# The use of spherical harmonics in convex geometry *

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## 1 Introduction

The following lectures concern only half the title of the summer school, namely 'Fourier analytic methods in convexity'. More precisely, they deal with the use of elementary harmonic analysis in convex geometry, and even more precisely, with applications of spherical harmonics to questions of uniqueness and stability for convex bodies. I will use some classical uniqueness results for convex bodies as a starting point for a brief introduction to spherical harmonics. Then I will present some more recent stability and approximation results, obtained with the aid of spherical harmonics.

We work in $d$-dimensional Euclidean space $\mathbb{R}^d$ with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The rotation group $SO_d$ acts transitively on the unit sphere

$$S^{d-1} := \{u \in \mathbb{R}^d : \|u\| = 1\},$$

a fact that is crucial for all what follows. We denote the unit ball $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ by $B^d$. All signed measures on $S^{d-1}$ or $\mathbb{R}^d$ that appear in the following are defined on the $\sigma$-algebra of Borel sets. The spherical Lebesgue measure on $S^{d-1}$ is denoted by $\sigma$.

Several uniqueness questions for convex bodies lead to a functional equation of the following type. Let $f$ be a real function and $\mu$ a finite signed measure on the sphere $S^{d-1}$. We consider the equation

$$\int_{S^{d-1}} f(\vartheta v)\, \mu(\mathrm{d}v) = 0 \qquad \text{for all } \vartheta \in SO_d \tag{1}$$

(assuming that all the integrals exist). Here, either the signed measure $\mu$ is given, and one has to determine all continuous functions $f$ satisfying (1), or the function $f$ is given (continuous, or nonnegative and measurable), and one has to determine all signed measures $\mu$ satisfying (1). In some cases, the solutions may be required to satisfy additional constraints (e.g., to be even (invariant under reflection in the origin) or odd.

The following collection of classical results about convex bodies exhibits various special cases of the equation (1) that have occurred in the literature.

**1. Aleksandrov's projection theorem.** Let $K, L$ be $d$-dimensional centrally symmetric convex bodies with the property that

$$V_{d-1}(K|u^\perp) = V_{d-1}(L|u^\perp) \qquad \text{for all } u \in S^{d-1}, \tag{2}$$

*Summer school on "Fourier analytic and probabilistic methods in geometric functional analysis and convexity", Kent State University, August 13–20, 2008

where $V_{d-1}$ denotes the $(d-1)$-dimensional volume and $K|u^\perp$ is the orthogonal projection of $K$ to the hyperplane $u^\perp$ through 0 orthogonal to $u$. We have

$$V_{d-1}(K|u^\perp) = \frac{1}{2}\int_{S^{d-1}} |\langle u, v\rangle|\, S_{d-1}(K, \mathrm{d}v),$$

where $S_{d-1}(K,\cdot)$ is the surface area measure of $K$, a finite measure on $S^{d-1}$. Hence, the condition leads to (1) with

$$f(v) := |\langle e, v\rangle|, \qquad \mu := S_{d-1}(K,\cdot) - S_{d-1}(L,\cdot),$$

with some fixed vector $e \in S^{d-1}$ (note that $f(\vartheta v) = |\langle \vartheta^{-1}e, v\rangle|$, and every $u \in S^{d-1}$ can be written as $u = \vartheta^{-1}e$ with a suitable rotation $\vartheta$). Due to the assumption of central symmetry, the signed measure $\mu$ is even. It turns out that the only even signed measure satisfying (1) for this special function $f$ is the zero measure. Two $d$-dimensional convex bodies $K, L$ with $S_{d-1}(K,\cdot) = S_{d-1}(L,\cdot)$ are translates of each other, by the Aleksandrov–Fenchel–Jessen theorem. Thus, we have to combine an analytic and a geometric uniqueness theorem to obtain Aleksandrov' projection theorem.

**2. A theorem of Minkowski.** This theorem says that every three-dimensional convex body of constant girth is of constant width. More generally, let $K$ be a $d$-dimensional convex body with the property that

$$V_1(K|u^\perp) = \text{const} \qquad \text{for all } u \in S^{d-1}. \tag{3}$$

Here $V_1$ is (up to a constant factor) the mean width. To reformulate condition (3), we introduce the support function

$$h(K, x) := \max\{\langle x, y\rangle : y \in K\}$$

and the great subsphere $S_e := \{u \in S^{d-1} : \langle u, e\rangle = 0\}$, for $e \in S^{d-1}$. We denote by $\sigma_e$ the $(d-2)$-dimensional spherical Lebesgue measure on $S_e$. The condition on $K$ can be written as

$$\int_{S_e} [h(K,v) + h(K,-v) - c]\, \sigma_e(\mathrm{d}v) = 0 \qquad \text{for all } e \in S^{d-1},$$

with a suitable number $c > 0$, or equivalently with some fixed $e \in S^{d-1}$,

$$\int_{S_e} [h(K,\vartheta v) + h(K,-\vartheta v) - c]\, \sigma_e(\mathrm{d}v) = 0 \qquad \text{for all } \vartheta \in SO_d.$$

This is of type (1) with $\mu = \sigma_e$, and it turns out that the only even solution is the zero function. Hence, $h(K,v) + h(K,-v) = c$ for $v \in S^{d-1}$, which means that $K$ is of constant width $c$.

**3. A theorem of Funk.** This theorem says that every convex body (or star body) $K$ with the property that every hyperplane through 0 divides the body into two parts of equal volume, must be centrally symmetric with respect to 0. Introducing the radial function of $K$,

$$\rho(K, v) := \max\{\lambda \geq 0 : \lambda v \in K\}, \qquad v \in S^{d-1},$$

the condition leads to an equation of type (1) with

$$f(v) := \rho(K,v)^d - \rho(K,-v)^d, \qquad \mu := \sigma \llcorner e^+,$$

where $e \in S^{d-1}$ is fixed, $e^+ := \{u \in S^{d-1} : \langle u, e\rangle \geq 0\}$, and $\llcorner$ denotes restriction of a measure. In this case, the only odd solution is the zero function, hence we obtain that $\rho(K, v) = \rho(K, -v)$ for all $v \in S^{d-1}$.

**4. A theorem of Blaschke.** As Monge has found, an ellipsoid has the following property: the vertices of all its circumscribed boxes (rectangular parallelepipeds) lie on a fixed sphere. Blaschke has proved that this property characterizes ellipsoids among all convex bodies. Suppose that the vertices of the boxes circumscribed to the convex body $K$ lie on the sphere with center 0 and radius $R$. Let $(e_1, \dots, e_d)$ be an orthonormal basis of $\mathbb{R}^d$. Then

$$\sum_{i=1}^{d} h(K, \vartheta e_i)^2 = R^2 \qquad \text{for all } \vartheta \in SO_d.$$

This is equivalent to (1) with

$$f(v) := h(K, v)^2 - \frac{1}{d}R^2, \qquad \mu := \sum_{i=1}^{d} \delta_{e_i},$$

where $\delta$ denotes the Dirac measure. For this measure $\mu$, the only continuous solutions of (1) are restrictions to $S^{d-1}$ of harmonic homogeneous polynomials of degree two, and this yields that $K$ must be an ellipsoid.

**5. A theorem of Meissner.** Let $T$ be a regular simplex. A convex body $K$ contained in $T$ is called a **rotor** of $T$ if it can be completely turned inside $T$, always gliding along its facets. More precisely, this condition demands that to every rotation $\vartheta \in SO_d$ there exists a translation vector $t$ such that $\vartheta K + t$ is contained in $T$ and touches all the facets of $T$. Which rotors exist besides a ball? This question was answered by Meissner in three-space. We shall later mention the answer for $d$-space. The question leads to the following equation of type (1). Let $u_1, \dots, u_{d+1}$ be the outer unit normal vectors of the facets of $T$, and let $R$ denote the inradius of $T$. Then the condition is equivalent to (1) with

$$f(v) := h(K, v) - R, \qquad \mu := \sum_{i=1}^{d+1} \delta_{u_i}.$$

These results are all very old. They serve us here to illuminate the role of the unifying equation

$$\int_{S^{d-1}} f(\vartheta v)\, \mu(\mathrm{d}v) = 0 \qquad \text{for all } \vartheta \in SO_d.$$

To give a first hint of how to approach it, suppose that the signed measure $\mu$ is given and we want to find all solutions $f \in \mathbf{C}(S^{d-1})$. Clearly, the set of all solutions is a vector subspace $V$ of $\mathbf{C}(S^{d-1})$, which is invariant under the action of the rotation group, that is, if $f \in V$, then for each $\vartheta \in SO_d$ also $\vartheta f \in V$ (where $(\vartheta f)(u) := f(\vartheta^{-1}u)$). Now the spherical harmonics enter the scene. They are, in a sense to be made precise, the elements of the simplest invariant subspaces of $\mathbf{C}(S^{d-1})$. From the spherical harmonics solving the equation, one can construct all solutions. We turn now to an introduction to the theory of spherical harmonics.

## 2 Spherical Harmonics

First, we define spherical harmonics as restrictions of homogeneous harmonic polynomials to the unit sphere and use this to establish their basic properties. After that, the connection of spherical harmonics with representations of the rotation group is explained.

To relate functions on the sphere $S^{d-1}$ to functions on $\mathbb{R}^d$, we define

$$\check{f}(x) = f^{\vee}(x) := f\left(\frac{x}{\|x\|}\right), \qquad x \in \mathbb{R}^d \setminus \{0\},$$

for $f : S^{d-1} \to \mathbb{R}$, and for $g : \mathbb{R}^d \to \mathbb{R}$ we denote by $\hat{g} = g^{\wedge} := g|S^{d-1}$ the restriction to $S^{d-1}$.

A function $F : \mathbb{R}^d \to \mathbb{R}$ is called **homogeneous of degree** $k$ if

$$F(tx) = t^k F(x)$$

holds for all $x \in \mathbb{R}^d$ and all $t > 0$. If this holds and $F$ is of class $C^1$, then it follows by partial differentiation that $\partial_i F$ is homogeneous of degree $k-1$.

The Laplace operator $\Delta$ on $\mathbb{R}^d$ is defined by

$$\Delta f := \sum_{i=1}^{d} \partial_{ii} f.$$

A function $f : \mathbb{R}^d \to \mathbb{R}$ of class $C^2$ with $\Delta f = 0$ is called **harmonic**.

In the following, $f$ and $g$ are real functions of class $C^2$ on the sphere $S^{d-1}$. The spherical Laplace operator can be defined by

$$\Delta_S f := (\Delta \check{f})|S^{d-1} = (\Delta f^{\vee})^{\wedge}.$$

**Lemma 2.1.**

$$\int_{S^{d-1}} f \Delta_S g \,\mathrm{d}\sigma = \int_{S^{d-1}} g \Delta_S f \,\mathrm{d}\sigma.$$

*Proof.* The following integrals over balls and spheres are with respect to the standard measures, and $\partial/\partial r$ denotes differentiation in radial direction. Using Green's formula in $\mathbb{R}^d$, we obtain

$$\int_{1 \le \|x\| \le 2} (\check{g}\Delta f - \check{f}\Delta g) = \int_{\|x\|=2} \left(\check{g}\frac{\partial \check{f}}{\partial r} - \check{f}\frac{\partial \check{g}}{\partial r}\right) - \int_{\|x\|=1} \left(\check{g}\frac{\partial \check{f}}{\partial r} - \check{f}\frac{\partial \check{g}}{\partial r}\right).$$

Since $\check{f}, \check{g}$ are homogeneous of degree 0, the right side vanishes; more explicitly,

$$\check{f}(tx) = \check{f}(x) \Rightarrow \sum_{i=1}^{d} \frac{\partial \check{f}}{\partial x_i}(tx) \cdot x = 0 \Rightarrow \langle \mathrm{grad}\check{f}(x), x \rangle = 0.$$

For the integration on the left side, we use spherical cordinates, that is, we write $x \in \mathbb{R}^d \setminus \{0\}$ in the form $x = rx_0$ with $r = \|x\|$ and $x_0 \in S^{d-1}$ and employ the transformation formula $\mathrm{d}V(x) = r^{d-1}\,\mathrm{d}\sigma(x_0)\,\mathrm{d}r$ for the volume element at $x$. Since $\Delta \check{f}$ is homogeneous of degree $-2$, we get

$$0 = \int_1^2 \int_{S^{d-1}} (g\Delta_S f - f\Delta_S g) r^{d-3} \,\mathrm{d}\sigma\,\mathrm{d}r$$

and thus the assertion. $\square$

For homogeneous functions, the Laplace operator can be expressed in terms of the spherical Laplace operator.

**Lemma 2.2.** *If $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is twice continuously differentiable and homogeneous of degree $k$, then*

$$\Delta f(x) = k(k+d-2)\|x\|^{k-2} f\left(\frac{x}{\|x\|}\right) + \|x\|^{k-2} \Delta_S \hat{f}\left(\frac{x}{\|x\|}\right).$$

*Proof.* We apply the formula

$$\Delta(HG) = H\Delta G + 2\langle \operatorname{grad} H, \operatorname{grad} G\rangle + G\Delta H$$

to the functions defined by $H(X) := \|x\|^k$ and $G := \hat{f}^\vee$ (thus, $G(x) = f(x/\|x\|)$). This gives

$$\Delta f(x) = \Delta(HG)(x) = \|x\|^k \Delta G(x) + 2\langle \operatorname{grad} \|x\|^k, \operatorname{grad} G(x)\rangle + G(x)\Delta H(x).$$

The scalar product of the two gradients vanishes, since $\operatorname{grad} H$ is orthogonal to the sphere $\|x\| = \text{const}$ and $\operatorname{grad} \mathrm{G}$ is tangential to it. Moreover, $\Delta G$ is homogeneous of degree $-2$, hence

$$\Delta G(x) = \|x\|^{-2} \Delta G\left(\frac{x}{\|x\|}\right) = \|x\|^{-2} \Delta_S \hat{f}\left(\frac{x}{\|x\|}\right).$$

Calculation gives

$$\Delta H(x) = k(k+d-2)\|x\|^{k-2},$$

which completes the proof. $\square$

We define the following finite-dimensional real vector spaces of polynomials on $\mathbb{R}^d$.

- $\mathcal{P}^k$ vector space of real polynomials of degree $\leq k$ on $\mathbb{R}^d$,
- $\mathcal{P}_h^k$ subspace of polynomials that are homogeneous of degree $k$,
- $\mathcal{Q}_h^k$ subspace of harmonic polynomials in $\mathcal{P}_h^k$.

**Lemma 2.3.** *If $p \in \mathcal{P}_h^k$ satisfies*

$$\int_{S^{d-1}} pq \, \mathrm{d}\sigma = 0 \qquad \text{for all } q \in \mathcal{P}_h^{k-2},$$

*then $p$ is harmonic.*

*Proof.* Suppose that the condition is satisfied, and let $q \in \mathcal{P}_h^{k-2}$. Using, in this order, Lemma 2.2, the assumption, Lemma 2.1, Lemma 2.2, the assumption, we get

$$\begin{aligned}\int_{S^{d-1}} q\Delta p \, \mathrm{d}\sigma &= \int_{S^{d-1}} q[k(k+d-2)p + \Delta_S p] \, \mathrm{d}\sigma = \int_{S^{d-1}} q\Delta_S p \, \mathrm{d}\sigma \\ &= \int_{S^{d-1}} p\Delta_S q \, \mathrm{d}\sigma = \int_{S^{d-1}} p[\Delta q - (k-2)(k+d-4)q] \, \mathrm{d}\sigma = 0.\end{aligned}$$

Here we have used that $q \in \mathcal{P}_h^{k-2}$ and $r^2\Delta q \in \mathcal{P}_h^{k-2}$, with $r(x) := \|x\|$; on the unit sphere, the two polynomials $\Delta q$ and $r^2\Delta q$ are the same. Choosing $q = \Delta p$, we obtain $\Delta p = 0$ on $S^{d-1}$ and hence $\Delta p = 0$ in general, by homogeneity. $\square$

Now we define spherical harmonics.

**Definition.** The elements of the vector space

$$\mathcal{H}_k^d := \{\hat{p} : p \in \mathcal{Q}_h^k\}$$

of functions on $S^{d-1}$ are called **spherical harmonics** of order $k$ ($k = 0, 1, 2, \dots$).

We denote by $\mathbf{C}(S^{d-1})$ the real vector space of real continuous functions on $S^{d-1}$ and introduce on it a scalar product $(\cdot,\cdot)$ by

$$(f, g) := \int_{S^{d-1}} fg \, \mathrm{d}\sigma.$$

The following theorem shows, among other results, that the spherical harmonics are eigenfunctions of the spherical Laplace operator.

**Theorem 2.1.**
(a) *For* $f \in \mathcal{H}_k^d$,

$$\Delta_S f = -k(k + d - 2)f.$$

(b) For $f \in \mathcal{H}_k^d$ and $g \in \mathcal{H}_j^d$ with $k \neq j$,

$$(f, g) = 0.$$

(c) $\mathcal{P}^k | S^{d-1} = \mathcal{H}_0^d \oplus \cdots \oplus \mathcal{H}_k^d$.

*Proof.* (a) Let $p \in \mathcal{Q}_h^k$. By Lemma 2.2,

$$0 = \Delta p(x) = k(k + d - 2)\|x\|^{k-2} p\left(\frac{x}{\|x\|}\right) + \|x\|^{k-2}\Delta_S \hat{p}\left(\frac{x}{\|x\|}\right).$$

For $\|x\| = 1$ we obtain the assertion.

(b) By Lemma 2.1,

$$\begin{aligned} 0 &= (\Delta_S f, g) - (f, \Delta_S, g) \\ &= [-k(k + d - 2) + j(j + d - 2)](f, g). \end{aligned}$$

For $k \neq j$ this gives $(f, g) = 0$.

(c) Let $p \in \mathcal{P}^k$. It suffices to show that $\hat{p} \in \mathcal{H}_0^d + \cdots + \mathcal{H}_k^d$. First let $p \in \mathcal{P}_h^k$. In the space $\mathcal{P}^k | S^{d-1} = \{\hat{p} : p \in \mathcal{P}^k\}$, the vector $\hat{p}$ has (with respect to the scalar product $(\cdot,\cdot)$) an orthogonal decomposition

$$\hat{p} = \hat{p}_{k-2} + \hat{h} \quad \text{with } \hat{p}_{k-2} \in \mathcal{P}_h^{k-2} | S^{d-1} \text{ and } (\hat{h}, \hat{q}) = 0 \text{ for all } q \in \mathcal{P}_h^{k-2}.$$

We can write

$$p(x) = h(x) + \|x\|^2 p_{k-2}(x),$$

then $h \in \mathcal{P}_h^k$. It follows from Lemma 2.3 that $h$ is harmonic. Repeating the procedure, we obtain

$$p(x) = h_k(x) + \|x\|^2 h_{k-2}(x) + \|x\|^4 h_{k-4}(x) + \cdots + \begin{cases} \|x\|^k h_0, & \text{k even,} \\ \|x\|^{k-1} h_1(x), & \text{k odd,} \end{cases}$$

with $h_j \in \mathcal{Q}_h^j$ for each $j$. This shows that

$$\hat{p} \in \mathcal{H}_k^d + \mathcal{H}_{k-2}^d + \cdots + \begin{cases} \mathcal{H}_0^d, & \text{if } k \text{ is even} \\ \mathcal{H}_1^d, & \text{if } k \text{ is odd.} \end{cases}$$

Since every polynomial is a sum of homogeneous polynomials, the assertion follows. □

Now we are in a position to determine the dimension of the vector space of spherical harmonics of a given order.

**Theorem 2.2.**

$$\begin{aligned} \dim \mathcal{H}_k^d =: N(d,k) &= \binom{k+d-1}{k} - \binom{k+d-3}{k-2} \\ &= \frac{2k+d-2}{k+d-2} \binom{k+d-2}{k}. \end{aligned}$$

*Proof.* Put $\dim \mathcal{P}_h^k =: d_{k,d}$. In $d$ variables, there are $d_{k,d-1}$ monomials of degree $k$ that do not contain $x_d$, and there are $d_{k-1,d}$ monomials that contain $x_d$ at least once, thus $d_{k,d} = d_{k-1,d} + d_{k,d-1}$. Since $d_{k,1} = 1$ and $d_{0,d} = 1$, we obtain

$$d_{k,d} = \binom{k+d-1}{k}.$$

We assert that

$$\mathcal{P}^k | S^{d-1} = \mathcal{P}_h^k | S^{d-1} \oplus \mathcal{P}_h^{k-1} | S^{d-1} \tag{4}$$

for $k \geq 1$. For the proof we note that the intersection of the two spaces on the right is $\{0\}$, because if $k$ is even (odd), then $\mathcal{P}_h^k | S^{d-1}$ contains only even (odd) functions and $\mathcal{P}_h^{k-1} | S^{d-1}$ contains only odd (even) functions. The polynomial $p \in \mathcal{P}^k$ is the sum of homogeneous polynomials. For $q \in \mathcal{P}_h^j$ we have $r^2 q \in \mathcal{P}_h^{j+2}$ with $r(x) = \|x\|$. This yields (4).

Let $k \geq 2$. Using Theorem 2.1, relation (4) and the equation $\dim \mathcal{P}_h^k | S^{d-1} = \dim \mathcal{P}_h^k$, which holds by homogeneity, we obtain

$$\begin{aligned} \dim \mathcal{H}_k^d &= \dim(\mathcal{H}_0^d \oplus \cdots \oplus \mathcal{H}_k^d) - \dim(\mathcal{H}_0^d \oplus \cdots \oplus \mathcal{H}_{k-1}^d) \\ &= \dim \mathcal{P}^k | S^{d-1} - \dim \mathcal{P}^{k-1} | S^{d-1} \\ &= \dim \mathcal{P}_h^k + \dim \mathcal{P}_h^{k-1} - (\dim \mathcal{P}_h^{k-1} + \dim \mathcal{P}_h^{k-2}) \\ &= d_{k,d} - d_{k-2,d}. \end{aligned}$$

This yields the assertion. The result holds also for $k = 0$ and $k = 1$ (for $k = 0$ and $d = 2$, the fraction has to be read as 1). □

**Remark.** For many purposes, the most important consequence of Theorem 2.2 is the estimate

$$\dim \mathcal{H}_k^d = O(k^{d-2}) \qquad \text{as } k \to \infty.$$

For the applications, the following result is fundamental.

**Theorem 2.3.** *Every function $f \in \mathbf{C}(S^{d-1})$ can be uniformly approximated by finite sums of spherical harmonics.*

*Proof.* Let $f : S^{d-1} \to \mathbb{R}$ be continuous. By the theorem of Stone–Weierstraß, the homogeneous extension $\check{f}$ can be approximated by polynomials, uniformly on the compact set $1 \le \|x\| \le 2$. Therefore, $f$ can be uniformly approximated by elements of $\bigcup_{k\in\mathbb{N}} \mathcal{P}^k|S^{d-1}$, and by Theorem 2.1(c) each element of $\mathcal{P}^k|S^{d-1}$ is a finite sum of spherical harmonics. $\square$

From this result, we shall deduce the completeness of the system of spherical harmonics.

In each space $\mathcal{H}_k^d$ we choose an orthonormal basis

$$(Y_{k1}, \dots, Y_{kN(d,k)});$$

then $\{Y_{kj} : k \in \mathbb{N}_0,\ j = 1, \dots, N(d,k)\}$ is an orthonormal system in $\mathbf{C}(S^{d-1})$, by Theorem 2.1(b). The **Fourier series** of a function $f \in \mathbf{C}(S^{d-1})$ with respect to this orthonormal system is given by

$$f \sim \sum_{k=0}^{\infty} \sum_{j=1}^{N(d,k)} (f, Y_{kj}) Y_{kj} = \sum_{k=0}^{\infty} \pi_k f.$$

Here

$$\pi_k f := \sum_{j=1}^{N(d,k)} (f, Y_{kj}) Y_{kj} \tag{5}$$

is independent of the choice of the basis, since it is the image of $f$ under orthogonal projection to the space $\mathcal{H}_k^d$. The series $\sum_{k=0}^{\infty} \pi_k f$ is sometimes called the **condensed harmonic expansion** of $f$.

We write the functions of $\{Y_{kj} : k \in \mathbb{N}_0,\ j = 1, \dots, N(d,k)\}$ into a single sequence, which we denote by $(b_j)_{j\in\mathbb{N}}$. This is an orthonormal sequence, which means that

$$(b_i, b_j) = \delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & \imath \neq j. \end{cases}$$

The $L_2$-norm of a function $f \in \mathbf{C}(S^{d-1})$ is defined by

$$\|f\|_2 := (f, f)^{1/2} = \left( \int_{S^{d-1}} f^2 \, \mathrm{d}\sigma \right)^{1/2}.$$

**Theorem 2.4.** *The sequence $(b_j)_{j\in\mathbb{N}}$ is* **complete***, that is, the* **Parseval relation**

$$\sum_{j=1}^{\infty} (f, b_j)^2 = \|f\|_2^2 \tag{6}$$

*holds for each* $f \in \mathbf{C}(S^{d-1})$*. Moreover,*

$$\lim_{n\to\infty} \left\| f - \sum_{j=1}^{n} (f, b_j) b_j \right\|_2 = 0 \tag{7}$$

*and* (*also known as* **Parseval relation**)

$$\sum_{j=1}^{\infty} (f, b_j)(g, b_j) = (f, g) \tag{8}$$

*for* $f, g \in \mathbf{C}(S^{d-1})$*.*

*Proof.* Let $f, g \in \mathbf{C}(S^{d-1})$. We write

$$c_j := (f, b_j)$$

for the **Fourier coefficients** of the function $f$ with respect to the orthonormal sequence $(b_j)_{j\in\mathbb{N}}$. Let $n \in \mathbb{N}$ and any numbers $\alpha_1, \dots, \alpha_n$ be given. A simple calculation, using the properties of the scalar product, gives

$$\left\| f - \sum_{j=1}^{n} \alpha_j b_j \right\|_2^2 = \|f\|_2^2 - \sum_{j=1}^{n} |c_j|^2 + \sum_{j=1}^{n} |c_j - \alpha_j|^2. \tag{9}$$

We deduce (with $\alpha_j = c_j$) that

$$\|f\|_2^2 - \sum_{j=1}^{n} |c_j|^2 = \left\| f - \sum_{j=1}^{n} c_j b_j \right\|_2^2 \tag{10}$$

and

$$\left\| f - \sum_{j=1}^{n} c_j b_j \right\|_2^2 \le \left\| f - \sum_{j=1}^{n} \alpha_j b_j \right\|_2^2 . \tag{11}$$

Let $\epsilon > 0$. It follows from Theorem 2.3 that there exists a function of the form $h = \sum_{j=1}^{n} \alpha_j b_j$ with $\|f - h\|_2 < \epsilon$. By (11),

$$\left\| f - \sum_{j=1}^{n} c_j b_j \right\|_2^2 \le \|f - h\|_2^2 < \epsilon^2.$$

This proves (7) and, in view of (10), also (6).

Relation (8) now follows from

$$\left( \sum_{j=1}^{n} (f, b_j) b_j, g \right) = \sum_{j=1}^{n} (f, b_j)(g, b_j)$$

and the continuity of the scalar product with respect to its induced norm. □

The Parseval relation has the immediate consequence that

$$(f, b_j) = 0 \quad \text{for all } j \in \mathbb{N} \quad \text{implies } f = 0.$$

We remark that the Parseval relation (8) can also be written in the form

$$\sum_{k=0}^{\infty} (\pi_k f, \pi_k g) = (f, g) \tag{12}$$

(for the proof, use (5), the orthogonality relation $(Y_{ki}, Y_{kj}) = \delta_{ij}$, and (8)).

The space $\mathcal{H}_k^d$ of spherical harmonics of order $k$ has the important property of being invariant under rotations. For a function $f : S^{d-1} \to \mathbb{R}$ or $f : \mathbb{R}^d \to \mathbb{R}$ and a rotation $\vartheta \in SO_d$, the function $\vartheta f$ is defined by $\vartheta f := f \circ \vartheta^{-1}$, thus

$$(\vartheta f)(x) := f(\vartheta^{-1} x)$$

for all $x$ in the domain of $f$. We have $(\vartheta_1 \vartheta_2) f = \vartheta_1(\vartheta_2 f)$ and $\mathrm{id} f = f$, thus the mapping $(\vartheta, f) \to \vartheta f$ is an **operation** of $SO_d$ on $\mathbf{C}(S^{d-1})$.

**Theorem 2.5.** *The space $\mathcal{H}_k^d$ of spherical harmonics of order $k$ is invariant under rotations, that is, if $f \in \mathcal{H}_k^d$ and $\vartheta \in SO_d$, then $\vartheta f \in SO_d$.*

*Proof.* The Laplace operator $\Delta$ is invariant under rotations, that is, $\Delta(f \circ \vartheta) = (\Delta f) \circ \vartheta$ holds for every function $f$ of class $C^2$ on $\mathbb{R}^d$ and all $\vartheta \in SO_d$. This follows by a simple calculation. Hence, if $p$ is a harmonic homogeneous polynomial of degree $k$ on $\mathbb{R}^d$, then $\vartheta p$ is also harmonic, for $\vartheta \in SO_d$, and it is clearly homogeneous of degree $k$. Now restriction to the sphere yields the assertion. □

It is important to note that also the scalar product $(\cdot, \cdot)$ is rotation invariant, that is, it satisfies

$$(\vartheta f, \vartheta g) = (f, g) \qquad \text{for } \vartheta \in SO_d.$$

This follows immediately from the rotation invariance of the spherical Lebesgue measure $\sigma$.

By Theorem 2.1, the spherical harmonics are eigenfunctions of an invariant differential operator on the sphere. Now we show that they are also eigenfunctions of a class of invariant integral operators. This gives us the opportunity to introduce the particularly important spherical harmonics with axial symmetry, that is, of the form $h(u) = f(\langle u, e \rangle)$ for fixed $e \in S^{d-1}$.

We consider an integral operator $A : \mathbf{C}(S^{d-1}) \to \mathbf{C}(S^{d-1})$ of the form

$$(Af)(u) = \int_{S^{d-1}} K(\langle u, v \rangle) f(v) \, \mathrm{d}\sigma(v),$$

with a given continuous function $K : [-1, 1] \to \mathbb{R}$. Thus, the kernel $K(\langle u, v \rangle)$ depends only on the spherical distance of the points $u$ and $v$. We want to investigate the effect of $A$ on a spherical harmonic.

For this, we define

$$f_m(u, v) := \sum_{j=1}^{N} Y_{mj}(u) Y_{mj}(v), \qquad u, v \in S^{d-1},$$

for $m \in \mathbb{N}_0$, where $N := N(d,m)$ and $(Y_{m1}, \dots, Y_{mN})$ is the previously chosen orthonormal basis of the space $\mathcal{H}^d_m$. The function $f_m$ is independent of the choice of this basis: if $(Y'_{m1}, \dots, Y'_{mN})$ is a second orthonormal basis of $\mathcal{H}^d_m$, then $Y_{mj} = \sum_{r=1}^N a_{rj} Y'_{mr}$ with an orthogonal matrix $(a_{rj})$, and inserting this we verify the statement (a similar argument is carried out in the proof of Lemma 2.10). For any rotation $\vartheta \in SO_d$, also $(\vartheta Y_{m1}, \dots, \vartheta Y_{mN})$ is an orthonormal basis. This follows from Theorem 2.5 and the rotation invariance of the scalar product $(\cdot,\cdot)$. We deduce that $f_m(\vartheta^{-1}u, \vartheta^{-1}v) = f_m(u,v)$ for $\vartheta \in SO_d$. Thus, $f_m(u,v)$ depends only of the scalar product $\langle u, v\rangle$, Therefore, there exists a function $C_m : [-1,1] \to \mathbb{R}$ with

$$C_m(\langle u, v\rangle) = \sum_{j=1}^N Y_{mj}(u) Y_{mj}(v).$$

We establish some properties of this function.

**Proposition 1.** For $Y \in \mathcal{H}^d_k$,

$$\begin{aligned}\int_{S^{d-1}} C_m(\langle u, v\rangle) Y(u)\, \mathrm{d}\sigma(u) &= \int_{S^{d-1}} \sum_{j=1}^N Y_{mj}(u) Y_{mj}(v) Y(u)\, \mathrm{d}\sigma(u) \\ &= \sum_{j=1}^N (Y_{mj}, Y) Y_{mj}(v) = \delta_{km} Y(v).\end{aligned}$$

**Proposition 2.** $C_m$ is a polynomial of degree $\leq m$.

*Proof.* For fixed $v \in S^{d-1}$, the function

$$C_m(\langle \cdot, v\rangle) = \sum_{j=1}^N Y_{mj}(v) Y_{mj}$$

is a spherical harmonic of order $m$, hence (choose for $v$ the first basis vector of $\mathbb{R}^d$)

$$\|x\|^m C_m\left(\frac{x_1}{\|x\|}\right), \qquad x = (x_1, \dots, x_m),$$

is a homogeneous harmonic polynomial of degree $m$. Therefore, we have

$$\|x\|^m C_m\left(\frac{x_1}{\|x\|}\right) = \sum_{|\alpha|=m} a_\alpha x^\alpha =: P(x),$$

with multi index notation, that is,

$$|\alpha| := \sum_{j=1}^d \alpha_j, \quad x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad a_\alpha := a_{\alpha_1 \dots \alpha_d}$$

for nonnegative integers $\alpha_1, \dots, \alpha_d$. With arbitrary $\gamma$, put

$$\begin{aligned} x &:= (\cos\gamma, \sin\gamma, 0, \dots, 0), \\ y &:= (\cos\gamma, -\sin\gamma, 0, \dots, 0), \end{aligned}$$

then $C_m(\cos\gamma) = P(x) = P(y)$, hence

$$\begin{aligned} C_m(\cos\gamma) &= \frac{1}{2}[P(x) + P(y)] \\ &= \frac{1}{2}\sum_{|\alpha|=m} a_\alpha (\cos\gamma)^{\alpha_1}[(\sin\gamma)^{\alpha_2} + (-\sin\gamma)^{\alpha_2}]. \end{aligned}$$

The function $(\sin\gamma)^{\alpha_2} + (-\sin\gamma)^{\alpha_2}$ is zero for odd $\alpha_2$, and for even $\alpha_2$ it is a polynomial of degree $\alpha_2$ in $\cos\gamma$. Therefore, $C_m(\cos\gamma)$ is a polynomial of degree at most $m$ in $\cos\gamma$, as stated.

In the following, we fix a vector $e \in S^{d-1}$. Any vector $u \in S^{d-1}$ can be decomposed in the form

$$u = te + \sqrt{1-t^2}\, u_0 \qquad \text{with } u_0 \perp e.$$

Thus, $t = \langle u, e\rangle$ and

$$u_0 \in S_e := \{x \in S^{d-1} : \langle x, e\rangle = 0\}.$$

We denote the $(d-2)$-dimensional spherical Lebesgue measure on the great subsphere $S_e$ by $\sigma_e$. With a suitable parametrization of the sphere $S^{d-1}$, one proves the transformation formula

$$\mathrm{d}\sigma(u) = (1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t\,\mathrm{d}\sigma_e(u_0).$$

**Proposition 3.** For $k \neq m$,

$$\int_{-1}^{1} C_k(t)C_m(t)(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t = 0.$$

*Proof.* Since $C_k(\langle\cdot, e\rangle)$ and $C_m(\langle\cdot, e\rangle)$ are orthogonal, we obtain

$$\begin{aligned} 0 &= \int_{S^{d-1}} C_k(\langle u, e\rangle)C_m(\langle u, e\rangle)\,\mathrm{d}\sigma(u) \\ &= \int_{S_e}\int_{-1}^{1} C_k(t)C_m(t)(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t\,\mathrm{d}\sigma_e(u_0), \end{aligned}$$

from which the assertion follows.

**Proposition 4.** The polynomial $C_m$ is of degree $m$.

*Proof.* By Proposition 1, no $C_k$ is identically zero, and by the orthogonality property of Proposition 3, the functions $C_0, \ldots, C_m$ are linearly independent. Since $C_k$ is of degree $\leq k$, induction with respect to $m$ yields that $C_m$ is precisely of degree $m$.

**Proposition 5.** With $\omega_{d-1} := \sigma_e(S_e)$,

$$C_m(1) = \omega_{d-1}\int_{-1}^{1} C_m(t)^2(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.$$

*Proof.* Since $C_m(\langle\cdot, e\rangle) \in \mathcal{H}_m^d$ for fixed $e$, Proposition 1 gives

$$\begin{aligned} C_m(1) = C_m(\langle e, e\rangle) &= \int_{S^{d-1}} C_m(\langle u, e\rangle) C_m(\langle u, e\rangle)\,\mathrm{d}\sigma(u) \\ &= \int_{S_e} \int_{-1}^{1} C_m(t)^2 (1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t\,\mathrm{d}\sigma_e(u_0), \end{aligned}$$

which gives the assertion.

Now we can prove a useful result.

**Theorem 2.6 (Funk–Hecke theorem).** *If $F : [-1,1] \to \mathbb{R}$ is a bounded measurable function and $Y_m$ is a spherical harmonic of order $m$, then*

$$\int_{S^{d-1}} F(\langle u, v\rangle) Y_m(v)\,\mathrm{d}\sigma(v) = \lambda_m Y_m(u)$$

*with*

$$\lambda_m = \omega_{d-1} C_m(1)^{-1} \int_{-1}^{1} F(t) C_m(t) (1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.$$

*Proof.* First we assume that $F$ is a polynomial. If $F$ is of degree $k$, it follows from Proposition 4 that there is a representation

$$F = \sum_{j=0}^{k} a_j C_j$$

with real coefficients $a_1, \ldots, a_k$. If we multiply this by $C_m$ and use Propositions 3 and 5, we obtain

$$a_m = \omega_{d-1} C_m(1)^{-1} \int_{-1}^{1} F(t) C_m(t) (1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.$$

Now Proposition 1 completes the proof in the case of a polynomial. Since every continuous function on $[-1,1]$ can be uniformly approximated by polynomials, the assertion extends to the case where $F$ is a continuous function. The further extension to bounded measurable functions uses standard arguments of integration theory (see [16, p. 99]). □

The polynomials $C_m$ appearing in the previous considerations are known as **Gegenbauer polynomials**. They are indispensable when one actually works with spherical harmonics. We use them in a renormalized form, with a notation showing also the dimension.

**Definition.** The polynomial defined by

$$P_k^d(t) := \frac{\omega_d}{N(d,k)} C_k(t)$$

with

$$\omega_d := \sigma(S^{d-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

is the **Legendre polynomial** of dimension $d$ and degree (or order) $k$.

Thus, by the definition of the polynomials $C_k$, with $N := N(d,k)$,

$$\sum_{j=1}^{N} Y_{kj}(u) Y_{kj}(v) = \frac{N}{\omega_d} P_k^d(\langle u, v\rangle) \tag{13}$$

for $u, v \in S^{d-1}$.

For $f \in \mathbf{C}(S^{d-1})$, we get from (13)

$$\begin{aligned}\frac{N}{\omega_d}\int_{S^{d-1}} f(v)P_k^d(\langle u,v\rangle)\,\mathrm{d}\sigma(v) &= \int_{S^{d-1}} f(v)\sum_{j=1}^{N} Y_{kj}(u)Y_{kj}(v)\,\mathrm{d}\sigma(v)\\ &= \sum_{j=1}^{N}(f, Y_{kj})Y_{kj}(u) = \pi_k f(u),\end{aligned}$$

by (5). Thus, the orthogonal projection to $\mathcal{H}_k^d$ can be written in the elegant form

$$\pi_k f = \frac{N}{\omega_d}\int_{S^{d-1}} f(v)P_k^d(\langle u,v\rangle)\,\mathrm{d}\sigma(v). \tag{14}$$

We list some properties of the Legendre polynomials, but we do not give all the proofs. These proofs can be found, for example, in the book by Groemer [16].

**Lemma 2.4.**

$$P_k^d(1) = 1, \tag{15}$$

$$|P_k^d(t)| \le 1 \quad \textit{for} \quad t \in [-1,1]. \tag{16}$$

*Proof.* Putting $u = v$ in (13) and integrating over $S^{d-1}$, we obtain (15). For given $t \in [-1, 1]$, we choose $u, v \in S^{d-1}$ with $t = \langle u, v\rangle$. From (13) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}|P_k^d(t)|^2 &= |P_k^d(\langle u,v\rangle)|^2 = \frac{\omega_d^2}{N^2}\left(\sum_{j=1}^{N} Y_{kj}(u)Y_{kj}(v)\right)^2\\ &\le \left(\frac{\omega_d}{N}\sum_{j=1}^{N} Y_{kj}(u)^2\right)\left(\frac{\omega_d}{N}\sum_{j=1}^{N} Y_{kj}(v)^2\right)\\ &= P_k^d(1)^2 = 1\end{aligned}$$

and thus (16). □

**Lemma 2.5 (Formula of Rodrigues).**

$$P_k^d(t) = \frac{(-1)^k}{2^k(\nu+1)(\nu+2)\cdots(\nu+k)}(1-t^2)^{-\nu}\frac{\mathrm{d}^k}{\mathrm{d}t^k}(1-t^2)^{\nu+k}$$

*with*

$$\nu := \frac{d-3}{2}.$$

**Lemma 2.6.** *If $k$ is odd, then $P_k^d(0) = 0$; if $k$ is even, then*

$$P_k^d(0) = (-1)^{\frac{k}{2}}\frac{1\cdot 3\cdots(k-1)}{(d-1)(d+1)\cdots(d+k-3)}.$$

The next lemmas provide recursion formulas, differential equations, and generating functions.

**Lemma 2.7.** *For $k \geq 0$ (with $P_{-1}^d := 0$),*

$$(k+d-2)P_{k+1}^d(t) - (2k+d-2)tP_k^d(t) + kP_{k-1}^d(t) = 0.$$

**Lemma 2.8.**

$$(1-t)^2 \frac{\mathrm{d}^2 P_k^d(t)}{\mathrm{d}t^2} - (d-1)t\frac{\mathrm{d}P_k^d(t)}{\mathrm{d}t} + k(k+d-2)P_k^d(t) = 0.$$

**Lemma 2.9.** *Let $|t| \leq 1$ and $|r| \leq 1$. If $d \geq 3$, then*

$$\frac{1}{(1+r^2-2rt)^{(d-2)/2}} = \sum_{k=0}^{\infty} \binom{k+d-3}{d-3} P_k^d(t) r^k,$$

*and for $d \geq 2$,*

$$\frac{1-r^2}{(1+r^2-2rt)^{d/2}} = \sum_{k=0}^{\infty} N(d,k) P_k^d(t) r^k.$$

*For every $\epsilon > 0$, the series converge absolutely and uniformly in $|t| \leq 1$, $|r| \leq 1-\epsilon$.*

The following theorem uses the Poisson integral to provide a summation method for the harmonic expansion. Let $F \in \mathbf{C}(S^{d-1})$, and let

$$F \sim \sum_{k=0}^{\infty} \pi_k F$$

be the condensed harmonic expansion. By Theorem 2.4, the series $\sum_{k=0}^{\infty} \pi_k F$ converges to $F$ in the $\|\cdot\|_2$ norm, but in general it does not converge in the maximum norm $\|\cdot\|_\infty$. However, the following theorem shows that for every $\epsilon > 0$ there exist numbers $n \in \mathbb{N}$ and $r \in (0,1)$ such that

$$\left\| F - \sum_{k=0}^{n} r^k \pi_k F \right\|_\infty \leq \epsilon.$$

**Theorem 2.7.** *Let $F$ be a continuous function on $S^{d-1}$. For $r \in (-1,1)$, define $F_r$ by*

$$F_r(u) := \frac{1}{\omega_d} \int_{S^{d-1}} \frac{1-r^2}{(1-2r\langle u,v\rangle + r^2)^{d/2}} F(v)\,\sigma(\mathrm{d}v).$$

*Then the following holds.*

(a) *If*

$$F \sim \sum_{k=0}^{\infty} \pi_k F,$$

*then*

$$F_r \sim \sum_{k=0}^{\infty} r^k \pi_k F.$$

*In particular,*

$$Y_r(u) = r^k Y(u)$$

*for* $Y \in \mathcal{H}_k^d$.

*For each* $r \in (-1, 1)$,

$$F_r(u) = \sum_{k=0}^{\infty} r^k \pi_k F(u)$$

*with uniform convergence for* $u \in S^{d-1}$.

(b) *The relation*

$$\lim_{r \to 1} F_r(u) = F(u)$$

*holds uniformly in* $u$.

(c) *For every continuous function* $G$ *on* $S^{d-1}$,

$$(F_r, G) = (F, G_r).$$

*Proof.* The second series expansion of Lemma 2.9 yields

$$\begin{aligned} F_r(u) &= \frac{1}{\omega_d} \int_{S^{d-1}} \frac{1 - r^2}{(1 - 2r\langle u, v\rangle + r^2)^{d/2}} F(v)\, \sigma(\mathrm{d}v) \\ &= \sum_{k=0}^{\infty} \frac{N(d,k)}{\omega_d} r^k \int_{S^{d-1}} F(v) P_k^d(\langle u, v\rangle)\, \sigma(\mathrm{d}v). \end{aligned}$$

For fixed $r$, the convergence is uniform in $u$. Inserting (13), we obtain

$$F_r(u) = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{N(d,k)} (F, Y_{kj}) Y_{kj}(u) = \sum_{k=0}^{\infty} r^k (\pi_k F)(u).$$

This proves (a).

In particular, with $F \equiv 1$ we get

$$\frac{1}{\omega_d} \int_{S^{d-1}} \frac{1 - r^2}{(1 - 2r\langle u, v\rangle + r^2)^{d/2}}\, \sigma(\mathrm{d}v) = 1.$$

For the proof of (b), let $\epsilon > 0$ be given. By Theorem 2.3 there is a finite sum of spherical harmonics, say

$$H = G_0 + \cdots + G_m, \qquad G_j \in \mathcal{H}_j^d,$$

with

$$|F(v) - H(v)| < \frac{\epsilon}{3} \qquad \text{for } v \in S^{d-1}.$$

It follows that

$$|F_r(v) - H_r(v)| \le \frac{1}{\omega_d} \int_{S^{d-1}} \frac{1 - r^2}{(1 - 2r\langle u, v\rangle + r^2)^{d/2}} |F(v) - H(v)|\, \sigma(\mathrm{d}v) \le \frac{\epsilon}{3}.$$

By (a) we have

$$H_r = G_0 + rG_1 \cdots + r^m G_m$$

and therefore, for $r \in (0,1)$,

$$\begin{aligned}|F_r(u) - F(u)| &\le |F_r(u) - H_r(u)| + |H_r(u) - H(u)| + |H(u) - F(u)| \\ &\le \frac{2}{3}\epsilon + |(1-r)G_1(u) + \cdots + (1-r^m)G_m(u)| \\ &\le \frac{2}{3}\epsilon + (1-r^m)mM \\ &\le \frac{2}{3}\epsilon + (1-r)m^2M\end{aligned}$$

with $M := \max\{|G_j(u)| : u \in S^{d-1},\ j = 1,\dots,m\}$. If $F$ is given, $m^2M$ depends only on $\epsilon$. Putting $\delta := \epsilon/3m^2M$, we get

$$|F_r(u) - F(u)| \le \epsilon \qquad \text{for } 1-\delta < r < 1.$$

This finishes the proof of (b). Assertion (c) is clear. □

Now we return to general spherical harmonics. Our next aim is to identify the spaces $\mathcal{H}_k^d$ of spherical harmonics of order $k$, $k \in \mathbb{N}_0$, as the irreducible subspaces of $\mathbf{C}(S^{d-1})$ for the operation of the rotation group. Recall that this operation is defined by $(\vartheta f)(u) = f(\vartheta^{-1}u)$.

**Definition.** Let $V$ be a vector subspace of $\mathbf{C}(S^{d-1})$. The space $V$ is called **invariant** if $f \in V$ implies $\vartheta f \in V$ for all $\vartheta \in SO_d$. The subspace $V$ is called **irreducible** if it is invariant and has no invariant subspace except $\{0\}$ and $V$.

By Theorem 2.5, each space $\mathcal{H}_k^d$ is invariant, and we want to show that it is irreducible.

We choose a vector $e \in S^{d-1}$ and put

$$G_e := \{\vartheta \in SO_d : \vartheta e = e\};$$

this is a subgroup of $SO_d$, called the stabilizer of $e$. Further, let

$$W_e := \{f \in \mathbf{C}(S^{d-1}) : \vartheta f = f \text{ for all } \vartheta \in G_e\}.$$

Thus, $W_e$ is the subspace of functions that are invariant under all rotations fixing $e$.

**Lemma 2.10.** *If $V \subset \mathbf{C}(S^{d-1})$ is an invariant subspace with $0 < \dim V < \infty$, then*

$$\dim(V \cap W_e) \ge 1.$$

*Proof.* In the finite-dimensional vector space $V$, say with $\dim V = n$, we can choose an orthonormal basis $(f_1,\dots,f_n)$ with respect to the scalar product $(\cdot,\cdot)$. For each $j$ and $\vartheta \in SO_d$, also $\vartheta f_j \in V$, hence there is a representation

$$\vartheta f_j = \sum_{i=1}^n t_{ij}(\vartheta) f_i$$

with real coefficients $t_{ij}(\vartheta)$. Since the scalar product is rotation invariant, also $(\vartheta f_1,\dots,\vartheta f_n)$ is an orthonormal basis. Therefore, the coefficient matrix $(t_{ij}(\vartheta))_{i,j=1}^n$ is orthogonal. We define a function $F : S^{d-1} \times S^{d-1} \to \mathbb{R}$ by

$$F(u,v) := \sum_{i=1}^n f_i(u) f_i(v).$$

Then, for $\vartheta \in SO_d$,

$$\begin{aligned} F(\vartheta^{-1}u, \vartheta^{-1}v) &= \sum_{i=1}^{n} (\vartheta f_i)(u)(\vartheta f_i)(v) = \sum_{i=1}^{n} \sum_{r=1}^{n} t_{ri}(\vartheta) f_r(u) \sum_{s=1}^{n} t_{si}(\vartheta) f_s(v) \\ &= \sum_{r,s=1}^{n} f_r(u) f_s(v) \sum_{i=1}^{n} t_{ri}(\vartheta) t_{si}(\vartheta) = \sum_{r=1}^{n} f_r(u) f_r(v) = F(u,v). \end{aligned}$$

In particular, the function $f$ defined by

$$f := F(e, \cdot) = \sum_{i=1}^{n} f_i(e) f_i$$

has the property that, for all $\vartheta \in G_e$ and all $x \in S^{d-1}$,

$$f(\vartheta x) = F(e, \vartheta x) = F(\vartheta e, \vartheta x) = F(e, x) = f(x).$$

This shows that $f \in W_e$. On the other hand, also $f \in V$. Suppose that $f = 0$. Then from the linear independence of $f_1, \dots, f_n$ it follows that $f_i(e) = 0$ for $i = 1, \dots, n$. Let $g \in V$. There is a representation $g = \sum_{i=1}^{n} a_i f_i$, which gives $g(e) = 0$. To $x \in S^{d-1}$ there exists a rotation $\vartheta \in SO_d$ with $\vartheta x = e$. It follows that $g(x) = g(\vartheta^{-1} e) = (\vartheta g)(e) = 0$, since also $\vartheta g \in V$. We conclude that $g = 0$. Since $g \in V$ was arbitrary, this gives $\dim V = 0$, a contradiction. Thus $f \neq 0$ and, therefore, $\dim(V \cap W_e) \geq 1$. □

**Lemma 2.11.** *If $V \subset \mathbf{C}(S^{d-1})$ is a finite-dimensional invariant subspace with*

$$\dim(V \cap W_e) \leq 1,$$

*then $V$ is irreducible.*

*Proof.* Let $\dim(V \cap W_e) \leq 1$, and suppose that $V$ is not irreducible. Then $V$ has an invariant subspace $U$ different from $\{0\}$ and $V$. Let $U^\perp$ be the orthogonal complement of $U$ in $V$ with respect to the scalar product $(\cdot, \cdot)$. Let $f \in U^\perp$. For $g \in U$ and $\vartheta \in SO_d$ we have, using the rotation invariance of the scalar product and the invariance of $U$,

$$(\vartheta f, g) = (f, \vartheta^{-1} g) = 0,$$

hence $\vartheta f \in U^\perp$. Thus, also the subspace $U^\perp$ is invariant, and it is different from $\{0\}$ and $V$. By Lemma 2.10,

$$\dim(U \cap W_e) \geq 1, \qquad \dim(U^\perp \cap W_e) \geq 1.$$

Since $\dim(V \cap W_e) \leq 1$ by assumption and $\dim(U \cap U^\perp) = 0$, this is a contradiction. □

**Theorem 2.8.** *The space $\mathcal{H}_k^d$ of spherical harmonics of order $k$ is irreducible.*

*Proof.* We take $e = e_d$, where $(e_1, \dots, e_d)$ is the standard basis of $\mathbb{R}^d$. Let $f \in \mathcal{H}_k^d \cap W_e$. The function $f$ is the restriction to $S^{d-1}$ of a homogeneous harmonic polynomial $p$ of degree $k$. We can write it in the form

$$p(x) = \sum_{j=0}^{k} x_d^{k-j} p_j(\overline{x}), \qquad x = (x_1, \dots, x_d), \qquad \overline{x} = (x_1, \dots, x_{d-1}),$$

where $p_j : \mathbb{R}^{d-1} \to \mathbb{R}$ is a homogeneous polynomial of degree $j$. For $\vartheta \in G_e$,

$$\sum_{j=0}^{k} x_d^{k-j} p_j(\overline{x}) = p(x) = p(\vartheta x) = \sum_{j=0}^{k} x_d^{k-j} p_j(\vartheta \overline{x}).$$

Since this holds for all real $x_d$, it follows that $p_j(\overline{x}) = p_j(\vartheta \overline{x})$. This being true for all $\vartheta \in G_e$, the function $p_j$ depends only on $x_1^2 + \cdots + x_{d-1}^2$. Since $p_j$ is homogeneous of degree $j$, we obtain

$$p_j(\overline{x}) = c_j(x_1^2 + \cdots + x_{d-1}^2)^{j/2}$$

with $c_j \in \mathbb{R}$. Since $p_j$ is a polynomial, we have $c_j = 0$ for odd $j$, thus

$$p(x) = \sum_{0 \le i \le k/2} c_{2i} x_d^{k-2i} (x_1^2 + \cdots + x_{d-1}^2)^i.$$

The polynomial $p$ is harmonic. The condition $\Delta p = 0$ yields recursion formulas for the coefficients. Up to a factor $c_0$, they have the unique solution

$$c_{2i} = (-1) \frac{k(k-1)\cdots(k-2i+1)}{2i!(d-1)(d+1)\cdots(d+2i-3)} c_0.$$

Thus, the polynomial $p$ is uniquely determined up to a constant factor. This shows that $\dim(\mathcal{H}_k^d \cap W_e) \le 1$. Now Lemma 2.11 proves the assertion. $\square$

From the irreducibility of the spaces of spherical harmonics we immediately deduce a fact that is basic for many applications. Let $A : V \to \mathbf{C}(S^{d-1})$ be a linear mapping, where $V$ is an invariant subspace of $\mathbf{C}(S^{d-1})$. This mapping is called **intertwining** if $A\vartheta f = \vartheta A f$ for all $f \in V$ and all $\vartheta \in SO_d$. For example, each projection $\pi_k$ is intertwining. We denote by $\mathcal{H}^d$ the vector space of all finite linear combinations of spherical harmonics.

**Theorem 2.9.** *Let $d \ge 3$. Let $A : \mathcal{H}^d \to \mathbf{C}(S^{d-1})$ be an intertwining linear map. Then for each $m \in \mathbb{N}_0$ there exists a real number $c_m$ such that*

$$\pi_m A = c_m \pi_m.$$

*Proof.* Let $m, k \in \mathbb{N}_0$, and let $A_m$ be the restriction of $A$ to $\mathcal{H}_m^d$. Then $\pi_k A_m$ is an intertwining linear map from $\mathcal{H}_m^d$ to $\mathcal{H}_k^d$. The kernel of this map is an invariant subspace of $\mathcal{H}_m^d$ and hence either equal to $\{0\}$ or to $\mathcal{H}_m^d$. Thus, $\pi_k A_m$ is either injective or the zero map. Similarly, the image of $\pi_k A_m$ is an invariant subspace of $\mathcal{H}_k^d$ and hence equal to either $\{0\}$ or $\mathcal{H}_k^d$. It follows that $\pi_k A_m$ is either 0 or bijective. The latter case is only possible if $m = k$, since $\dim \mathcal{H}_m^d \ne \dim \mathcal{H}_k^d$ for $m \ne k$ and $d \ge 3$. From $\pi_k A_m = 0$ for $k \ne m$ and the completeness of the system of spherical harmonics it follows that $A_m$ maps $\mathcal{H}_m^d$ into itself.

Let $e \in S^{d-1}$. The function $P_m^d(\langle e, \cdot \rangle) \in \mathcal{H}_m^d$ is invariant under the group $G_e$ and is, up to a constant factor, the only element of $\mathcal{H}_m^d$ with this property, as shown in the proof of Theorem 2.8. For $\vartheta \in G_e$ we have $\vartheta \pi_m A_m P_m^d(\langle e, \cdot \rangle) = \pi_m A_m \vartheta P_m^d(\langle e, \cdot \rangle) = \pi_m A_m P_m^d(\langle e, \cdot \rangle)$, hence $\pi_m A_m P_m^d(\langle e, \cdot \rangle) = c_m(e) P_m^d(\langle e, \cdot \rangle)$ with a real constant $c_m(e)$. Replacing $e$ be $\vartheta e$ with $\vartheta \in SO_d$, we see that $c_m$ does not depend on $e$. The functions $P_m^d(\langle e, \cdot \rangle)$, $e \in S^{d-1}$, linearly span $\mathcal{H}_m^d$ (since their span is an invariant subspace), hence it follows that

$$\pi_m A Y = \begin{cases} c_m Y, & \text{if } Y \in \mathcal{H}_m^d, \\ 0, & \text{if } Y \in \mathcal{H}_k^d \text{ and } k \ne m. \end{cases}$$

By linearity, we obtain

$$\pi_m A f = c_m \pi_m f,$$

if $f$ is a finite sum of spherical harmonics. □

There are versions of Theorem 2.9 for intertwining maps between other suitable vector spaces of functions (or signed measures) on $S^{d-1}$, possibly under continuity assumptions on $A$. For obvious reasons, maps with the properties of Theorem 2.9 are known as **multiplier maps**.

We add a brief remark about representations of the rotation group. Let $V \subset \mathbf{C}(S^{d-1})$ be an invariant subspace. Then $f \in V$ and $\vartheta \in SO_d$ implies $\vartheta f \in V$. We now write $\vartheta f =: T(\vartheta) f$, then

$$T(\vartheta) : V \to V$$

is obviously a linear mapping of $V$ into itself. It is bijective and thus an automorphism of $V$. In this way, a mapping

$$\begin{aligned} T: \quad SO_d &\to \operatorname{Aut} V \\ \vartheta &\mapsto T(\vartheta) \end{aligned}$$

from the rotation group $SO_d$ into the automorphism group $\operatorname{Aut} V$ of the vector space $V$ is defined. It satisfies

$$T(\vartheta_1)T(\vartheta_2)f = \vartheta_1(\vartheta_2 f) = (\vartheta_1\vartheta_2)f = T(\vartheta_1\vartheta_2)f$$

for all $f \in V$, hence $T(\vartheta_1\vartheta_2) = T(\vartheta_1)T(\vartheta_2)$. Thus, $T$ is a homomorphism. Generally, a homomorphism of a group $G$ into the automorphism group of a vector space is called a **representation** of the group. A representation $T : G \to \operatorname{Aut} V$ is called **irreducible** if $V$ does not have a subspace $U \neq \{0\}, V$ with $T(\vartheta)u \in U$ for all $\vartheta \in G$ and all $u \in U$. In this sense, the spherical harmonics are closely tied together with irreducible representations of the rotation group.

Let $k \in \mathbb{N}$ and recall that $(Y_{k1}, \dots, Y_{kN})$ with $N = N(d, k)$ is an orthonormal basis of $\mathcal{H}_k^d$. As in the proof of Lemma 2.10, for $\vartheta \in SO_d$ we have

$$T(\vartheta)Y_{kj} = \vartheta Y_{kj} = \sum_{i=1}^{N} t_{ij}^k(\vartheta) Y_{ki}, \qquad j = 1, \dots, N, \tag{17}$$

and the matrix $M(\vartheta) := (t_{ij}^k(\vartheta))_{i,j=1}^N$ is orthogonal. The relation $T(\vartheta_1\vartheta_2) = T(\vartheta_1)T(\vartheta_2)$ translates into $M(\vartheta_1\vartheta_2) = M(\vartheta_1)M(\vartheta_2)$ (matrix product). Thus, $M$ is a homomorphism of the group $SO_d$ into the group of orthogonal $N \times N$ matrices, a **matrix valued orthogonal representation** of the rotation group.

Each function $t_{ij}^k$ defined by (17) is a continuous function on the topological group $SO_d$. These functions satisfy orthogonality relations. On the compact group $SO_d$ there is a unique invariant measure $\nu$ with $\nu(SO_d) = 1$, its normalized Haar measure.

**Lemma 2.12.** *For all $k, m \in \mathbb{N}_0$, $i, j \in \{1, \dots, N(d, k)\}$, $r, s \in \{1, \dots, N(d, m)\}$,*

$$N(d, k) \int_{SO_d} t_{ij}^k t_{rs}^m \, \mathrm{d}\nu = \delta_{km}\delta_{ir}\delta_{js}.$$

A proof can be found, e.g., in [35].

Using the functions $t_{ij}^k$, we derive, for later application, a relation that is similar in spirit to the Funk–Hecke formula, but involves integration over the rotation group.

**Lemma 2.13.** *If $f \in \mathbf{C}(S^{d-1})$, $k \in \mathbb{N}_0$ and $i, j = 1, \dots, N(d,k)$, then*

$$\int_{SO_d} f(\vartheta^{-1}u) t_{ij}^k(\vartheta)\, \nu(\mathrm{d}\vartheta) = N(d,k)^{-1}(f, Y_{kj})Y_{ki}(u) \tag{18}$$

*for $u \in S^{d-1}$.*

*Proof.* First we consider the case where $f$ is a spherical harmonic from a basis, say $f = Y_{mr}$. Then

$$\begin{aligned}
\int_{SO_d} Y_{mr}(\vartheta^{-1}u) t_{ij}^k(\vartheta)\, \nu(\mathrm{d}\vartheta) &= \int_{SO_d} \sum_{s=1}^{N(d,m)} t_{sr}^m(\vartheta) Y_{ms}(u) t_{ij}^k(\vartheta)\, \nu(\mathrm{d}\vartheta) \\
&= \sum_{s=1}^{N(d,m)} N(d,m)^{-1} \delta_{mk}\delta_{si}\delta_{rj} Y_{ms}(u) \\
&= N(d,m)^{-1}\delta_{mk}\delta_{rj} Y_{ki}(u) \\
&= N(d,n)^{-1}(Y_{mr}, Y_{kj}) Y_{ki}(u).
\end{aligned}$$

Thus, (18) holds for $f = Y_{mr}$. By linearity, (18) is true if $f$ is a finite sum of spherical harmonics. By Theorem 2.3, every continuous function on the sphere can be uniformly approximated by finite sums of spherical harmonics. This proves the assertion. $\square$

**Hints to the literature.** The book by Groemer [16], aiming at geometric applications, gives also an introduction to the theory of spherical harmonics. Older introductions, still recommended, are by Müller [23] and, in a brief, elegant article from which we have much profited, by Seeley [33]. The connections between group representations and spherical functions are presented, for example, in Vilenkin [35] and Coifman and Weiss [6]. Lemma 2.13 is taken from [29].

# 3 Rotation Invariant Equations and Uniqueness Problems

We are now in a position to treat the equation

$$\int_{S^{d-1}} f(\vartheta v)\, \mu(\mathrm{d}v) = 0 \qquad \text{for all } \vartheta \in SO_d. \tag{19}$$

Here $\mu$ is a finite signed measure and $f$ is a continuous real function on the sphere $S^{d-1}$. The continuity assumption can be relaxed, when necessary. We recall that

$$(f,g) = \int_{S^{d-1}} fg\, \mathrm{d}\sigma$$

and

$$\pi_k f = \sum_{j=1}^{N(d,k)} (f, Y_{kj}) Y_{kj}.$$

Similarly, we put

$$(\mu, f) = (f, \mu) := \int_{S^{d-1}} f \, d\mu$$

and

$$\pi_k \mu := \sum_{j=1}^{N(d,k)} (\mu, Y_{kj}) Y_{kj}.$$

Equation (19) can now be written in the form

$$(\vartheta f, \mu) = 0 \qquad \text{for all } \vartheta \in SO_d. \tag{20}$$

**Definition.** Let $k \in \mathbb{N}_0$. We say that the space $\mathcal{H}_k^d$ **occurs in** $f$ if $\pi_k f \neq 0$. Similarly, $\mathcal{H}_k^d$ **occurs in** $\mu$ if $\pi_k \mu \neq 0$.

Thus, $\mathcal{H}_k^d$ **occurs in** $f$ if and only if there exists a spherical harmonic $Y_k$ of order $k$ with $(f, Y_k) \neq 0$, and analogously for $\mu$.

**Theorem 3.1.** *Relation* (19) *holds if and only if no space* $\mathcal{H}_k^d$, $k \in \mathbb{N}_0$, *occurs in both* $f$ *and* $\mu$.

*Proof.* Using Fubini's theorem and (18), we obtain

$$\begin{aligned} N(d,k) \int_{SO_d} (\vartheta f, \mu) t_{ij}^k(\vartheta) \, \nu(d\vartheta) &= N(d,k) \int_{S^{d-1}} \int_{SO_d} f(\vartheta^{-1} u) t_{ij}^k(\vartheta) \, \nu(d\vartheta) \, \mu(du) \\ &= (f, Y_{kj})(\mu, Y_{ki}). \end{aligned}$$

If now (20) holds, then $(f, Y_{kj})(\mu, Y_{ki}) = 0$ for $k \in \mathbb{N}_0$ and $i, j = 1, \dots, N(d,k)$. Suppose that, say, $\mathcal{H}_k^d$ occurs in $\mu$. Then there exists a number $i \in \{1, \dots, N(d,k)\}$ with $(\mu, Y_{ki}) \neq 0$. It follows that $(f, Y_{kj}) = 0$ for $j = 1, \dots, N(d,k)$, thus $\mathcal{H}_k^d$ does not occur in $f$.

Suppose, conversely, that no space $\mathcal{H}_k^d$, $k \in \mathbb{N}_0$, occurs in both $f$ and $\mu$. The function $f \in \mathbf{C}(S^{d-1})$ can be uniformly approximated by a sequence $(f_n)_{n \in \mathbb{N}}$, where each $f_n$ is a finite sum of spherical harmonics. By Theorem 2.7, we can assume that in each $f_n$ only those spaces $\mathcal{H}_k^d$ occur that occur also in $f$.

A given $f_n$ is a finite sum of spherical harmonics,

$$f_n = \sum_{k=0}^{m} \sum_{j=1}^{N(d,k)} (f_n, Y_{kj}) Y_{kj}.$$

This gives

$$(\vartheta f_n, \mu) = \sum_{k=0}^{m} \sum_{i,j=1}^{N(d,k)} t_{ij}^k(\vartheta)(f_n, Y_{kj})(\mu, Y_{ki}). \tag{21}$$

If $\mathcal{H}_k^d$ occurs in $\mu$, then it does not occur in $f$ and hence not in $f_n$. Therefore, $(f_n, Y_{kj})(\mu, Y_{ki}) = 0$ for all $i, j$. If $\mathcal{H}_k^d$ does not occur in $\mu$, then this also holds. We deduce that $(\vartheta f_n, \mu) = 0$. By approximation, we get $(\vartheta f, \mu) = 0$. □

The general principle expressed in Theorem 3.1 can be applied to various geometric situations, as we now demonstrate with the classical examples from the introduction and a few others.

Aleksandrov's projection theorem, as mentioned, combines an analytic and a geometric uniqueness result. The analytic one, which interests us here, is the following.

**Theorem 3.2.** *If the even finite signed measure $\mu$ on $S^{d-1}$ satisfies*

$$\int_{S^{d-1}} |\langle u, v\rangle|\, \mu(\mathrm{d}v) = 0 \qquad \textit{for all } u \in S^{d-1}, \tag{22}$$

*then $\mu = 0$.*

Before the proof, we consider a more general integral transform, of which some other cases have found geometric applications. Let $\Phi : [-1, 1] \to \mathbb{R}$ be a bounded measurable function. For a finite signed measure $\mu$ on $S^{d-1}$, define the function $T_\Phi \mu$ by

$$(T_\Phi \mu)(u) := \int_{S^{d-1}} \Phi(\langle u, v\rangle)\, \mu(\mathrm{d}v) \qquad \text{for } u \in S^{d-1}.$$

For a bounded measurable function $f$ on $S^{d-1}$, the transform $T_\Phi f$ is defined as $T_\Phi(f\sigma)$, where $f\sigma := \int_{(\cdot)} f\, \mathrm{d}\sigma$. From Fubini's theorem, we immediately get the symmetry relation

$$(T_\Phi \mu, f) = (\mu, T_\Phi f). \tag{23}$$

If $Y_m$ is a spherical harmonic of order $m$, then the Funk–Hecke formula of Theorem 2.6 gives

$$T_\Phi Y_m = a_{d,m}(\Phi) Y_m$$

with

$$a_{d,m}(\Phi) = \omega_{d-1} \int_{-1}^{1} \Phi(t) P_m^d(t)(1 - t^2)^{\frac{d-3}{2}}\, \mathrm{d}t.$$

Since $(T_\Phi \mu, f) = (\mu, T_\Phi f) = a_{d,m}(\Phi)(\mu, Y_m)$, from the condensed harmonic expansion

$$\mu \sim \sum_{k=0}^{\infty} \pi_k \mu$$

of $\mu$ we obtain the condensed harmonic expansion of $T_\Phi \mu$ as

$$T_\Phi \mu \sim \sum_{k=0}^{\infty} a_{d,m}(\Phi) \pi_k \mu.$$

Therefore, the transformation $T_\Phi$ is called a **multiplier transform**, and the numbers $a_{d,m}(\Phi)$ are called the **multipliers** of $T_\Phi$.

For the transformations $T_\Phi$ we formulate the crucial consequence of Theorem 3.1 in an alternative version and with a different proof.

**Theorem 3.3.** *Let $\Phi : [-1, 1] \to \mathbb{R}$ be a bounded measurable function. If $\mu$ is a finite signed measure on $S^{d-1}$ with*

$$T_\Phi \mu = 0 \tag{24}$$

*and*

$$\pi_m \mu = 0 \qquad \textit{for the } m \in \mathbb{N}_0 \textit{ with} \qquad a_{d,m}(\Phi) = 0, \tag{25}$$

*then*

$$\mu = 0.$$

*Proof.* The assumption (24) gives $0 = (T_\Phi \mu, Y_m) = (\mu, T_\Phi Y_m) = a_{d,m}(\Phi)(\mu, Y_m)$ for $Y_m \in \mathcal{H}^d_m$. Hence, if $a_{d,m}(\Phi) \neq 0$, then $(\mu, Y_m) = 0$. If $a_{d,m}(\Phi) = 0$, then $(\mu, Y_m) = 0$ holds by assumption. It follows that $(\mu, f) = 0$ if $f$ is a finite sum of spherical harmonics. Since every function $f \in \mathbf{C}(S^{d-1})$ can be uniformly approximated by finite sums of spherical harmonics, we conclude that $(\mu, f) = 0$ for all $f \in \mathbf{C}(S^{d-1})$. This implies $\mu = 0$. □

In the special case of Theorem 3.2 we have $\Phi(t) = |t|$ and therefore

$$a_{d,m}(\Phi) = \omega_{d-1} \int_{-1}^{1} |t| P^d_m(t)(1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t.$$

For odd $m$, the Legendre polynomial $P^d_m$ is an odd function, hence $a_{d,m}(\Phi) = 0$. For even $m$, the formula of Rodrigues (Lemma 2.5) together with partial integration can be used to compute the integral, and one obtains

$$a_{d,m}(\Phi) = \frac{(-1)^{(m-2)/2} \pi^{(d-1)/2} \Gamma(m-1)}{2^{m-2} \Gamma(m/2) \Gamma((m+d+1)/2)}.$$

Therefore, $a_{d,m}(\Phi) \neq 0$ for even $m$, and if the signed measure $\mu$ is even, then $\pi_m \mu = 0$ for odd $m$. Hence, Theorem 3.3 gives $\mu = 0$, which proveds Theorem 2.2.

The transform $T_\Phi$ with $\Phi(t) = |t|$ is known as the **cosine transform**. For the function $\Phi(t) = \sqrt{1-t^2}$, one calls $T_\Phi$ the **sine transform**. Also in this case, the multipliers can be computed, and one finds that $a_{d,m}(\Phi) \neq 0$ for all even $m$. Therefore, an even finite signed measure $\mu$ on $S^{d-1}$ satisfying

$$\int_{S^{d-1}} \sqrt{1 - \langle u, v \rangle^2} \, \mu(\mathrm{d}v) = 0 \qquad \text{for all } u \in S^{d-1} \tag{26}$$

is the zero measure. A geometric application (besides applications in theoretical stereology, see [30]) reads as follows. For a convex body $K$, consider the direction-dependent functional

$$V^{(d-1)}(K, u) := \int_{\infty}^{\infty} V_{d-2}(K \cap (u^\perp + tu)) \, \mathrm{d}t, \qquad u \in S^{d-1},$$

where the intrinsic volume $V_{d-2}(K \cap H)$ is (up to a constant factor) the surface area of the intersection of $K$ with the hyperplane $H$. Then

$$V^{(d-1)}(K, u) = \frac{1}{2(d+1)} T_\Phi S_{d-1}(K, \cdot),$$

where $S_{d-1}(K, \cdot)$ is the surface area measure of $K$. Hence, the uniqueness theorem for the sine transform leads to the following: if $K, L$ are $d$-dimensional convex bodies with 0 as center of symmetry and satisfying $V^{(d-1)}(K, u) = V^{(d-1)}(L, u)$ for all $u$, then $K = L$.

To formulate a different counterpart to Theorem 3.2, we denote by

$$u^+ := \{v \in S^{d-1} : \langle u, v \rangle \geq 0\}$$

the hemisphere with center $u \in S^{d-1}$.

**Theorem 3.4.** *If the odd finite signed measure $\mu$ on $S^{d-1}$ satisfies*

$$\mu(u^+) = 0 \qquad \textit{for all } u \in S^{d-1}, \tag{27}$$

*then* $\mu = 0$.

For the proof, we take the function $\Phi$ defined by

$$\Phi(t) := \begin{cases} 1, & 0 \le t \le 1, \\ 0, & -1 \le t < 0. \end{cases}$$

then $(T_\Phi \mu)(u) = \mu(u^+)$. This $T_\Phi$ is known as the **hemispherical transform**. One computes that $a_{d,m}(\Phi) \neq 0$ for odd $m$. If the signed measure $\mu$ is odd, then $\pi_m \mu = 0$ for even $m$. Hence, Theorem 3.3. gives $\mu = 0$.

The third example from the introduction can be subsumed here. Let $K$ be a $d$-dimensional convex body with the property that every hyperplane through 0 halves the volume of $K$. Defining an odd signed measure $\mu$ by

$$\mu(B) := \int_B \left[\rho(K,v)^d - \rho(K,-v)^d\right] \sigma(\mathrm{d}v)$$

for Borel sets $B \subset S^{d-1}$, we deduce from Theorem 3.4 that $\mu = 0$, and this gives $\rho(K,v) = \rho(K,-v)$ for all $v \in S^{d-1}$.

For the mentioned functions $\Phi$, the multipliers of the transform $T_\Phi$ could be computed explicitly. This need not always be the case. For example, a question from stereology in [30] involves the transform $T_\Phi$ for the function

$$\Phi(t) := \mathbf{1}_{[0,1]}(t)\sqrt{1-t^2}.$$

In that case, the recursion formulas of Lemma 2.7 were used to establish recursion formulas and inequalities for the multipliers, from which it could be deduced that $a_{d,m}(\Phi) \neq 0$ for all $m$. Thus, the transform $T_\Phi$, which can be called the **hemispherical sine transform**, is injective.

The second example from the introduction involves the **spherical Radon transform**, defined by

$$(\mathcal{R}f)(u) := \int_{S_u} f \, \mathrm{d}\sigma_u, \qquad u \in S^{d-1}.$$

Recall that $\sigma_u$ denotes the $(d-2)$-dimensional spherical Lebesgue measure on the great subsphere $S_u$ with pole $u$. For a spherical harmonic $Y_m$ of order $m$ it is easy to compute that

$$\mathcal{R}Y_m = \omega_{d-1} \frac{P_m^d(0)}{P_m^d(1)} Y_m.$$

For even $m$, we have $P_m^d(0) \neq 0$ by Lemma 2.6, hence (for given $e \in S^{d-1}$) $(\sigma_e, Y_{mi}) \neq 0$ for suitable $i$. If $\mathcal{R}f = 0$ for $f \in \mathbf{C}(S^{d-1})$, it follows that $\mathcal{H}_m^d$ does not occur in $f$ for even $m$. If $f$ is an even function, then $\mathcal{H}_m^d$ also does not occur in $f$ for odd $m$, and it follows that $f = 0$. We can state this in the following form.

**Theorem 3.5.** *The spherical Radon transform is injective on even functions.*

A geometric consequence different from Minkowski's result mentioned in the introduction concerns central sections. Let $K, L$ be $d$-dimensional convex bodies (or star bodies) which are centrally symmetric with respect to 0 and satisfy

$$V_{d-1}(K \cap u^\perp) = V_{d-1}(L \cap u^\perp) \qquad \text{for all } u \in S^{d-1}.$$

Then Theorem 3.5 with $f := \rho(K,\cdot)^{d-1} - \rho(L,\cdot)^{d-1}$ shows that $K = L$.

The preceding results on the unique determination of convex bodies from the volumes of projections or sections are restricted to centrally symmetric convex bodies. There have been several attempts to find natural data, involving projections or sections, by which a general convex body is uniquely determined up to translations. An early example is an investigation by Anikonov and Stepanov [2], who have shown that a linear combination of the projection volume $V_{d-1}(K|u^\perp)$ and the surface area of the part of $K$ that is illuminated under illumination in direction $u$, determines a convex body uniquely, up to a translation. Note that the functional that they consider can be written in the form

$$T_\Phi S_{d-1}(K,\cdot) \qquad \text{with} \qquad \Phi(t) = p|t| + q\mathbf{1}_{[0,1]}(t).$$

As a special case of a more general construction, Goodey and Weil [11] introduced the second mean section body $M_2(K)$ of a convex body $K \subset \mathbb{R}^d$ by

$$h(M_2(K),\cdot) = \int_{A(d,2)} h(K \cap E,\cdot)\,\mu_2(\mathrm{d}E).$$

Here $A(d,2)$ is the affine Grassmannian of two-dimensional planes in $\mathbb{R}^d$ and $\mu_2$ is its motion invariant measure, suitably normalized. Thus, $M_2(K)$ comprises information about the two-dimensional sections of $K$, in integrated form. Goodey and Weil showed that two convex bodies with the same second mean section bodies differ only by a translation. This follows from the injectivity of the integral transform $T_\Phi$ with

$$\Phi(t) = (\arccos t)\sqrt{1-t^2},$$

since, with $c_d := \binom{d}{2}\kappa_d/\kappa_2\kappa_{d-2}$ and a suitable vector $z_{d-1}(K)$,

$$h(c_d M_2(K) - z_{d-1}(K),\cdot) = T_\Phi S_{d-1}(K,\cdot).$$

In analogy to adding up sections of convex bodies, one can add up (by integrating support functions) projections of convex bodies. For various results on the determination of convex bodies from Minkowski sums of projections, we refer to Schneider [27], Goodey [8], Spriestersbach [34], Kiderlen [20].

Other data determining a three-dimensional convex body were suggested by Groemer [17, 18], in the form of semi-girths of projections and intersections with half-planes. This was widely generalized, to higher dimensions and various directed projection and section data and mean values derived from them, in deep work of Goodey and Weil [12, 13, 14]. Their investigation involves various linear operators on function spaces on $S^{d-1}$ that intertwine the action of the rotation group and, therefore, act by multiplication on the spherical harmonics. To decide which multipliers are non-zero, turned out to be a very formidable task in some cases.

Two recent uniqueness results for general convex bodies are closer to the classical projection and section theorems. Let $K, L$ be two $d$-dimensional convex bodies with the property that, for each $u \in S^{d-1}$, the projections $K|u^\perp$ and $L|u^\perp$ have the same mean width and the same Steiner point. Then $K = L$. Similarly, suppose that $0 \in \operatorname{int} K, \operatorname{int} L$ and that, for each $u \in S^{d-1}$, the sections $K \cap u^\perp$ and $L \cap u^\perp$ have (in dimension $d-1$) the same volume and the same center of gravity. Then $K = L$. The uniqueness is easily derived from Theorem 3.5, applying it twice in each case. In the next section, we shall deal with corresponding stability results, taken from [31] and [3].

We turn to the fourth example from the introduction, Blaschke's characterization of ellipsoids. This leads to the equation

$$\sum_{i=1}^{d} f(\vartheta u_i) = 0 \qquad \text{for all } \vartheta \in S^{d-1}, \tag{28}$$

for the function $f = h(K,\cdot)^2 - \text{const}$, where $(u_1, \dots, u_d)$ is an orthonormal basis of $\mathbb{R}^d$. This equation is of the form (19), with $\mu = \sum_{i=1}^{d} \delta_{u_i}$. For the spherical harmonic $Y_m = P_m^d(\langle u_1, \cdot\rangle)$ we have

$$(Y_m, \mu) = \sum_{i=1}^{d} Y_m(\vartheta u_i) = P_m^d(1) + (d-1)P_m^d(0) \neq 0 \qquad \text{for } m \neq 2.$$

Hence, the only solutions of (28) are spherical harmonics of order two (and they are solutions). We deduce that the convex body $K$ must be an ellipsoid.

The fifth example of the introduction asks for the rotors of a regular simplex. More generally, we consider a polytope $P$ and ask for its rotors. By definition, a convex body $K$ is a rotor of $P$ if to each rotation $\vartheta \in SO_d$ there is a vector $t \in \mathbb{R}^d$ such such $\vartheta K + t$ is contained in $P$ and touches all the facets of $P$. If $P$ has a rotor of positive dimension, then it admits an inscribed ball (a ball touching all the facets). Suppose that the polytope $P$ has the ball $RB^d$ as inscribed ball. Let $u_1, \dots, u_n$ be the outer unit normal vectors of the facets of $P$. Then it can be shown that the convex body $K$ is a rotor of $P$ if and only if for any linear relation

$$\sum_{i=1}^{n} \alpha_i u_i = 0$$

the equations

$$\sum_{i=1}^{n} \alpha_i [h(K, \vartheta u_i) - R] = 0 \qquad \text{for all } \vartheta \in SO_d$$

are satisfied. Thus, in order to find non-spherical rotors of $P$, we have to solve a whole system of equations of type (19). This task amounts to finding all spherical harmonics that satisfy the system. Non-trivial solutions exist only for special polytopes $P$. The following theorem gives a complete classification. In its formulation, $Y_m$ denotes a spherical harmonic of order $m$, and support functions are restricted to the sphere $S^{d-1}$. We say that a polytope $Q$ is derived from the polytope $P$ if each facet of $Q$ contains a facet of $P$.

**Theorem 3.6.** *Let $d \geq 3$, let $P \subset \mathbb{R}^d$ be a polytope, and suppose that $K$ is a non-spherical rotor of $P$. Then we have one of the following cases.*

(1) *$P$ is a parallelepiped with equal heights; $K$ is a body of constant width.*

(2) *$d = 2$; $P$ is derived from a regular $k$-gon; the $n$th Fourier coefficients of the support function of $K$ are zero if $n \not\equiv \pm 1 \mod k$, $n \neq 0$,*

(3) *$d = 3$; $P$ is a regular tetrahedron; the support function of $K$ is of the form $Y_0 + Y_1 + Y_2 + Y_5$.*

(4) *$d = 3$; $P$ is derived from a regular octahedron, but is not a tetrahedron; the support function of $K$ is of the form $Y_0 + Y_1 + Y_5$.*

(5) *$d \geq 4$; $P$ is a regular simplex; the support function of $K$ is of the form $Y_0 + Y_1 + Y_2$.*

The proof in [26], which is reproduced in Groemer's book [16], uses most of the results on Legendre polynomials listed in Section 2, and several further properties of spherical harmonics. It is already clear from the formulation of the theorem that spherical harmonics are an indispensable tool for such a result. We remark that the survey article by Goldberg [7] contains photographs of models of non-trivial rotors of the regular tetrahedron, octahedron, and cube.

**Hints to the literature.** The general reference is Groemer's book [16]. As mentioned in the introduction, the presented examples are all very old; they served us for introducing spherical harmonics and demonstrating typical applications. For some more history about these and other examples, see [24, 25, 26].

# 4 Stability Results

The strength of the use of spherical harmonics in the treatment of rotation invariant uniqueness problems lies in the fact that very often the obtained uniqueness can be improved to a stability result. By this we mean explicit quantitative estimates, showing that small perturbations of the condition enforcing uniqueness result in only small deviations from the uniqueness situation. We shall present three examples of different approaches.

The first example is a rather simple, but very useful application of the Parseval relation. In the following, all integrations are over the unit sphere $S^{d-1}$. First we state an analytical lemma, which can be viewed as a variant of the Poincaré inequality.

**Lemma 4.1.** *Let $f$ be a real function of class $C^2$ on the unit sphere $S^{d-1}$ satisfying*

$$\int f \,\mathrm{d}\sigma = 0 \qquad \textit{and} \qquad \int f(u) u \,\sigma(\mathrm{d}u) = 0.$$

*Then*

$$\int \left( f^2 + \frac{1}{d-1} f \Delta_S f \right) \mathrm{d}\sigma + \frac{d+1}{d-1} \int f^2 \,\mathrm{d}\sigma \le 0.$$

*Proof.* Let

$$f \sim \sum_{m=2}^{\infty} Y_m, \qquad Y_m := \pi_m f,$$

be the condensed harmonic expansion of $f$ (note that $\pi_0 f = \pi_1 f = 0$ by the assumptions). By Green's formula on the sphere (Lemma 2.1) and Theorem 2.1(a),

$$(\Delta_S f, Y_m) = (f, \Delta_S Y_m) = -m(m+d-2)(f, Y_m).$$

Therefore, the condensed harmonic expansion of $\Delta_S f$ is given by

$$\Delta_S f \sim -\sum_{m=2}^{\infty} m(m+d-2) Y_m.$$

Now the Parseval relation in the form (8) gives

$$\int f^2 \,\mathrm{d}\sigma = \sum_{m=2}^{\infty} \int Y_m^2 \,\mathrm{d}\sigma,$$

$$\int f \Delta_S f \,\mathrm{d}\sigma = -\sum_{m=2}^{\infty} m(m+d-2) \int Y_m^2 \,\mathrm{d}\sigma.$$

This yields

$$\int f\left(f+\frac{1}{d-1}\Delta_S f\right) \mathrm{d}\sigma + \frac{d+1}{d-1}\int f^2 \,\mathrm{d}\sigma$$

$$= \frac{1}{d-1}\sum_{m=2}^{\infty}[2d-m(m+d-2)]\int Y_m^2 \,\mathrm{d}\sigma \le 0,$$

as stated. □

We apply this to an inequality for mixed volumes. Recall that the mixed volume $V(K_1,\dots,K_d)$ of convex bodies $K_1,\dots,K_d \subset \mathbb{R}^d$ is the symmetric function defined by

$$V(\lambda_1 K_1 + \cdots + \lambda_d K_d) = \sum_{i_1,\dots,i_d=1}^{d} \lambda_{i_1}\cdots\lambda_{i_d} V(K_{i_1},\dots,K_{i_d}),$$

where $\lambda_1,\dots,\lambda_d \ge 0$, and that

$$W_i(K) := V(\underbrace{K,\dots,K}_{d-i},\underbrace{B^d,\dots,B^d}_{i})$$

is the $i$th quermassintegral. The quermassintegrals satisfy the important inequalities

$$W_i(K)^2 \ge W_{i-1}(K)W_{i+1}(K), \qquad i=1,\dots,d-1, \tag{29}$$

which are special cases of the Aleksandrov–Fenchel inequality. For one of these inequalities, we derive an improved version. For this, we assume first that $K$ is a convex body with a support function $h_K$ of class $C^2$. We apply Lemma 4.1 to the function

$$f = h_K - h_{B(K)},$$

where $B(K)$ is the ball which has the same mean width and the same Steiner point as $K$. This function satisfies the assumptions of the lemma. The key to its applicability is the fact that the quermassintegral $W_{d-2}$ has the representation

$$W_{d-2}(K) = \frac{1}{d}\int \left(h_K^2 + \frac{1}{d-1}h_K\Delta_S h_K\right) \mathrm{d}\sigma.$$

Moreover,

$$W_{d-1}(K) = \frac{1}{d}\int h_K \,\mathrm{d}\sigma, \qquad W_d(K) = \kappa_d.$$

Using these facts, one obtains from the lemma that

$$W_{d-1}^2(K) - W_{d-2}(K)W_d(K) \ge \frac{(d+1)\kappa_d}{d(d-1)}\,\delta_2(K,B(K))^2, \tag{30}$$

where the $L_2$-distance $\delta_2(K,L)$ of two convex bodies $K,L$ is defined by

$$\delta_2(K,L)^2 := \frac{1}{\omega_d}\int |h_K - h_L|^2 \,\mathrm{d}\sigma$$

($\kappa_d$ denotes the volume and $\omega_d$ the surface area of $B^d$). This distance can be compared with the Hausdorff distance $\delta$, according to

$$c\delta(K,L)^{(d+1)/2} \le \delta_2(K,L) \le \delta(K,L),$$

where the constant $c$ depends only on the diameters of $K$ and $L$.

By approximation, the inequality (30) is extended to general convex bodies. This inequality is a typical stability result. It gives an explicit estimate for the deviation of the convex body $K$ from a suitable ball if one knows that $W_{d-1}^2(K) - W_{d-2}(K)W_d(K) \le \epsilon$.

The inequality (30) can be combined with known inequalities to obtain several results of geometric interest. As an example, we mention an improvement of the isoperimetric inequality for convex bodies. Let $V$ denote the volume and $S$ the surface area of a convex body $K$ in $\mathbb{R}^d$. Then one can deduce from (30) and (29) that

$$\left(\frac{S}{\omega_d}\right)^d - \left(\frac{V}{\kappa_d}\right)^{d-1} \ge c\delta(K, B_K)^{(d+3)/2},$$

where $B_K$ is a suitable ball and the constant $c$ depends on the dimension and bounds for the inradius and circumradius of $K$. The exponent on the right side is close to optimal, since such an inequality cannot hold with an exponent less than $(d+1)/2$.

Also the other two methods that we are going to explain make use of the Parseval relation, but in a more sophisticated way. We describe the next approach in general terms, for a transform $T_\Phi$ as introduced in Section 3. We abbreviate its multipliers by $a_m := a_{d,m}(\Phi)$. Thus, if $f \in \mathbf{C}(S^{d-1})$ has the condensed harmonic expansion

$$f \sim \sum Y_m$$

(summations here and in the following are from 0 to $\infty$), then $T_\Phi f$ has the condensed harmonic expansion

$$T_\Phi f \sim \sum a_m Y_m.$$

Now let

$$g = T_\Phi f$$

and suppose that the transform $T_\Phi$ has an inverse on a space of functions under consideration. A stability result for the inverse would be an assertion telling us that $\|f\|_2$ must be small if $\|g\|_2$ is small. This would be easy if $|a_m| \ge 1/c$ for all $m$ with a positive constant $c$. Namely, in that case the Parseval relation would give

$$\|g\|_2^2 = \sum |a_m|^2 \|Y_m\|_2^2 \ge \frac{1}{c^2} \sum \|Y_m\|_2^2 = \frac{1}{c^2}\|f\|_2^2$$

and thus $\|f\|_2 \le c\|g\|_2$. However, in the cases of interest, the sequence $a = (a_m)$ of multipliers tends to zero, and the faster it does, the more delicate is the stability problem. We sketch two different ways out of this dilemma.

Let

$$f \sim \sum Y_m, \qquad g \sim \sum a_m Y_m,$$

and assume that $Y_m = 0$ whenever $a_m = 0$. For any $\beta > 0$, an application of Hölder's inequality gives

$$\begin{aligned}
\|f\|_2^2 &= \sum \|Y_m\|_2^2 = \sum_{a_m \ne 0} \|Y_m\|_2^2 \\
&= \sum_{a_m \ne 0} \left(|a_m|^{-\frac{2\beta}{\beta+2}} \|Y_m\|^{\frac{4}{\beta+2}}\right) \left(|a_m|^{\frac{2\beta}{\beta+2}} \|Y_m\|^{\frac{2\beta}{\beta+2}}\right) \\
&\le \left(\sum_{a_m \ne 0} |a_m|^{-\beta} \|Y_m\|^2\right)^{\frac{2}{\beta+2}} \left(\sum |a_m|^2 \|Y_m\|^2\right)^{\frac{\beta}{\beta+2}},
\end{aligned}$$

hence

$$\|f\|_2 \le C(f,a,\beta)^{\frac{1}{\beta+2}} \|g\|_2^{\frac{\beta}{\beta+2}}$$

with

$$C(f,a,\beta) := \sum_{a_m \neq 0} |a_m|^{-\beta} \|Y_m\|^2.$$

The next step consists in estimating $|a_m|$ in the form

$$|a_m|^{-\beta} \le c(d,\beta) m(m+d-2)$$

and using

$$\Delta_S f \sim \sum m(m+d-2) Y_m.$$

The spherical gradient $\nabla_S$ is defined by

$$\nabla_S f := (\nabla \check{f})|S^{d-1} = (\nabla \check{f})^{\wedge},$$

where $\nabla$ is the gradient on $\mathbb{R}^d$. Using $(f, \Delta_S f) = -\|\nabla_S f\|_2^2$ and the Parseval relation, we get

$$\|\nabla_S f\|_2^2 = \sum m(m+d-2)\|Y_m\|_2^2.$$

If $\beta$ is chosen suitably in dependence on the sequence $a = (a_m)$, one obtains an estimate of the form

$$\|f\|_2 \le c_1(d,\beta) \|\nabla_S f\|_2^{\frac{2}{\beta+2}} \|g\|_2^{\frac{\beta}{\beta+2}}.$$

It will then depend on the particular geometric situation whether one is able to estimate $\|\nabla_S f\|_2$ reasonably. For example, if $f$ is the support function $h_K$ of a convex body $K$, the formula

$$W_{d-2}(K) = \frac{1}{d(d-1)} \left[ (d-1)\|h_K\|_2^2 - \|\nabla_S h_K\|_2^2 \right]$$

together with $W_{d-2}(K) \ge 0$ can be used to obtain an estimate

$$\|\nabla_S h_K\|_2 \le \sqrt{d-1} \|h_K\|_2 \le c(d,R),$$

if $K \subset RB^d$.

Carrying out this general program for the special case of the spherical Radon transform $\mathcal{R}$, Groemer [16, Th. 3.4.14] obtained the following lemma.

**Lemma 4.2.** *Let $F_1$ and $F_2$ be twice continuously differentiable functions on $S^{d-1}$ ($d \ge 3$), and let $F_i^+$ denote the even part of $F_i$. Then*

$$\|F_1^+ - F_2^+\|_2 \le h_d(F_1,F_2) \, \|\mathcal{R}F_1 - \mathcal{R}F_2\|_2^{2/d}$$

*with*

$$h_d(F_1,F_2) = \frac{1}{\omega_{d-1}} \left( 2\omega_{d-1}^2 \beta_d^{-\frac{4}{d-2}} \left( \|\nabla_S F_1\|_2^2 + \|\nabla_S F_2\|_2^2 \right) + \|\mathcal{R}F_1 - \mathcal{R}F_2\|_2^2 \right)^{\frac{d-2}{2d}},$$

*where $\beta_d$ is an explicit constant.*

Hence, if the difference of the spherical Radon transforms of two functions is small, then the difference of the even parts of the functions is small. There can, of course, be no information on the odd parts of the functions, since the spherical Radon transform of an odd function is zero. In concrete cases, the value of the explicit estimate of the lemma depends on the possibility to bound the quantities $\|\nabla_S F_i\|$ appearing in the factor $h_d(F_1, F_2)$.

We want to apply this method to prove a stability version (obtained in [3]) of a new uniqueness result. This result says that a star body is uniquely determined by the volumes and centroids of its hyperplane sections through a fixed interior point. Let $K$ be a star body in $\mathbb{R}^d$, that is, a nonempty compact set which is starshaped with respect to 0 and has a continuous positive radial function, defined by $\rho_K(v) := \max\{\lambda \geq 0 : \lambda v \in K\}$, $v \in S^{d-1}$. For $u \in S^{d-1}$, we denote the $(d-1)$-dimensional volume of the intersection $K \cap u^\perp$ by $v_{d-1}(K, u)$ and its center of gravity by $c_{d-1}(K, u)$; thus

$$\begin{aligned} v_{d-1}(K,u) &= \frac{1}{d-1}\int_{S_u} \rho_K^{d-1}\,\mathrm{d}\sigma_u, \\ c_{d-1}(K,u) &= \frac{1}{d}\int_{S_u} \rho_K(v)^d v\,\sigma_u(\mathrm{d}v). \end{aligned}$$

Let $K, L$ be star bodies satisfying

$$v_{d-1}(K,u) = v_{d-1}(L,u) \qquad \text{for } u \in S^{d-1} \tag{31}$$

and

$$c_{d-1}(K,u) = c_{d-1}(L,u) \qquad \text{for } u \in S^{d-1}. \tag{32}$$

Since the spherical Radon transform of a continuous function on $S^{d-1}$ uniquely determines the even part of the function, assumption (31) implies that the even part of $\rho_K^{d-1} - \rho_L^{d-1}$ vanishes, thus

$$\rho_K^{d-1}(v) - \rho_L^{d-1}(v) = -\rho_K^{d-1}(-v) + \rho_L^{d-1}(-v) \qquad \text{for } v \in S^{d-1}. \tag{33}$$

Similarly, assumption (32) yields (if the result on the spherical Radon transform is applied coordinate-wise) that the even part of the function $v \mapsto [\rho_K^d(v) - \rho_L^d(v)]v$ vanishes, and this gives

$$\rho_K^d(v) - \rho_L^d(v) = \rho_K^d(-v) - \rho_L^d(-v) \qquad \text{for } v \in S^{d-1}. \tag{34}$$

Suppose now that there exists some $v \in S^{d-1}$ with $\rho_K(v) \neq \rho_L(v)$, say $\rho_K(v) < \rho_L(v)$. Then $\rho_K^{d-1}(v) < \rho_L^{d-1}(v)$, hence (33) gives $\rho_L^{d-1}(-v) < \rho_K^{d-1}(-v)$. This yields $\rho_L^d(-v) < \rho_K^d(-v)$, and now (34) gives $\rho_K^d(v) > \rho_L^d(v)$, a contradiction. We conclude that $\rho_K(v) = \rho_L(v)$ for all $v$, hence $K = L$.

While this uniqueness theorem holds for star bodies, our stability result requires convexity assumptions.

**Theorem 4.1.** *Let $d \geq 3$, let $r, R, \epsilon_0 > 0$, let $K, L$ be convex bodies with $rB^d \subset K, L \subset RB^d$, and let $0 \leq \epsilon \leq \epsilon_0$. If*

$$\|v_{d-1}(K,\cdot) - v_{d-1}(L,\cdot)\|_2 \leq \epsilon \tag{35}$$

*and*

$$\|c_{d-1}(K,\cdot) - c_{d-1}(L,\cdot)\|_2 \leq \epsilon, \tag{36}$$

*then*

$$\delta_2(K,L) \leq c(d,r,R,\epsilon_0)\epsilon^{2/d}, \tag{37}$$

*with an explicit constant $c(d,r,R,\epsilon_0)$ depending only on $d, r, R, \epsilon_0$.*

For the proof, one applies the lemma twice, first to the functions $F_1 := \rho_K^{d-1}$, $F_2 := \rho_L^{d-1}$, and then to the functions defined by $G_1(v) := \rho_K^d(v)\langle v, e\rangle$ and $G_2(v) := \rho_L^d(v)\langle v, e\rangle$ for $v \in S^{d-1}$, with fixed $e \in S^{d-1}$. From (35) and (36) this leads to the estimates

$$\begin{aligned} \|(\rho_K^{d-1})^+ - (\rho_L^{d-1})^+\|_2 &\leq h_d(F_1, F_2)((d-1)\epsilon)^{2/d} =: A_1, \\ \|(\rho_K^d)^- - (\rho_L^d)^-\|_2 &\leq \sqrt{d} h_d(G_1, G_2)(d\epsilon)^{2/d} =: A_2, \end{aligned}$$

where $F^-$ denotes the odd part of a function $F$ on $S^{d-1}$. Even and odd parts are taken here of different powers of the radial functions. This causes a complication which, however, can be dealt with, and one obtains an estimate

$$\|\rho_K - \rho_L\|_2^2 \leq \frac{4A_1^2}{(d-1)^2 r^{2(d-1)}} + \frac{4A_2^2}{d^2 r^{2d}}.$$

It remains to bound the constants $h_d(F_1, F_2)$ and $h_d(G_1, G_2)$ and thus $\|\nabla_S \rho_K^m\|_2$, $\|\nabla_S \rho_L^m\|_2$ for $m = d-1, d$. It is here where the convexity and the further assumptions on $K$ and $L$ are needed. One obtains, for example,

$$\|\nabla_S \rho_K^m\|_2 \leq m\sqrt{(d-1)\omega_d}\, \frac{R^{m+1}}{r},$$

and with the help of such estimates, the proof of Theorem 4.1 can be completed.

A result similar to Theorem 4.1, with section, volume, centroid replaced by projection, mean width, Steiner point, was proved in [31].

In several applications, a transform of type $T_\Phi$ is applied to the surface area measure $S_{d-1}(K, \cdot)$ of a convex body. We abbreviate this now by $S_K$ and recall its meaning. For a Borel set $A \subset S^{d-1}$, the value $S_K(A)$ is the surface area (the $(d-1)$-dimensional Hausdorff measure) of the set of all boundary points of $K$ with an outer unit normal vector falling in $A$. We describe a method to obtain stability results for $T_\Phi S_K$, which goes back to Bourgain and Lindenstrauss [4] and was slightly extended in [19].

**Theorem 4.2.** *Assume that the multipliers of $T_\Phi$ satisfy*

$$a_{d,0}(\Phi) \neq 0, \qquad |a_{d,n}(\Phi)^{-1}| \leq bn^\beta \quad \text{for } n \in \mathbb{N} \tag{38}$$

*with suitable $b, \beta > 0$. Let $0 < r < R$, and let $K, M \subset \mathbb{R}^d$ be convex bodies satisfying $rB^d \subset K, M \subset RB^d$.*

*For $\alpha \in (0, 1/d(1+\beta))$, there is a constant $c = c(d, \Phi, \alpha, r, R)$ such that*

$$\delta(K, M + x) \leq c\|T_\Phi(S_K - S_M)\|^\alpha$$

*for suitable $x \in \mathbb{R}$.*

*If $K$ and $M$ are centrally symmetric and* (38) *holds for even $n$, then the same conclusion can be drawn.*

We sketch the main steps of the proof. Let $F$ be a continuous real function and $\mu$ a finite signed measure on $S^{d-1}$. The aim is to find an estimate of the form

$$\left| \int F \,\mathrm{d}\mu \right| \leq c(d, F, \mu)\|T_\Phi \mu\|_2^\alpha$$

(all integrations are over $S^{d-1}$).

Let $\|\mu\|_{TV}$ be the total variation norm of $\mu$, and let

$$\|F\|_L := \max_{x\neq y} \frac{|F(x)-F(y)|}{\|x-y\|}, \qquad \|F\|_\infty := \max_x |F(x)|,$$

$$\|F\|_{BL} := \|F\|_L + \|F\|_\infty.$$

Recall that (in terms of condensed harmonic expansions)

$$\mu \sim \sum Y_n \qquad \text{implies} \qquad T_\Phi \mu \sim \sum a_{d,n}(\Phi) Y_n.$$

The basic idea of Bourgain and Lindenstrauss was to compare $T_\Phi\mu$ with the Poisson transform

$$\mu_\tau(u) = \frac{1}{\omega_d}\int_{S^{d-1}} \frac{1-\tau^2}{(1+\tau^2-2\tau\langle u,v\rangle)^{d/2}}\,\mu(\mathrm{d}v)$$

for $0<\tau<1$, which has the condensed harmonic expansion

$$\mu_\tau \sim \sum \tau^n Y_n.$$

One can then exploit that $\tau^n$ tends to zero more rapidly than $a_{d,n}(\Phi)$, as $n\to\infty$.

In a first step, one can estimate, comparing $F$ with its Poisson transform $F_\tau$ and using $\int F_\tau\,\mathrm{d}\mu = \int F\mu_\tau\,\mathrm{d}\sigma$ and several intermediate inequalities, that

$$\left|\int F\,\mathrm{d}\mu\right| \le c_1\|F\|_L\,\|\mu\|_{TV}(1-\tau)\log\frac{2}{1-\tau} + \|F\|_2\,\|\mu_\tau\|_2$$

for $\tau\in[\frac{1}{4},1)$.

The second step uses the assumption (38). Since $n^\beta\tau^n(1-\tau)^\beta \le (\beta/e)^\beta$, it follows from (38) that

$$\tau^n \le c_2(1-\tau)^{-\beta}|a_{d,n}(\Phi)|.$$

With Parseval's relation, this gives

$$\begin{aligned}\|\mu_\tau\|_2 &= \left(\sum \tau^{2n}\|Y_n\|_2^2\right)^{\frac{1}{2}} \le c_2(1-\tau)^{-\beta}\left(\sum |a_{d,n}(\Phi)|^2\|Y_n\|_2^2\right)^{\frac{1}{2}}\\ &\le c_2(1-\tau)^{-\beta}\|T_\Phi\mu\|_2.\end{aligned}$$

Together with the previous estimate, this yields

$$\left|\int F\,\mathrm{d}\mu\right| \le c_3\|F\|_{BL}\left[\underbrace{\|\mu\|_{TV}(1-\tau)\log\frac{2}{1-\tau}}_{A} + \underbrace{\|T_\Phi\mu\|_2(1-\tau)^{-\beta}}_{B}\right].$$

For suitable $\tau\in[\frac{1}{4},1)$ and a constant $c_4$ we have $c_4A=B$. For $\alpha<1/(1+\beta)$ this leads to

$$\left|\int F\,\mathrm{d}\mu\right| \le c_5\|F\|_{BL}\,\|\mu\|_{TV}^{1-\alpha}\,\|T_\Phi\mu\|_2^\alpha,$$

which is of the desired form.

This is now applied with

$$F = h_K \text{ or } h_M, \qquad \mu = S_K - S_M.$$

Since $K, M \subset RB^d$, we can then estimate

$$\|F\|_{BL} \leq c(d,R), \qquad \|\mu\|_{TV} \leq c(d,R).$$

Since

$$\frac{1}{d}\int h_K \,\mathrm{d}\mu = V_d(K) - V_1(M,K),$$

where $V_1(M,K)$ denotes the mixed volume $V_1(M,\ldots,M,K)$, we obtain

$$\begin{aligned} |V_d(K) - V_1(M,K)| &\leq c_6\|T_\Phi(S_K - S_M)\|^\alpha, \\ |V_d(M) - V_1(K,M)| &\leq c_6\|T_\Phi(S_K - S_M)\|^\alpha. \end{aligned}$$

Now geometry has to take over. Knowing that the right side is small, there are methods from Brunn–Minkowski theory to estimate the Hausdorff distance between $K$ and a suitable translate of $M$. In this way, the proof of Theorem 4.2 can be completed.

Theorem 4.2 gives a number of concrete stability results for transformations $T_\Phi$ that have been considered in the geometry of convex bodies or in stochastic geometry. They are not restricted to centrally symmetric convex bodies. We give here a list of the transforms which have been mentioned in Section 3 and for which we now have corresponding stability results.

1) $\Phi(t) = \frac{1}{2}|t|$, the cosine transform. This is the case of Aleksandrov's projection theorem and the stability result of Bourgain and Lindenstrauss.

Assumption (38) holds for even $n$ with $\beta = (d+2)/2$.

2) $\Phi(t) = \sqrt{1-t^2}$, the sine transform. It appears in work of Berwald, Schneider, and Goodey.

Assumption (38) holds for even $n$ with $\beta = d$.

3) $\Phi(t) = p|t| + q\mathbf{1}_{[0,1]}$, with constants $p, q$. The corresponding transform was studied by Anikonov and Stepanov. They obtained a stability result in $\mathbb{R}^3$, but a weak one only, as it requires bounds on derivatives up to the fifth order.

Assumption (38) holds for all $n$ with $\beta = (d+2)/2$.

4) $\Phi(t) = (\arccos t)\sqrt{1-t^2}$. The corresponding transform appeared in work of Goodey and Weil on mean section bodies.

Assumption (38) holds for all $n$ with $\beta = d$.

5) $\Phi(t) = \mathbf{1}_{[0,1]}\sqrt{1-t^2}$. This was used in a contribution to stochastic geometry by Schneider.

Assumption (38) holds with $\beta = d$.

Recently, the described approaches to stability results on convex bodies involving tomographic data have been considerably unified, extended and improved by Kiderlen [21]. Goodey, Kiderlen and Weil [10] present a comprehensive survey of integral transforms in geometric tomography.

**Hints to the literature.** The book by Groemer [16] presents a thorough treatment of applications of spherical harmonics to stability problems in convex geometry. The following contributions have appeared after the publication of this book: [3], [8], [9], [10], [12], [13], [14], [15], [17], [18], [19],[20], [21], [31], [34]. A stability version of the fourth example from the introduction appears in [5]. Some stability results can also be found in [28].

An interesting application of spherical harmonics to Minkowski symmetrizations was made by Klartag [22].

# 5 Universal Convex Bodies

For the treatment of the functional equation (3.1) it was, according to Theorem 3.1, essential for which numbers $k$ the space $\mathcal{H}_k^d$ of spherical harmonics of order $k$ occurs in a function $f$, that is, $\pi_k f \neq 0$. In this section, we will treat a question which leads us to convex bodies $K$ with the property that their support function satisfies $\pi_k h_K \neq 0$ for all $k \neq 1$ (the case $k = 1$ is excluded, since $\pi_1 h_K$ is not invariant under translations). Such a convex body is called **universal**.

We describe the question leading to these bodies. The space $\mathcal{K}^d$ of convex bodies in $\mathbb{R}^d$ admits two basic operations: Minkowski addition, defined by

$$K + L := \{x + y : x \in K,\, y \in L\}, \qquad K, L \in \mathcal{K}^d,$$

and dilatation, given by

$$\alpha K := \{\alpha x : x \in K\}, \qquad K \in \mathcal{K}^d,\, \alpha \geq 0.$$

Combined, they yield Minkowski linear combinations. Thinking of the role that bases of all kinds play in various part of mathematics, the following question seems natural. Can general convex bodies be obtained as linear combinations of 'a few special convex bodies'? We think, in particular, of only one convex body and its congruent copies (images under rigid motions).

For dealing with this question, the support function is the natural tool. We denote the support function of the convex body $K \in \mathcal{K}^d$ by $h_K$, thus

$$h_K(u) := \max\{\langle u, x\rangle : x \in K\}, \qquad u \in \mathbb{R}^d.$$

It is adapted to linear combinations and rotations, since $h_{K+L} = h_K + h_L$, $h_{\alpha K} = \alpha h_K$ for $\alpha \geq 0$, and $h_{\vartheta K} = \vartheta h_K$ for every rotation $\vartheta \in SO_d$. Moreover, for the Hausdorff metric $\delta$, we have $\delta(K, L) = \max\{|h_K(u) - h_L(u)| : u \in S^{d-1}\}$.

Let us first see how far we get when we start with the simplest non-trivial convex body, a segment (closed line segment) $S$. The support function of the segment $S$ with endpoints $\pm\alpha v$ ($v \in S^{d-1}$, $\alpha > 0$) is given by

$$h_S(u) = |\langle u, v\rangle|\alpha.$$

The sum of finitely many segments is called a **zonotope**. The support function of a zonotope $Z$ with center 0 is of the form

$$h_Z(u) = \sum_{i=1}^{k} |\langle u, v_i\rangle|\alpha_i$$

with $v_i \in S^{d-1}$, $\alpha_i > 0$. A limit of zonotopes is called a **zonoid**. The support function of a zonoid $Z$ with center 0 is given by

$$h_Z(u) = \int_{S^{d-1}} |\langle u, v\rangle| \, \rho(\mathrm{d}v)$$

with a finite Borel measure $\rho$. A convex body $K$ with a support function of the form

$$h_K(u) = \int_{S^{d-1}} |\langle u, v\rangle| \, \rho(\mathrm{d}v),$$

where $\rho$ is a finite signed Borel measure, is call a **generalized zonoid**. Thus, $K$ is a generalized zonoid if and only if there exist zonoids $Z_1, Z_2$ with $K + Z_1 = Z_2$.

Let $\mathcal{K}_s^d$ denote the set of centrally symmetric convex bodies. The following well-known facts exhibit the essential difference between zonoids and generalized zonoids.

- The zonoids are nowhere dense in $\mathcal{K}_s^d$.
- The generalized zonoids are dense in $\mathcal{K}_s^d$.

The second fact has often been useful in the investigation of centrally symmetric convex bodies. We mention that the standard proof of this fact is a typical application of spherical harmonics. For given $K \in \mathcal{K}_s^d$ with center 0, one tries to solve the integral equation

$$h_K(u) = \int_{S^{d-1}} |\langle u, v\rangle| \, f(v) \, \sigma(\mathrm{d}v)$$

by a function $f$, say continuous. If the equation holds, if further

$$f \sim \sum_{2|k} Y_k, \qquad h_K \sim \sum_{2|k} X_k$$

are the condensed harmonic expansions, and $a_0, a_2, \dots$ is the sequence of even-order multipliers of the cosine transform, then we know that $X_k = a_k Y_k$ for all even $k$. Conversely, for given $h_K$ one can try to define a function by $f = \sum_{2|k} a_k^{-1} X_k$. If $h_K$ is sufficiently often differentiable (the required differentiability increases with the dimension), one can indeed show that this series converges absolutely and uniformly. The continuous function $f$ that it defines is then a solution of the integral equation. This shows that every sufficiently smooth centrally symmetric convex body is a generalized zonoid.

Now we want to get rid of the central symmetry, and we ask whether the segment $S$ can be replaced by a non-symmetric convex body, to obtain a dense class in $\mathcal{K}^d$ instead of $\mathcal{K}_s^d$. In other words: suppose we have only one convex body $B$ at our hands and want to produce other convex bodies from it by taking Minkowski linear combinations of congruent copies of $B$, limits, and differences (of support functions). How big a class of convex bodies can we obtain? Some definitions are now in order.

**Definition.** A **Minkowski class** is a subset of $\mathcal{K}^d$ that is closed in the Hausdorff metric, closed under Minkowski linear combinations, and closed under translations.

Let $G$ be a subgroup of $GL(d)$, for example $SO_d$. The Minkowski class $\mathcal{M}$ is $G$**-invariant** if $K \in \mathcal{M}$ implies $gK \in \mathcal{M}$ for all $g \in G$.

If $B \in \mathcal{K}^d$ and $G \subset GL(d)$ are given, then $\mathcal{M}_{B,G}$ is defined as the smallest $G$-invariant Minkowski class containing $B$.

**Examples:** If $S$ is a segment and $B$ is a ball, then

$$\begin{aligned}
\mathcal{M}_{S,SO_d} &= \mathcal{M}_{S,GL(d)} = \{\text{zonoids}\},\\
\mathcal{M}_{B,SO_d} &= \{\text{balls}\},\\
\mathcal{M}_{B,GL(d)} &= \{\text{zonoids}\}.
\end{aligned}$$

**Definition.** The convex body $K$ is called an **$\mathcal{M}$-body** if $K \in \mathcal{M}$. The body $K$ is called a **generalized $\mathcal{M}$-body** if there exist $\mathcal{M}$-bodies $M_1, M_2$ with $K + M_1 = M_2$.

For $d \geq 3$ it follows from known results that for each $B \in \mathcal{K}^d$, the $\mathcal{M}_{B,GL(d)}$-bodies are nowhere dense in $\mathcal{K}^d$. What can be said about generalized $\mathcal{M}_{B,GL(d)}$-bodies? First answers are given by the following special results.

**Theorem 5.1.** (Schneider [29]) *Let $T \subset \mathbb{R}^d$ be a triangle with an irrational angle. Then the set of generalized $\mathcal{M}_{T,SO_d}$-bodies is dense in $\mathcal{K}^d$.*

(By an irrational angle we mean an angle which is an irrational multiple of $\pi$.)

**Theorem 5.2.** (Alesker [1]) *Let $K$ be a non-symmetric convex body. Then the set of generalized $\mathcal{M}_{K,GL(d)}$-bodies is dense in $\mathcal{K}^d$.*

Alesker's proof uses representation theory for the group $GL(d)$. Of course, Alesker's result does not hold if the general linear group $GL(d)$ is replaced by the rotation group $SO_d$. For example, if $K$ is a body of constant width, then all generalized $\mathcal{M}_{K,SO_d}$-bodies are of constant width.

The following results were obtained jointly with Franz Schuster, in [32].

**Theorem 5.3.** *Let $B \in \mathcal{K}^d$ be non-symmetric. Then every neighborhood of $B$ contains an affine image $B'$ of $B$ such that the set of generalized $\mathcal{M}_{B',SO_d}$-bodies is dense in $\mathcal{K}^d$.*

This is a consequence of the following two theorems. Recall that a convex body $B \in \mathcal{K}^d$ is called **universal** if the expansion of its support function $h_B$ in spherical harmonics contains non-zero harmonics of all orders $\neq 1$.

**Theorem 5.4.** *Let $B \in \mathcal{K}^d$. The set of generalized $\mathcal{M}_{B,SO_d}$-bodies is dense in $\mathcal{K}^d$ if and only if $B$ is universal.*

**Theorem 5.5.** *Let $B \in \mathcal{K}^d$ be non-symmetric. Then every neighborhood of $B$ contains an affine image of $B$ that is universal.*

Theorem 3 has a counterpart for symmetric bodies.

We sketch the proof of Theorem 5.4. Suppose first that $B$ is not universal. Then there exists a number $m \neq 1$ with $\pi_m h_B = 0$. This implies $\pi_m h_K = 0$ for all generalized $\mathcal{M}_{B,SO_d}$-bodies $K$ and their limits. But there exists a body $M \in \mathcal{K}^d$ with $\pi_m h_M \neq 0$. Hence, the set of generalized $\mathcal{M}_{B,SO_d}$-bodies is not dense in $\mathcal{K}^d$.

Conversely, assume that $B$ is universal.

Recall that for showing that a sufficiently smooth body $K$ with center 0 is a generalized zonoid, we have solved the integral equation

$$h_K(u) = \int_{S^{d-1}} |\langle u, v\rangle| \, f(v) \, \sigma(\mathrm{d}v).$$

We try to solve a corresponding integral equation on the group $SO_d$, with respect to its Haar measure $\nu$. Suppose we can solve the integral equation

$$h_K(u) = \int_{SO_d} h_{\vartheta B}(u) \, f(\vartheta) \, \nu(\mathrm{d}\vartheta),$$

say, by a continuous function $f$. Then we decompose $f$ into its positive and negative part, $f = f^+ - f^-$, and get

$$h_K(u) + \underbrace{\int_{SO_d} h_{\vartheta B}(u) f^-(v) \, \nu(\mathrm{d}\vartheta)}_{h_{M_1}(u)} = \underbrace{\int_{SO_d} h_{\vartheta B}(u) f^+(v) \, \nu(\mathrm{d}\vartheta)}_{h_{M_2}(u)},$$

where $M_1, M_2 \in \mathcal{M}_{B,SO_d}$ (approximate $\nu$ by discrete measures). Since $K + M_1 = M_2$, the body $K$ is a generalized $\mathcal{M}_{B,SO_d}$-body.

To solve the integral equation, it is sufficient to assume that $h_K$ is a sum of finitely many spherical harmonics, because the set of such bodies is dense in $\mathcal{K}^d$. Thus, for

$$h_K = \sum_{m=0}^{k} \sum_{j=1}^{N(d,m)} a_{mj} Y_{mj}$$

we have to find a continuous function $f$ on the group $SO_d$ satisfying

$$h_K(u) = \int_{SO_d} h_{\vartheta B}(u) \, f(\vartheta) \, \nu(\mathrm{d}\vartheta), \qquad u \in S^{d-1}.$$

Let $b_{mj} := (h_B, Y_{mj})$. Since $B$ is universal, for each $m \in \mathbb{N}_0$ there is an index $j_m$ such that $b_{mj_m} \neq 0$. With the functions $t_{ij}^m$ from (17), we put

$$f := N(d,m) \sum_{m=0}^{k} \frac{1}{b_{mj_m}} \sum_{i=1}^{N(d,m)} a_{mi} t_{ij_m}^m.$$

Using formula (18), we obtain

$$\begin{aligned}
\int_{SO_d} h_{\vartheta B}(u) f(\vartheta) \, \nu(\mathrm{d}\vartheta) &= \sum_{m=0}^{k} \frac{1}{b_{mj_m}} \sum_{i=1}^{N(d,m)} a_{mi} N(d,m) \int_{SO_d} h_{\vartheta B}(u) t_{ij_m}^m(\vartheta) \, \nu(\mathrm{d}\vartheta) \\
&= \sum_{m=0}^{k} \frac{1}{b_{mj_m}} \sum_{i=1}^{N(d,m)} a_{mi} (h_B, Y_{mj_m}) Y_{mi}(u) \\
&= \sum_{m=0}^{k} \sum_{i=1}^{N(d,m)} a_{mi} Y_{mi}(u) = h_K(u),
\end{aligned}$$

which shows that $f$ is a solution of the integral equation. Thus, Theorem 5.4 is proved.

We turn to Theorem 5.5, the main result. We explain the idea of its proof by demonstrating first an easier part of the proof.

**Proposition.** *Let $B \in \mathcal{K}^d$ be not one-pointed. Then there exists a linear transformation $g \in GL(d)$, arbitrarily close to the identity, such that $\pi_m h_{gB} \neq 0$ for all even numbers $m \in \mathbb{N}_0$.*

We reduce the proof to the fact that a segment $S$ satisfies

$$\pi_m h_S \neq 0 \qquad \text{for all even } m$$

(which was the reason for the injectivity of the cosine transform on even functions). In Cartesian coordinates, let $\Pi_1$ be the projection onto the $x_1$-axis, and suppose, without loss of generality, that $\Pi_1 B =: S$ is a non-degenerate segment. Define $g(\lambda) \in GL(d)$ by

$$g(\lambda) : (x_1, \dots, x_n) \mapsto (x_1, \lambda x_2, \dots, \lambda x_n).$$

For $\lambda \to 0$, the map $g(\lambda)$ converges to $\Pi_1$. It follows that

$$\lim_{\lambda \to 0} (h_{g(\lambda)B}, Y_{mj}) = (h_S, Y_{mj}).$$

If $m$ is even, then $(h_S, Y_{mj_m}) \neq 0$ for some $j_m$. Hence, the function

$$F(\lambda) := (h_{g(\lambda)B}, Y_{mj_m}), \qquad \lambda \in (0, 1],$$

does not vanish identically. This function is real analytic. Therefore, the set

$$Z_m := \{\lambda \in (0, 1] : \pi_m h_{g(\lambda)B} = 0\}$$

is countable. This holds for each even $m$. It follows that every neighborhood of 1 contains some $\lambda$ with

$$\pi_m h_{g(\lambda)B} \neq 0 \qquad \text{for all even } m.$$

This completes the proof of the Proposition.

For the remaining part of the proof of Theorem 5.5, that is, the case of non-symmetric $B$ and arbitrary $m$, we sketch only the strategy. We recall what we have to prove.

**Theorem 5.5.** *Let $B \in \mathcal{K}^d$ be non-symmetric. Then there exists $g \in GL(d)$, arbitrarily close to the identity, such $\pi_m h_{gB} \neq 0$ for all $m$.*

The following are the essential steps:

**1.)** Prove the two-dimensional case of the Theorem 5.5.

**2.)** Lemma. *If $B \subset \mathbb{R}^2 \subset \mathbb{R}^d$ and $B$ is universal in $\mathbb{R}^2$, then $B$ is universal in $\mathbb{R}^d$.*

**3.)** Similarly as before, use linear maps converging to the projection onto $\mathbb{R}^2$.

Step 2 requires only the use of suitable bases of the spaces of spherical harmonics and some direct calculations. The idea of Step 3 is similar to the argument sketched before. Therefore, we indicate here only Step 1.

**The two-dimensional case**

Let $B \subset \mathbb{R}^2$ be a non-symmetric convex body. Write

$$h_B((\cos\varphi, \sin\varphi)) =: h_B(\varphi).$$

The space $\mathcal{H}_m^2$ is spanned by the functions $\cos m\varphi$ and $\sin m\varphi$. Therefore, in complex notation

$$\pi_m h_{gB} = 0 \;\Leftrightarrow\; \int_0^{2\pi} h_{gB}(\varphi)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{d}\varphi = 0.$$

Define a map $F_{B,m} : GL(2)^+ \to \mathbb{C}$ (where $GL(2)^+$ is the connected component of the identity) by

$$F_{B,m}(g) := \int_0^{2\pi} h_{gB}(\varphi)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{d}\varphi \qquad \text{for } g \in GL(2)^+.$$

This map is real analytic.

**Proposition.** *The relation*

$$F_{B,m}(g) = \int_0^{2\pi} h_{gB}(\varphi)\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{d}\varphi = 0 \qquad \textit{for all } g \in GL(2)^+$$

*cannot hold for any odd integer* $m \geq 1$.

Suppose this were false; then there exists a smallest number $m$ for which there is a counterexample. Let $B$ be such a counterexample. We use

$$g(\lambda) \sim \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad R(\alpha) \sim \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Consider the first map. From $h_{g(\lambda)B}(\varphi) = \sqrt{\cos^2\varphi + \lambda^2\sin^2\varphi}\, h_B(\psi)$ and a substitution we get

$$F_{B,m}(g(\lambda)) = \lambda^2 \int_0^{2\pi} h_B(\psi) \frac{(\lambda\cos\psi + \mathrm{i}\sin\psi)^m}{(\lambda^2\cos^2\psi + \sin^2\psi)^{\frac{m+3}{2}}}\, d\psi.$$

Since this vanishes for all $\lambda \in (0, 1]$, the derivative with respect to $\lambda$ at 1 vanishes. This yields

$$\int_0^{2\pi} h_B(\psi)[(3-m)\, \mathrm{e}^{\mathrm{i}(m-2)\psi} + (3+m)\, \mathrm{e}^{\mathrm{i}(m+2)\psi}]\, d\psi = 0.$$

Now we use the second map. Since $F_{B,m}(R(\alpha)) = 0$ for $\alpha$ in a neighborhood of 0, the preceding holds with $\psi + \alpha$ instead of $\psi$ in the exponents. This yields

$$\int_0^{2\pi} h_B(\psi)\, \mathrm{e}^{\mathrm{i}(m-2)\psi}\, d\psi = 0 \quad \text{for } m \neq 3,$$

$$\int_0^{2\pi} h_B(\psi)\, \mathrm{e}^{\mathrm{i}(m+2)\psi}\, d\psi = 0.$$

This can be used to find a number smaller than $m$ for which a counterexample exists, which is a contradiction.

These are the main ideas; for the details we refer to [32]

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