$-tuples of points in general position in $\mathbb{R}^d$. This so-called "Grassmannian approach" was proposed independently by Vershik and by Goodman & Pollack.

Later Baryshnikov & Vitale proved that an affine-invariant function on $(n+1)$-tuples with the described distribution is stochastically equivalent to the same function taken at an i.i.d. $(n+1)$-tuple of standard normal points in $\mathbb{R}^d$. Therefore, the results I am going to explain can also be considered as results about convex hulls of independent standard normal points.

What we want to study in the following is the number, f_k, of k-dimensional faces of the convex hull of $n+1$ random points. We use the probability distribution described above.

Thus, let T^n be an n-dimensional regular simplex in Euclidean space $\mathbb{R}^n$. We project T^n orthogonally into a randomly chosen d-dimensional linear subspace. The distribution of this subspace shall be given by the unique rotation invariant probability measure on the Grassmannian $G(n, d)$. Let

$$\mathbb{E} f_k(\Pi_d T^n)$$

denote the expected value of the number of k-faces of the projection. We will derive an asymptotic formula for this for $n \to \infty$.

Firt, we use some integral geometry to obtain an explicit expression involving higher dimensional angles. This can be done more generally for arbitrary convex polytopes.

The (unique) rotation-invariant probability measure on the Grassmannian $G(n, d)$ is denoted by ν_d. For $L \in G(n, d)$, let Π_L be the orthogonal projection from $\mathbb{R}^n$ onto L. Let Λ be an isotropic random d-subspace, that is, a random variable with values in $G(n, d)$ and with distribution ν_d. We denote the projection Π_Λ by Π_d, thus for a convex polytope P, $\Pi_d P$ is a random polytope and $f_k(\Pi_d P)$ is an integer-valued random variable, for $k \in \{0, 1, \ldots, d-1\}$. Its expectation is given by

$$\mathbb{E} f_k(\Pi_d P) = \int\limits_{G(n,d)} f_k(\Pi_L P)\, d\nu_d(L).$$

The right-hand side can be computed by means of integral geometry. It is clear that a given k-face F_k of P contributes to the k-faces of the projection $\Pi_L P$ if and only if the following is true: For a relatively interior point x of F_k, the affine subspace $L^\perp + x$ does not intersect the interior of P. The situation can now be described by considering a small sphere with centre x. The polytope P intersects it in a spherical polytope P_x. The subspace $L^\perp + x$ intersects the sphere in a great sphere of dimension $n-k-1$. We have to determine (essentially) the measure of all rotations which bring this great sphere into a non-hitting position with the polytope P_x. This is a typical task of spherical integral geometry. The answer can be given in terms of interior and exterior angles.

For a polytope P and a face F of P, let $\beta(F,P)$ be the internal and $\gamma(F,P)$ the external angle of P at F. By definition, $\beta(F,F) = \gamma(F,F) = 1$ and $\beta(F,P) = \gamma(F,P) = 0$ if $F \not\subset P$. Let $\mathcal{F}_r(P)$ denote the set of r-faces of the polytope P.

The result given by spherical integral geometry can then be expressed in the form

$$\boxed{\mathbb{E} f_k(\Pi_d P) = 2 \sum_{S \geq 0} \sum_{F \in \mathcal{F}_k(P)} \sum_{G \in \mathcal{F}_{d-1-2s}(P)} \beta(F,G)\gamma(G,P).}$$

Let us specialize this to the case where $P \subset \mathbb{R}^3$ and $d = 2$. For $k = 0$ we get

$$\begin{aligned} \mathbb{E} f_0(\Pi_2 P) &= 2 \sum_{F_0 \in \mathcal{F}_0(P)} \sum_{G_1 \in \mathcal{F}_1(P)} \underbrace{\beta(F_0, G_1)}_{\frac{1}{2},\ \text{if } F_0 \subset G_1} \gamma(G_1, P) \\ &= 2 \sum_{G_1 \in \mathcal{F}_1(P)} \gamma(G_1, P). \end{aligned}$$

For example, for the cube we have $\gamma(G_1, P) = \frac{1}{4}$ and hence $\mathbb{E} f_0(\Pi_2 P) = 6$, which is, however, not surprising, since in this case $f_0(\Pi_2 P) = 6$ almost surely. For $\mathbb{E} f_1(\Pi_2 P)$ we get, of course, the same number.

Now back to the general case. The internal and external angles appearing there are spherical volumes of spherical polytopes, hence there is little hope for explicit calculations. Let us now turn to the case of the regular simplex T^n in which we are interested.

Let $F \in \mathcal{F}_k(T^n)$ be a k-dimensional face of T^n. The set of exterior unit normal vectors to T^n at some relatively interior point of F is an $(n-k-1)$-dimensional regular spherical simplex lying in an $(n-k-1)$-dimensional great sphere S^{n-k-1} of the unit sphere S^{n-1}. Its spherical edge length is equal to the angle between the exterior normal vectors of two distinct facets of T^n, which is given by

$$\arccos\left(-\frac{1}{n}\right).$$

Let $v(m,\alpha)$ denote the m-dimensional spherical measure of a regular spherical simplex in S^m of edge length α, and let ω_m be the total measure of S^m. Thus the external angle of T^n at ist k-face F_k is given by

$$\gamma(F_k,T^n) = \frac{v\left(n-k-1, \arccos\left(-\frac{1}{n}\right)\right)}{\omega_{n-k-1}}.$$

For the internal angle of T^n at F_k one finds

$$\beta(F_k,T^n) = \frac{v\left(n-k-1, \arccos\frac{1}{k+2}\right)}{\omega_{n-k-1}}.$$

Now we consider first the case $k=d-1$. In this case, the general formula for $\mathbb{E}f_k(\Pi_d P)$ reduces (for $P=T^n$) to

$$\mathbb{E}f_{d-1}(\Pi_d T^n) = 2\binom{n+1}{d}\gamma(T^{d-1},T^n)$$

(we consider T^k, for $k<n$, as a face of T^n). We need a general integral formula for the volume of regular spherical simplices. Let $m\geq 2$. If

$$-\frac{1}{m-1} < \cos\alpha \leq 0,$$

then this formula says that

$$\frac{v(m-1,\alpha)}{\omega_{m-1}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{A_m(\alpha)t} e^{-s^2}\,ds\right)^m dt,$$

where

$$A_m(\alpha) := \left(\frac{-\cos\alpha}{1+(m-1)\cos\alpha}\right)^{\frac{1}{2}}.$$

We use this for $m=n-d+1$ and $\cos\alpha = -\frac{1}{n}$ and obtain

$$\gamma(T^{d-1},T^n) = \sqrt{\frac{d}{\pi}}\int_{-\infty}^{\infty} e^{-dt^2}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} e^{-s^2}\,ds\right)^{n-d+1} dt.$$

With

$$\varphi(t) := \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} e^{-s^2}\,ds,$$

our expectation $\mathbb{E}f_{d-1}(\Pi_d T^n)$ can be written in the form

$$\mathbb{E}f_{d-1}(\Pi_d T^n) = 2\binom{n+1}{d}\sqrt{\frac{d}{\pi}}\int_{-\infty}^{\infty} e^{-dt^2}\varphi(t)^{n-d+1}\,dt.$$

For $n\to\infty$, an asymptotic estimation of this integral is possible (Raynaud, Affentranger), and one finds that

$$\boxed{\mathbb{E} f_{d-1}(\Pi_d T^n) \sim \frac{2^d}{\sqrt{d}} (\pi \log n)^{\frac{d-1}{2}}}$$

as $n \to \infty$.

For arbitrary k, we have from the general formula that

$$\mathbb{E} f_k(\Pi_d T^n) 0 2 \sum_{s \geq 0} \binom{n+1}{d-2s} \binom{d-2s}{k+1} \beta(T^k, T^{d-2s-1}) \gamma(T^{d-2s-1}, T^n).$$

Now we observe that here the number of nonzero summands does not depend on n and that the already obtained result, applied to $d - 2s$ instead of s, gives

$$2\binom{n+1}{d-2s} \gamma(T^{d-2s-1}, T^n) \sim \frac{2^{d-2s}}{\sqrt{d-2s}} (\pi \log n)^{\frac{d-2s-1}{2}}$$

for $n \to \infty$. It follows that the term with $s = 0$ is dominating, and we obtain

$$\boxed{\mathbb{E} f_k(\Pi_d T^n) \sim \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta(T^k, T^{d-1}) (\pi \log n)^{\frac{d-1}{2}}}$$

This result has also a counterpart in which the projection is onto subspaces of fixed codimension:

For any given integers $0 \leq k < n - d$,

$$\boxed{\mathbb{E} f_k(\Pi_{n-d} T^n) \sim \binom{n+1}{k+1} = f_k(T^n)}$$

as n tends to infinity.

I would like to add some remarks on the topic. What I presented here was joint work with Affentranger, some years ago. We noticed that the obtained value for the expected number of facets of the orthogonal projection of an n-dimensional regular simplex onto an isotropic d-dimensional random subspace of $\mathbb{R}^n$, namely

$$\mathbb{E} f_{d-1}(\Pi_d T^n) = 2\binom{n+1}{d} \sqrt{\frac{d}{\pi}} \int_{-\infty}^{\infty} e^{-dt^2} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} e^{s^2} \, ds \right)^{n-d+1} dt,$$

coincided with the expected number of facets of the convex hull of $n+1$ i.i.d. normally distributed random points in $\mathbb{R}^n$. The explanation came only later, by Baryshnikov and Vitale, who established the equivalence that I mendioned earlier. This equivalence allowed then to transscribe our results on k-faces into results of convex hulls of standard Gaussian samples.

The regular simplex is one of the three types of regular polytopes that exist in all dimensions. The other two are the cube and the crosspolytope. For the cube W^n the internal and external angles are easily determined, and one finds that

$$\begin{aligned} \mathbb{E} f_k(\Pi_d W^n)) &= 2\binom{n}{k} \sum_{s \geq 0} \binom{n-k}{d-1-2s-k} \\ &\sim 2 \frac{n^{d-1}}{(d-1-k)!k!} \end{aligned}$$

Böröczky and Henk have treated the case of the crosspolytope C^n. They found that

$$\mathbb{E} f_k(\Pi_d C^n) \sim \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta(T^k, T^{d-1}) (\pi \log n)^{\frac{d-1}{2}}.$$

This is surprisingly the same result as for the regular simplex. As the authors remark, they are not aware of any direct argument leading to this coincidence. The authors remark further that the results for simplex and crosspolytope still contain the “unknown” internal angles $\beta(T^k, T^{d-1})$. For this, they obtain an asymptotic representation for each fixed k and for $d \to \infty$, namely

$$\beta(T^k, T^{d-1}) = \frac{(k+1)^{\frac{d-k-2}{2}} e^{\frac{d-3k-2}{2}}}{\sqrt{2}^{d-k} \sqrt{\pi}^{d-k-1} d^{\frac{d-k-2}{2}}} \left(1 + O\left(\frac{k^2+1}{d}\right)\right).$$