# Recent Results on Random Polytopes

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**Summary.** - *This is a survey over recent asymptotic results on random polytopes in d-dimensional Euclidean space. Three ways of generating a random polytope are considered: convex hulls of finitely many random points, projections of a fixed high-dimensional polytope into a random d-dimensional subspace, intersections of random closed halfspaces. The type of problems for which asymptotic results are described is different in each case.*

## 1. - Introduction

Randomly generated convex polytopes, briefly random polytopes, have found increasing interest during the last decades. Their study combines convex geometry and geometric probability. Geometric and analytic methods often go hand in hand in their investigation. First impressions of the subject can be obtained from the articles [39, 51, 41], but an up-to-date and comprehensive survey is not available, unfortunately. Our goal in the following is very restricted: we want to describe recent asymptotic results for three different models of random polytopes. These are

- convex hulls of random points,
- random projections of high-dimensional polytopes,
- intersections of random halfspaces.

The character of the asymptotic results is distinctly different in each of the cases. In the first case, they concern volume approximation of convex bodies by random polytopes if the number of generating points tends to infinity. For this topic, we also recommend the recent introductory article by Bárány [4]. The second case deals with purely combinatorial properties of random polytopes, namely neighborliness. Here, regular simplices or crosspolytopes of increasing dimensions are projected into lower dimensional random subspaces. The third case is devoted to asymptotic shapes of random polytopes, under the condition that their size (interpreted in different ways) is large. Our motivation for presenting these three topics is different. In the first two parts, we feel that important breakthroughs in these subjects have been achieved by several authors in the last few years, and we want to give a description, though very brief, of their main results. In the third part, we present recent joint work with Daniel Hug and Matthias Reitzner on extensions and generalizations of David Kendall's problem on the shape of large cells in random tessellations.

All random polytopes to be considered are in $d$-dimensional Euclidean space $\mathbb{R}^d$ ($d \geq 2$). By 'polytope' we always mean a compact convex polytope.

Throughout this paper, we denote the underlying probability by $\mathbb{P}$ and mathematical expectation by $\mathbb{E}$.

## 2. - Convex Hulls of Random Points

Every polytope is the convex hull of its vertices. A natural way to generate a random convex polytope is, therefore, to take the convex hull of finitely many random points. A situation that has been extensively studied is that of a given convex body $K$ (a compact convex set with nonempty interior) and $n$ stochastically independent uniform random points in $K$. Here, a random point in $\mathbb{R}^d$ (a Borel measurable map from some probability space into $\mathbb{R}^d$) is *uniform in K* if its distribution is obtained from the Lebesgue measure, restricted to $K$ and normalized to a probability measure. For a given convex body $K$, we denote by $K_n$ the convex hull of $n$ independent uniform random points in $K$. Then $K_n$ is a random polytope contained in $K$, and $V(K_n)$, where $V$ denotes the volume, is a real random variable. Other functions of $K_n$ that have been studied are intrinsic volumes, numbers of $i$-dimensional faces, or Hausdorff distance from $K$; we restrict ourselves in this section to the case of the volume. Other distributions of random points that have found interest are distributions concentrated on the boundary of a convex body, or normal distributions. All this began with Sylvester's [46] well-known 'four point problem' of 1864 and received particular impetus from the seminal papers of Rényi and Sulanke [35, 36, 37], which directed the attention towards the asymptotic behavior, as $n \to \infty$. From the vigorous development afterwards, we mention only two typical examples.

For $n \in \mathbb{N}$, let $X_1, \dots, X_n$ be independent uniform random points in a given convex body $K$. Then

$$\lim_{n\to\infty} [V(K) - \mathbb{E}\, V(K_n)] n^{1/(d+1)} = c(d) \int_{\partial K} \kappa^{1/(d+1)}\, dS. \tag{1}$$

Here $\kappa$ denotes the Gauss–Kronecker curvature, suitably generalized to arbitrary convex bodies; it exists $S$-almost everywhere on $\partial K$, where $S$ denotes the area measure (the $(d-1)$-dimensional Hausdorff measure on $\partial K$). The constant $c(d)$ depends only on the dimension. The limit relation (1) is due to Bárány [3] for convex bodies with a boundary of class $C^3$, and to Schütt [43] for general convex bodies.

A similar relation has been proved for random points on the boundary of a convex body $K$. Let $X_1, \dots, X_n$ be independent, identically distributed random points on $\partial K$, and suppose their distribution has a continuous density $h$ with respect to the area measure. Let $K_n^h$ denote the convex hull of $X_1, \dots, X_n$. Under a mild condition on the generalized curvatures of $\partial K$, Schütt and Werner [44] have shown that

$$\lim_{n\to\infty} [V(K) - \mathbb{E}\, V(K_n^h)] n^{1/(d-1)} = b(d) \left( \int_{\partial K} h^{-1/(d-1)} \kappa^{1/(d-1)}\, dS \right), \tag{2}$$

where the constant $b(d)$ depends only on the dimension. Although 'only' expectations, and not distributions, of geometric random variables are the subject here, the difficulties are formidable (note the length of the paper [44]); they are mainly on the geometric side.

Results on variances and asymptotic distributions were for a long time restricted to the case of the plane or to special convex bodies, like balls. A result of Cabo and Groeneboom [11] holds for a polygon $K \subset \mathbb{R}^2$ with $r$ vertices and concerns the normalized difference $D_n := n[V(K) - V(K_n)]/V(K)$. It states that

$$\frac{D_n - \frac{2}{3} r \log n}{\sqrt{\frac{28}{27} r \log n}} \xrightarrow{\mathcal{D}} N(0,1)$$

for $n \to \infty$, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $N(0,1)$ is the standard normal distribution. Actually, [11] exhibited a different constant in the denominator and a different scaling factor, but Buchta [10] pointed out (since 2000, published in [10]) that this was in conflict with another limit theorem of Groeneboom together with results that Buchta had obtained on the variance of $D_n$. For some time, nobody seemed able to put his/her finger on an erroneous step. The situation was clarified in Groeneboom [18, Section 2].

In higher dimensions, and for convex bodies more general than balls, the recent years have seen a breakthrough. It began with a paper by Reitzner [33], in which he obtained for smooth convex bodies $K$ (that is, with a boundary of class $C^2$ and positive Gauss–Kronecker curvature), an upper estimate for the variance, namely

$$\mathrm{Var}[V(K) - V(K_n)] \le c_1(K) n^{-(d+3)/(d+1)}. \tag{3}$$

A probabilistic tool in the proof is the Efron–Stein jackknife inequality for the variance of symmetric statistics. As a consequence, Reitzner obtained a strong law of large numbers for the random variable $V(K) - V(K_n)$.

Essential subsequent progress came with another paper by Reitzner [34]. For smooth convex bodies $K$ he obtained a lower estimate for the variance of $V(K)-V(K_n)$, of the same order as in (3). Further, instead of $K_n$, he considered the convex hull of the points inside $K$ of a stationary Poisson point process $\Pi$ of intensity $n$ in $\mathbb{R}^d$. Let $K_n^\Pi$ denote this convex hull. Reitzner proved that

$$\left| \mathbb{P}\left( \frac{V(K_n^\Pi) - \mathbb{E}V(K_n^\Pi)}{\sqrt{\mathrm{Var} V(K_n^\Pi)}} \le x \right) - \Phi(x) \right| \le c_2(K) n^{-(d+3)/(d+1)} (\log n)^{2d+4}.$$

Here $\Phi$ is the distribution function of the standard normal distribution. The crucial stochastic tool for the proof is a central limit theorem for weakly dependent random variables with information on their dependency graphs, due to Rinott [38]. Remarkably, already Avram and Bertsimas [2] had suggested the application of such a result for obtaining central limit theorems for $K_n$, but it took almost ten years before this idea was carried out. Reitzner's result for $V(K_n^\Pi)$ can be transferred to a central limit theorem for $V(K_n)$, in the form

$$\left| \mathbb{P}\left( \frac{V(K_n) - \mathbb{E}V(K_n)}{\sqrt{\mathrm{Var} V(K_n)}} \le x \right) - \Phi(x) \right| \le \epsilon(n)$$

for all $x$, with $\epsilon(n) \to 0$ for $n \to \infty$. In full generality, this was proved by Vu [50]. He used Reitzner's result and the strong tail estimates for geometric random variables like $V(K) - V(K_n)$ that he had obtained in [49], by a new powerful probabilistic and combinatorial method.

In these new developments, in particular in Vu's concentration results, there is now a balance between equally deep stochastic and geometric techniques. Essential for the success on the geometric side is the development beginning with the pathbreaking paper of Bárány and Larman [5], establishing a connection between the random polytope $K_n$ and certain floating bodies of $K$. The technique is well described in Bárány's article [4]. Here we only mention how the floating body enters the formulation of Vu's deviation estimate. For a convex body $K$, a closed halfspace H, and a (small) number $\epsilon > 0$, the intersection $H \cap K$ is called an *$\epsilon$-cap* if $V(H \cap K) = \epsilon$. The union of all $\epsilon$-caps is the *$\epsilon$-wet part* of $K$, and and its complement in $K$ is the *$\epsilon$-floating body* of $K$. Let $\rho_\epsilon$ denote the volume of the $\epsilon$-wet part of $K$, and for a point $x$ in the boundary of the $\epsilon$-floating body, let $S_{x,\epsilon}$ be the union of all $\epsilon$-caps containing $x$. Let $g(\epsilon)$ be the supremum of $V(S_{x,\epsilon})$ over all $x$ in the boundary of the $\epsilon$-floating body. Put $A := 3g(\epsilon)$ and $B := 36ng(\epsilon)^2\rho_\epsilon$. Vu [49, Th. 2.1] proved that there are positive constants $\alpha$, $c$ and $\epsilon_0$ such that the following holds. For any $\alpha \log n/n < \epsilon \leq \epsilon_0$ and $0 < \lambda \leq B/4A^2 = n\rho_\epsilon$, we have

$$\mathbb{P}(|V(K) - \mathbb{E}V(K_n)| \geq \sqrt{B\lambda}) \leq 2\exp(-\lambda/4) + \exp(-c\epsilon n).$$

Employing similar methods as in [33], [34], [50] (and overcoming new difficulties), Bárány and Reitzner [6] proved a central limit theorem for $V(K) - V(K_n)$ in the case where $K$ is a polytope. Bárány and Vu [7] obtained a CLT for the volume of the convex hull of $n$ independent Gaussian random points.

Corresponding results exist for other functionals instead of the volume, most often for the number of vertices.

## 3. - Random Projections of High-dimensional Polytopes

Every convex polytope can be represented as a projection of a simplex of sufficiently high dimension. This remark suggests a natural way of generating a class of random $d$-polytopes: take some polytope, say a regular simplex, in $\mathbb{R}^n$ (where $n > d$) and project it orthogonally to a random $d$-dimensional subspace, say with uniform distribution. This so-called 'Grassmann approach' was proposed independently by Vershik and by Goodman and Pollack; it was studied by Vershik and Sporyshev [47, 48], Affentranger and Schneider [1], and others. We will briefly describe some basic facts and then explain why this type of random polytopes has recently found fresh interest.

Let $P$ be a fixed polytope in $n$-dimensional Euclidean space $\mathbb{R}^n$, where $n > d$. Let $L$ be a random $d$-dimensional subspace of $\mathbb{R}^n$, whose distribution is given by the normalized rotation invariant Haar measure on the Grassmannian $G(n, d)$. The orthogonal projection from $\mathbb{R}^n$ to a subspace $S$ is denoted by $\Pi_S$. Thus, $\Pi_L P$ is a $d$-dimensional random polytope. Its combinatorial properties are of interest, in particular the expectation of $f_k(\Pi_L P)$, the number of $k$-dimensional faces of $\Pi_L P$. Two general formulas derived in [1], by means of spherical integral geometry, say that

$$\mathbb{E} f_k(\Pi_L P) = 2 \sum_{s\geq 0} \sum_{F\in\mathcal{F}_k(P)} \sum_{G\in\mathcal{F}_{d-1-2s}(P)} \beta(F,G)\gamma(G,P). \tag{4}$$

and

$$\mathbb{E} f_k(\Pi_L P) = f_k(P) - 2 \sum_{s\geq 0} \sum_{F\in\mathcal{F}_k(P)} \sum_{G\in\mathcal{F}_{d+1+2s}(P)} \beta(F,G)\gamma(G,P). \tag{5}$$

Here, $\mathcal{F}_j(P)$ denotes the set of $j$-dimensional faces of $P$, $\beta(F,G)$ is the internal angle of the polytope $G$ at its face $F$ (equal to 0 by definition if $F$ is not a face of $G$), and $\gamma(G,P)$ is the external angle of the polytope $P$ at its face $G$.

Formulas (4) and (5) express the expectation $\mathbb{E} f_k(\Pi_L P)$ in terms of internal and external angles of the polytope $P$. These angles can in general not be computed explicitly, since they are defined via volumes of polytopes in spherical space. Further progress is possible if $P$ is one of the regular polytopes in $\mathbb{R}^n$. In particular, let $T^n$ be a regular simplex in $\mathbb{R}^n$. An asymptotic formula, as $n$ tends to infinity, was proved in [1], namely

$$\mathbb{E} f_k(\Pi_L T^n) \sim \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta(T^k, T^{d-1}) (\pi \log n)^{(d-1)/2}. \tag{6}$$

Here $\beta(T^k, T^{d-1})$ denotes the internal angle of a regular $(d-1)$-simplex at one of its $k$-faces. An asymptotic formula for $\beta(T^k, T^{d-1})$, for fixed $k$ and $d \to \infty$, was provided by Böröczky and Henk [9]. A counterpart to (6), for projections to random subspaces of fixed codimension, was also proved in [1], namely

$$\mathbb{E} f_k(\Pi_{L^\perp} T^n) \sim f_k(T^n), \tag{7}$$

as $n$ tends to infinity.

In the asymptotic relations (6) and (7), either the dimension or the codimension of the subspaces onto which is projected remains fixed, while the dimension $n$ of the regular simplex tends to infinity. More interesting is the case of a linearly coordinated growth of the parameters $n, d, k$, which was investigated by Vershik and Sporyshev [48].

Random projections of regular simplices can also be looked at from a different point of view. Let $T^n$ be a regular simplex in $\mathbb{R}^n$, and let $S$ be a fixed $d$-dimensional subspace of $\mathbb{R}^n$. Let $\vartheta$ be a random rotation of $\mathbb{R}^n$, with distribution given by the normalized Haar measure on the rotation group $SO(n)$. Then $\Pi_S(\vartheta \operatorname{vert} T^n)$, where vert denotes the set of vertices, is a set of $n+1$ random points in $S$. If $f$ is any measurable function on sets of $n+1$ points in the $d$-dimensional space $S$ that is invariant under affine transformations (for example, the number of $k$-faces of the convex hull), then $f(\Pi_S(\vartheta \operatorname{vert} T^n))$ has the same distribution as $f(X_{n+1})$, where $X_{n+1}$ is the set of $n+1$ independent, identically distributed random points in $S$ with standard normal distribution. This was proved by Baryshnikov and Vitale [8], who explained and considerably extended an observation made in [1]. This allowed to carry over some results on random projections of regular simplices to convex hulls of Gaussian samples. Only much later were these results on convex hulls of Gaussian samples attacked directly, and stronger results were obtained, by Hug, Munsonius and Reitzner [19] and by Hug and Reitzner [20].

Let $C^n$ denote a regular crosspolytope in $\mathbb{R}^n$. Böröczky and Henk [9] have proved a counterpart to (6), namely

$$\mathbb{E}f_k(\Pi_L C^n) \sim \frac{2^d}{\sqrt{d}}\binom{d}{k+1}\beta(T^k, T^{d-1})(\pi \log n)^{(d-1)/2}, \tag{8}$$

as $n$ tends to infinity.

Now we explain why random projections of regular crosspolytopes have recently become of interest, mainly in the work of David Donoho. Before that, we recall the definition of centrally neighborly polytopes, which will play a role in this context.

**Definition.** A centrally symmetric polytope $P$ is called *centrally $k$-neighborly* if every subset of $k$ vertices of $P$, not containing a pair of opposite vertices, is the set of vertices of a $(k-1)$-face of $P$ (necessarily a $(k-1)$-dimensional simplex).

Consider the following reconstruction problem (coming from coding theory). Let $n, d, k \in \mathbb{N}$ be given numbers with $d, k < n$. Let $Y$ be an $(n-d)$-dimensional linear subspace of $\mathbb{R}^n$. Suppose we are given a vector $y' \in \mathbb{R}^n$, and we have to find a vector $y \in Y$ that differs from $y'$ in at most $k$ coordinates, or a good approximation of such a vector. A practicable way of approximating $y$, starting from the known $y'$, consists in finding a point $y$ in $Y$ nearest to $y'$. However, it would not be a good idea to interpret 'nearest' in the sense of the Euclidean norm, as simple examples show. It is much better to take the $L_1$-norm, since then one will often, that is, for many subspaces $Y$, get a correct solution. To investigate this phenomenon more closely, we formulate the following property.

**Definition.** The subspace $Y$ has property $U_k$ if the following holds. Whenever $y \in Y$ and $y' \in \mathbb{R}^n$ are such that they differ in at most $k$ coordinates, then the optimization problem

$$\text{minimize } \|x - y'\|_1 \text{ subject to the condition } x \in Y$$

has a unique solution and this is equal to $y$.

The unit ball of the $L_1$-norm in $\mathbb{R}^n$ is the $n$-dimensional regular crosspolytope $C^n$. Let $F$ be a $(k-1)$-face of $C^n$, and choose $z \in F$. Then $z$ lies in the intersection of $C^n$ with some $k$-dimensional coordinate subspace. Therefore, any point $y \in Y$ and the point $y' := y + z$ differ in at most $k$ coordinates. Suppose that the linear subspace $Y$ has property $U_k$. Then $Y$ touches the crosspolytope $C^n + y'$ at the unique point $y$. Equivalently, $Y + z$ touches $C^n$ only at $z$. Thus, to any point $z$ in a $(k-1)$-face of $C^n$, there exists a translate of the subspace $Y$ that touches $C^n$ only at $z$. It follows that the projection $\Pi_{Y^\perp} F$ is a $(k-1)$-face of the centrally symmetric polytope $\Pi_{Y^\perp} C^n$. Since $F$ was an arbitrary $(k-1)$-face of $C^n$, we deduce that $f_{k-1}(\Pi_{Y^\perp} C^n) = f_{k-1}(C^n) = 2^k\binom{n}{k}$. The centrally symmetric polytope $\Pi_{Y^\perp} C^n$ has $2v \leq 2n$ vertices and, therefore, at most $2^k\binom{v}{k}$ faces of dimension $k-1$. Hence, $\Pi_{Y^\perp} C^n$ has $2n$ vertices, and any $k$ non-opposite of these vertices determine a $(k-1)$-face of $\Pi_{Y^\perp} C^n$. The arguments can be reversed (replacing $C^n$ by $\|y' - y\|_1 C^n$), therefore we have the following result.

**Proposition 1.** *The subspace $Y$ has property $U_k$ if and only if the polytope $\Pi_{Y^\perp}C^n$ satisfies $f_{k-1}(\Pi_{Y^\perp}C^n) = f_{k-1}(C^n)$, equivalently, it has $2n$ vertices and is centrally $k$-neighborly.*

**Example.** $n = 3$, $d = 2$, $k = 1$: the one-dimensional subspace spanned by $(1,1,1)$ has property $U_1$.

A similar result can also be formulated in terms of sparse solutions of underdetermined systems of linear equations. The following formulation is taken from the work of Donoho [12].

**Proposition 2.** *Let $A$ be a $d \times n$ matrix, let $d, k < n$. The following conditions* (a) *and* (b) *are equivalent.*
(a) *Whenever, for given $y$, the system $Ax = y$ has a solution $x_0$ with at most $k$ nonzeros, then $x_0$ is the unique solution of the optimization problem*

$$\textit{minimize } \|x\|_1 \textit{ subject to the condition } Ax = y.$$

(b) *The polytope $AC^n$ has $2n$ vertices and is centrally $k$-neighborly.*

This equivalence will be useful if one knows that condition (b) is satisfied for 'many' matrices $A$. In the case of Proposition 1, 'many' subspaces $Y$ should have property $U_k$. In particular, one hopes that for a random subspace with uniform distribution, the required condition is satisfied with high probability.

Therefore, one considers a $d$-dimensional uniform random subspace $L$ of $\mathbb{R}^n$. One is interested in those realizations $L$ for which the polytope $\Pi_L C^n$ has the same number of $k$-faces as $C^n$, and hence is centrally $(k+1)$-neighborly (for convenience, we switched here from $k$ to $k+1$). Donoho [13] succeeded in showing that this holds with high probability, asymptotically for $n \to \infty$, where $d$ and $k$ are allowed to grow linearly with $n$, in a prescribed way. Before going into more detail, we note the following connection with the initially considered question. Since

$$\begin{aligned}\mathbb{E}[f_k(C^n) - f_k(\Pi_L C^n)] &= \int \mathbf{1}\{f_k(C^n) > f_k(\Pi_L C^n)\}[f_k(C^n) - f_k(\Pi_L C^n)]\, \mathrm{d}\mathbb{P} \\ &\geq \mathbb{P}[f_k(C^n) > f_k(\Pi_L C^n)],\end{aligned}$$

we have

$$\mathbb{P}[f_k(\Pi_L C^n) < f_k(C^n)] \leq f_k(C^n) - \mathbb{E} f_k(\Pi_L C^n). \tag{9}$$

This shows that for obtaining an upper estimate for the probability that the random subspace $L$ does not have property $U_k$, we need information on the expected number $\mathbb{E} f_k(\Pi_L C^n)$ of $k$-faces of the random polytope $\Pi_L C^n$. From (9) and (5), we get

$$\begin{aligned}\mathbb{P}[f_k(\Pi_L C^n) < f_k(C^n)] &\leq 2 \sum_{s \geq 0} \sum_{F \in \mathcal{F}_k(C^n)} \sum_{G \in \mathcal{F}_{d+1+2s}(C^n)} \beta(F,G)\gamma(G,C^n) & (10) \\ &=: \Delta(k,d,n). & (11)\end{aligned}$$

It is clear that the right side of (10) depends, in fact, only on the numbers $k, d, n$. The aim are good upper estimates for $\Delta(k, d, n)$, asymptotically for large $n$, but in the interesting cases where $d$ and $k$ are proportional to $n$.

Here is one of Donoho's [13] main results. He establishes the existence of a function $\mathsf{P} : (0, 1) \to (0, 1]$ with the following properties. This function is defined implicitly, but can be computed numerically with sufficient accuracy to obtain strong consequences. The crucial property of $\mathsf{P}$ is this. Let $\delta \in (0, 1)$ be given and put $d := \lfloor \delta n \rfloor$. Let $\rho < \mathsf{P}(\delta)$ and put $k := \lfloor \rho d \rfloor$. Then, for sufficiently small $\epsilon > 0$,

$$\mathbb{P}[f_k(\Pi_L C^n) < f_k(C^n)] \leq \Delta(k, d, n) \leq n\mathrm{e}^{-n\epsilon}$$

for $n > n_0(\delta, \rho, \epsilon)$.

As a little 'test' example, Donoho's result (supplemented by numerical calculations) shows that there exist centrally $k$-neighborly $d$-polytopes with $4d$ vertices for which $k \geq 0.089\, d$. Before that, the best result of this kind was due to Linial and Novik [30], who had achieved $k \geq 0.0025\, d$.

Donoho in his proof writes

$$\Delta(k, d, n) = \sum_{s \geq 0} D_s$$

with

$$D_s := 2 \sum_{F \in \mathcal{F}_k(C^n)} \sum_{G \in \mathcal{F}_{d+1+2s}(C^n)} \beta(F, G) \gamma(G, C^n)$$

and shows the following. If $(k, d) = (k_n, d_n)$ is a sequence with

$$\frac{k_n}{d_n} \to \rho, \qquad \frac{d_n}{n} \to \delta, \qquad \text{where } \rho < \mathsf{P}(\delta),$$

then, for sufficiently small $\epsilon > 0$ and for $n > n_0(\delta, \rho, \epsilon)$,

$$n^{-1} \log D_s \leq -\epsilon, \qquad s = 0, 1, 2, \ldots$$

For this, he first notes that by properties of the crosspolytope $C^n$ one has, setting $d + 1 + 2s =: \ell$,

$$D_s = 2 \cdot 2^\ell \binom{n}{k+1} \binom{n-k-1}{\ell-k} \beta(T^k, T^\ell) \gamma(T^\ell, C^n),$$

where $T^\ell$ is any proper face of the regular crosspolytope $C^n$ (which is a regular simplex) and $T^k$ is any $k$-face of the regular $\ell$-simplex $T^\ell$. He then proceeds with carefully investigating the asymptotic behavior, as $n \to \infty$, separately for the combinatorial factor, the internal angle, and the external angle. One essential tool for this is a new probabilistic interpretation of the internal angle, which then allows the application of large deviation techniques.

More recently, Donoho and Tanner [14] have applied this approach to random projections of simplices, with applications to sparse nonnegative solutions of underdetermined linear equations. In [15] they studied neighborliness properties of random projections of regular simplices, and in [16] they give a broad picture of various applications of (weak) neighborliness of random projections of high-dimensional polytopes.

## 4. - Intersections of Random Halfspaces

Every convex polytope can be represented as the intersection of finitely many closed halfspaces. Choosing these halfspaces at random, we get a random polytope. A particularly manageable model is obtained if sufficiently strong independence properties are built in. For that reason, we use a Poisson process of hyperplanes to generate the random halfspaces, in the following way.

For a set $X$ of hyperplanes in $\mathbb{R}^d$ not containing 0, let

$$Z_0 := \bigcap_{H \in X} H^-,$$

where $H^-$ is the closed halfspace bounded by $H$ that contains 0. If $X$ is such that it generates a tessellation of $\mathbb{R}^d$ into polytopes, then $Z_0$ is known as the *zero cell* of this tessellation.

We apply this to the realizations of a Poisson process in the space of hyperplanes. Let $\mathcal{H}^d$ denote the space of hyperplanes of $\mathbb{R}^d$, with its usual topology; then $\mathcal{H}^d$ is locally compact and has a countable base. Let $X$ be a Poisson process in $\mathcal{H}^d$, with a locally finite intensity measure $\Theta$. Thus, $\Theta$ is a measure on the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{H}^d)$, and it is finite on compact sets. The Poisson process $X$ in $\mathcal{H}^d$ is a random variable on some probability space $(\Omega, \mathsf{A}, \mathbb{P})$ with values in the system $\mathsf{N}(\mathcal{H}^d)$ of locally finite subsets of $\mathcal{H}^d$, with a suitable $\sigma$-algebra, such that, for Borel sets $A \in \mathcal{B}(\mathcal{H}^d)$ and for $k \in \mathbb{N}_0$,

$$\mathbb{P}(\#(X \cap A) = k) = \mathrm{e}^{-\Theta(A)} \frac{\Theta(A)^k}{k!}, \tag{12}$$

thus

$$\Theta(A) = \mathbb{E}\,\#(X \cap A). \tag{13}$$

The $\sigma$-algebra on $\mathsf{N}(\mathcal{H}^d)$ is the smallest one for which the counting variables $\#(X \cap A)$, $A \in \mathcal{B}(\mathcal{H}^d)$, are measurable. We refer to [42] for more information about hyperplane processes and the random tessellations that they induce.

We can neglect hyperplanes passing through 0. Every other hyperplane has a unique representation of the form

$$H_{u,t} := \{x \in \mathbb{R}^d : \langle u, x\rangle = t\} \qquad \text{with } u \in S^{d-1},\ t > 0.$$

We assume that the intensity measure $\Theta$ is of the form

$$\Theta(A) = 2\gamma \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(H_{u,t})\, t^{r-1}\, \mathrm{d}t\, \varphi(\mathrm{d}u), \qquad A \in \mathcal{B}(\mathcal{H}^d), \tag{14}$$

with a number $\gamma > 0$, called the *the intensity*, a number $r \geq 1$, called the *distance exponent*, and a Borel probability measure $\varphi$ on the unit sphere $S^{d-1}$, which is not concentrated on a closed hemisphere; it is called the *directional distribution.* If $r = 1$, then the Poisson hyperplane process $X$ is *stationary* (its distribution is invariant under translations), and if $\varphi$ is rotation invariant, then $X$ is *isotropic* (its distribution is invariant under rotations). The form (14) includes two important particular cases, as described below, but it is far more general, requiring neither stationarity nor isotropy. Only some homogeneity property with respect to dilatations is built into the definition, via the term $t^{r-1}$ in the integrand. The directional distribution $\varphi$ governs, roughly speaking, the frequency of the occurrence of hyperplanes $H_{u,t}$ in $X$ in dependence on their normal vector $u$.

The intersection $Z_0 = \bigcap_{H \in X} H^-$ is almost surely a polytope, and this random polytope is the object of our investigation. In particular, we want to study the approximate shape of $Z_0$ under the condition that is large, in some sense.

First we explain the background of this topic. Its origin is a conjecture of David Kendall from the 1940s, which was popularized in his forward to the first edition (1987) of the book by Stoyan, Kendall and Mecke [45]. Kendall was led to consider a stationary, isotropic Poisson line process in the plane and its zero cell $Z_0$. He wrote (in equivalent words): "One would have preferred to be able to say something about ... my conjecture that the conditional law for the shape of $Z_0$, given the area $A(Z_0)$ of $Z_0$, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the circular shape. Unfortunately nothing substantial is known ... even today." After the appearance of the book [45], Kendall's problem became widely known. Contributions by Miles [32] and Goldman [17] supported the conjecture. Finally, a proof was given by Kovalenko [27], and a simpler one in [29]. Kovalenko [28] also obtained a similar result for the typical cell of a stationary Poisson–Voronoï tessellation in the plane. In higher dimensions, Mecke and Osburg [31] treated the special case of large Crofton parallelotopes, generated by a stationary Poisson hyperplane process with a directional distribution concentrated on pairwise orthogonal directions and their opposites.

The immediate generalization of Kendall's original question to higher dimensions would ask for the asymptotic shape of the zero cell of a stationary and isotropic Poisson hyperplane process, under the condition that its volume is large. The properties 'stationary', 'isotropic' are equivalent (for Poisson processes) to the invariance of the intensity measure $\Theta$ under translations, respectively rotations. The intensity measure of a stationary, isotropic Poisson hyperplane process $X$ in $\mathbb{R}^d$ is given by (14) with $r = 1$ and $\varphi = \sigma$, the normalized spherical Lebesgue measure on $S^{d-1}$; in this case, $\gamma$ is the usual intensity of $X$. A natural extension of Kendall's problem consists in dropping the isotropy assumption. For a stationary Poisson hyperplane process $X$ of intensity $\gamma$, the intensity measure has the form (14), with $r = 1$ and $\varphi$ as described above (and, without loss of generality, invariant under reflection in the origin).

Also Kovalenko's variant of Kendall's problem for the typical cell of a stationary Poisson–Voronoï tessellation is of interest in higher dimensions. Let $S$ be a locally finite set in $\mathbb{R}^d$. For $x \in S$, the set

$$C(x,S) := \{y \in \mathbb{R}^d : \|y - x\| \le \|y - s\| \text{ for all } s \in S\}$$

consists of all points of $\mathbb{R}^d$ for which $x$ is the nearest point in $S$. It is the *Voronoï cell* (or *Dirichlet cell*) of $x$ with respect to $S$. Let $Y$ be a stationary Poisson point process in $\mathbb{R}^d$. Then

$$X := \{C(x,Y) : x \in Y\}$$

is a stationary random tessellation, called the *Poisson–Voronoï tessellation* induced by $Y$ (see [42], also for the following) There exists a natural notion of 'average cell', called the *typical cell* of the Poisson–Voronoï tessellation $X$. This is the random polytope with distribution $\mathbb{Q}$ given by

$$\mathbb{Q}(A) = \frac{\mathbb{E}\,\#\{x \in\ Y \cap B^d : C(x,Y) - x \in A\}}{\mathbb{E}\,\#(Y \cap B^d)}$$

for Borel sets $A \subset \mathcal{K}^d$ (the space of compact convex subsets of $\mathbb{R}^d$, equipped with the Hausdorff metric). Slivnyak's theorem for Poisson processes implies that the typical cell of the Poisson–Voronoï mosaic $X$ is stochastically equivalent to the random polytope

$$C(0, Y \cup \{0\}) = \bigcap_{x \in Y} H^-_{x/\|x\|,\|x\|/2}.$$

Thus, the typical cell of the Poisson–Voronoï mosaic $X$ is the zero cell $Z_0$ induced by the hyperplane process $\{H^-_{x/\|x\|,\|x\|/2} : x \in Y\}$, which is formed by the mid-hyperplanes of the points of $Y$ and the origin 0. This is a (non-stationary) Poisson hyperplane process with intensity measure given by

$$\Theta(A) = 2\gamma \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(H_{u,t})\, t^{d-1}\, \mathrm{d}t\, \sigma(\mathrm{d}u),$$

where $\gamma$ is $2^{d-1}$ times the intensity of the stationary point process $Y$.

Thus, we see that the zero cell $Z_0$ induced by our general Poisson hyperplane process $X$ with intensity measure given by (14) includes two classical random polytopes:

• the zero cell of a stationary Poisson hyperplane tessellation, not necessarily isotropic ($r = 1$, $\varphi$ even),

• the typical cell of a stationary Poisson–Voronoï tessellation ($r = d$, $\varphi$ = normalized spherical Lebesgue measure).

Kendall's original problem concerns the first of these cases, for $d = 2$ and rotation invariant $\varphi$.

We can now ask a very general version of Kendall's problem: What is the asymptotic shape of $Z_0$, under the condition that this zero cell is large (in some sense)? At first sight, none of the following questions seems to have an obvious answer: Does an asymptotic shape exist? What is a candidate for this shape? How does it depend on the interpretation of 'large'? The quintessence of the following considerations is the insight that the answers depend on the study of a certain inequality of isoperimetric type for two functionals of convex bodies, in particular its extremal bodies and corresponding stability estimates. In this way, the general version of Kendall's problem on certain random polytopes is closely connected with convex geometry. We describe this connection in the general version that was presented in Hug and Schneider [25].

The first of the two crucial functionals is the one which is used to measure the size of the random polytope $Z_0$. For this, we can use any real function $\Sigma$ on $\mathcal{K}^d$ that satisfies the following axioms: it is

• increasing under set inclusion,

• homogeneous of some degree $k \geq 0$,

• continuous,

• $\neq 0$.

We call any such function $\Sigma$ a *size functional*. Familiar examples are: volume, surface area, mean width, diameter, thickness, inradius, circumradius, volume of the John ellipsoid, width in a given direction, and many others.

The second functional is determined by the intensity measure of the given Poisson hyperplane process $X$. For $K \in \mathcal{K}^d$, we define

$$\mathcal{H}_K := \{H \in \mathcal{H}^d : H \cap K \neq \emptyset\}$$

and

$$\Phi(K) := \frac{1}{2\gamma}\,\mathbb{E}\,\#\,(X \cap \mathcal{H}_K). \tag{15}$$

We call $\Phi$ the *hitting functional*, since $\Phi(K)$ is, up to a normalizing factor, the expected number of hyperplanes of the process that hit the convex body $K$. Due to the assumed form (14) of the intensity measure $\Theta$ of $X$, the hitting functional is given by

$$\Phi(K) = \frac{1}{r}\int_{S^{d-1}} h(K,u)^r\,\varphi(\mathrm{d}u).$$

where

$$h(K,u) := \max\{\langle x,u\rangle : x \in K\}, \qquad u \in \mathbb{R}^d,$$

is the support function.

By continuity and homogeneity, the hitting functional $\Phi$ and the size functional $\Sigma$ satisfy a sharp isoperimetric inequality,

$$\Phi(K) \geq \tau\Sigma(K)^{r/k}. \tag{16}$$

'Sharp' means that there are *extremal bodies*, that is, bodies $K$ with more than one point for which equality holds in (16) (this determines the factor $\tau$). They play a crucial role: *the extremal bodies determine the asymptotic shapes of zero cells with large* $\Sigma$*-size.*

To make this precise, the notion of 'shape' must be specified. Let $\mathsf{G}$ be one of the groups of: similarities, homotheties, positive dilatations of $\mathbb{R}^d$. We define the $\mathsf{G}$-*shape* of $K \in \mathcal{K}^d$ as the class $s_{\mathsf{G}}(K) := \{gK : g \in \mathsf{G}\}$. By $\mathcal{S}_{\mathsf{G}}$ we denote the space of all $\mathsf{G}$-shapes, with the quotient topology. Now we can give a precise meaning to conditional laws of shapes.

**Definition.** The conditional law of the $\mathsf{G}$-shape of $Z_0$, given the lower bound $a > 0$ for the size $\Sigma(Z_0)$, is the probability measure $\mu_a$ on $\mathcal{S}_{\mathsf{G}}$ defined by

$$\mu_a(A) := \mathbb{P}(s_{\mathsf{G}}(Z_0) \in A \mid \Sigma(Z_0) \geq a)$$

for Borel sets $A \subset \mathcal{S}_{\mathsf{G}}$.

The following theorem, on weak convergence of conditional laws of shapes, provides a first answer to a considerably generalized version of Kendall's problem.

**Theorem 1.** *Suppose that the extremal bodies of* (16) *belong to a unique* $\mathsf{G}$*-shape* $s_{\mathsf{G}}(B)$*. Then* $s_{\mathsf{G}}(B)$ *is the limit shape of* $Z_0$ *for increasing* $\Sigma$*, in the sense that*

$$\lim_{a\to\infty} \mu_a = \delta_{s_{\mathsf{G}}(B)} \quad \textit{weakly},$$

*where* $\delta_{s_{\mathsf{G}}(B)}$ *denotes the Dirac measure concentrated at* $s_{\mathsf{G}}(B)$*.*

This follows from a stronger result, estimating the probability of *large deviations* from an asymptotic shape. Its proof requires a *stability improvement* of the isoperimetric inequality (16). We need a suitable function to measure deviations from extremal bodies. Again, this can be introduced axiomatically. For given $\Phi$ and $\Sigma$, let $\vartheta$ be a function on $\{K \in \mathcal{K}_o^d : \Sigma(K) > 0\}$ (where $\mathcal{K}_o^d := \{K \in \mathcal{K}^d : 0 \in K\}$) with the following properties:

- $\vartheta$ is continuous,
- nonnegative,
- homogeneous of degree zero,
- $\vartheta(K) = 0$ for $K \in \mathcal{K}_o^d \iff K$ is extremal.

We call $\vartheta$ a *deviation functional.* It is easy to see that deviation functionals always exist. In concrete cases, deviation functionals of intuitive geometric meaning will be preferable.

It is also not difficult to see that the isoperimetric type inequality (16) admits a stability improvement: there exists a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(\epsilon) > 0$ for $\epsilon > 0$ and $f(0) = 0$ such that

$$\vartheta(K) \geq \epsilon \quad \Rightarrow \quad \Phi(K) \geq (1 + f(\epsilon))\tau\Sigma(K)^{r/k} \tag{17}$$

for $K \in \mathcal{K}_o^d$. Any such function $f$ is called a *stability function* for $\Phi, \Sigma, \vartheta$. In concrete cases, explicit stability functions of optimal order are of interest.

**Theorem 2.** *Let* $X, \Sigma, \vartheta, f$ *be given. For* $\epsilon > 0$ *and* $a > 0$*,*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \leq c \exp\left(-c_0 f(\epsilon) a^{r/k}\gamma\right) \tag{18}$$

*with positive constants* $c$ *(depending on* $X, \Sigma, f, \epsilon$*) and* $c_0$ *(depending only on* $\tau$*).*

The reader will have noticed that in our considerations the condition $\Sigma(Z_0) \geq a$ is used, whereas Kendall's original question involved the condition $\Sigma(Z_0) = a$. Our results are strong enough to include also such conditional distributions. The random variable $Z_0$ takes its values in $\mathcal{K}^d$, which is a Polish space. Hence, the regular conditional probability distribution of $Z_0$ with respect to $\Sigma(Z_0)$ exists.

**Theorem 3.** *Under the same conditions, and with similar constants,*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) = a) \leq c \exp\left(-c_0 f(\epsilon) a^{r/k}\gamma\right).$$

From this, also a counterpart to Theorem 1 can be deduced, which is closer to Kendall's original conjecture.

**Special Cases**

We consider some concrete examples.

**(1)** The zero cell of a *stationary Poisson hyperplane tessellation*; the size measured by the *volume*. Here,

$$\Phi(K) = \int_{S^{d-1}} h(K,u)\,\varphi(\mathrm{d}u).$$

To find the extremal bodies, one has to apply Minkowski's existence theorem from the theory of convex bodies to the directional distribution $\varphi$. This measure can be assumed to be even (symmetric under reflection in the origin). By Minkowski's theorem (see [40, Th. 7.1.2]), there exists a convex body $B$ with center 0 for which $\varphi$ is the area measure. With this body, the hitting functional can be expressed as a mixed volume, $\Phi(K) = dV(K, B, \dots, B)$, and the crucial inequality (16) becomes *Minkowski's inequality*

$$V(K, B, \dots, B) \geq V_d(B)^{1-1/d} V_d(K)^{1/d}.$$

It is a classical result that here equality holds if and only if $K$ is homothetic to $B$ (see [40, Th. 6.2.1]). Hence, *the homothety class of* $B$ is the *limit shape* of $Z_0$ with respect to the volume. This case was treated in [21].

**(2)** The typical cell of a *stationary Poisson–Voronoï tessellation*; the size measured by the *kth intrinsic volume* $V_k$ (see [40] for this notion). Here,

$$\Phi(K) = \frac{1}{d}\int_{S^{d-1}} h(K,u)^d \, d\sigma(u).$$

Hölder's inequality and the Aleksandrov–Fenchel inequality give

$$\Phi(K) \geq \tau V_k(K)^{d/k}$$

with explicit $\tau$, where equality holds if and only if K is a centered ball (i.e., a ball with center at the origin). Hence, the *class of centered balls* is the *limit shape* of the typical cell with respect to $V_k$. This case was treated in [22], where also an explicit stability estimate was obtained.

**(3)** The zero cell of a *stationary non-isotropic Poisson hyperplane tessellation*; the size measured by the *inradius*. The *limit shape* of $Z_0$ with respect to the inradius turns out to be the homothety class of the convex body

$$B_\varphi := \bigcap_{u \in \operatorname{supp} \varphi} H^-_{u,1},$$

where supp denotes the support of a measure.

**(4)** The zero cell of a *stationary isotropic Poisson hyperplane tessellation*; the size measured by the *circumradius*. The *limit shape* is the class of *segments*.

**(5)** The zero cell of a *stationary isotropic Poisson hyperplane tessellation*; the size measured by the *diameter*. The *limit shape* is the class of *segments*.

**(6)** The zero cell of a *stationary isotropic Poisson hyperplane tessellation*; the size measured by the *thickness*. There is *no limit shape* in the sense of Theorem 1, since the set of extremal bodies of the crucial isoperimetric inequality is the set of all bodies of constant width. Theorem 2 estimates the probability of large deviations of the zero cell $Z_0$ from the class of bodies of constant width; hence, the class of bodies of constant width can be considered as the *asymptotic shape*.

**(7)** The zero cell of a *stationary isotropic Poisson hyperplane tessellation*; the size measured by the *width in a given direction*. The *limit shape* is the class of *segments of the given direction*.

In most cases, the estimation of the probability of large deviations from limit shapes can be done with intuitive deviation functionals and explicit stability functions.

### The Idea of the Proof - a Rough Sketch

We give a rough sketch of the idea of the proof of Theorem 2, to show the role of the isoperimetric type inequality

$$\Phi(K) \geq \tau \Sigma(K)^{r/k} \tag{19}$$

and the corresponding stability improvement

$$\vartheta(K) \geq \epsilon \quad \Rightarrow \quad \Phi(K) \geq (1 + f(\epsilon))\tau \Sigma(K)^{r/k}. \tag{20}$$

We have to estimate the conditional probability

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) = \frac{\mathbb{P}(\vartheta(Z_0) \geq \epsilon, \, \Sigma(Z_0) \geq a)}{\mathbb{P}(\Sigma(Z_0) \geq a)} \tag{21}$$

from above. Let $B$ be an extremal body of (19) with $0 \in B$. Since only the shape of $B$ plays a role, we may replace $B$ by a dilate and assume that $\Sigma(B) = a$; then, since $B$ satisfies (19) with equality,

$$\Phi(B) = \tau a^{r/k}.$$

The estimation of the denominator of (21) is easy. If no hyperplane of the process $X$ hits $B$, then $B \subset Z_0$ and thus $\Sigma(Z_0) \geq \Sigma(B) = a$, by the monotonicity of $\Sigma$. Hence, observing (12), (13), (15),

$$\mathbb{P}(\Sigma(Z_0) \geq a) \geq \mathbb{P}(\#(X \cap \mathcal{H}_B) = 0) = \exp\left(-\Phi(B) 2\gamma\right) = \exp\left(-\tau a^{r/k} 2\gamma\right).$$

For the estimation of the numerator, we first give a heuristic idea. It consists in comparing the zero cell $Z_0$ with a fixed convex body with similar properties, that is, not cut by hyperplanes of the process, with $\Sigma$-size at least $a$ and deviation from $B$ at least $\epsilon$. Let $K$ be a convex body satisfying

$$\vartheta(K) \geq \epsilon \quad \text{and} \quad \Sigma(K) \geq a.$$

Then, by the stability estimate (21),

$$\mathbb{P}(\#(X \cap \mathcal{H}_K) = 0) = \exp\left(-\Phi(K) 2\gamma\right) \leq \exp\left(-(1 + f(\epsilon))\tau a^{r/k} 2\gamma\right).$$

Heuristically, we replace $K$ satisfying

$$\#(X \cap \mathcal{H}_K) = 0, \qquad \vartheta(K) \geq \epsilon, \qquad \Sigma(K) \geq a$$

by the random zero cell $Z_0$ satisfying

$$\#(X \cap \mathcal{H}_{\alpha Z_0}) = 0 \ \forall\, \alpha \in (0,1), \quad \vartheta(Z_0) \geq \epsilon, \quad \Sigma(Z_0) \geq a,$$

and hope that this costs only an inessential weakening of the inequality, say

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\, \Sigma(Z_0) \geq a) \leq c_1 \exp\left(-(1 + c_2 f(\epsilon))\tau a^{r/k} 2\gamma\right)$$

with positive constants $c_1, c_2$. Division then gives

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \leq c_1 \exp\left(-c_2 f(\epsilon)\tau a^{r/k} 2\gamma\right),$$

which is of the required form.

This strategy works indeed, though with considerable effort. We give a brief description of the essential steps.

It is necessary to prove the following stronger version of Theorem 2.

**Theorem 2'.** *Let $X, \Sigma, \vartheta, f$ be given. For $\epsilon > 0$ and $0 < a < b \leq \infty$,*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \in [a,b)) \leq c \exp\left(-c_0 f(\epsilon) a^{r/k} \gamma\right) \tag{22}$$

*with positive constants $c$ (depending on $X, \Sigma, f, \epsilon$) and $c_0$ (depending only on $\tau$).*

For the proof, we have to estimate probabilities of the type

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \in a(1, 1+h))$$

for $h > 0$. This is first done for $h = 1$, then extended to small $h$, then to all $h > 0$.

The random polytope $Z_0$ can have an arbitrarily large diameter. To deal with this, we introduce the *relative diameter*

$$\delta(K) := \frac{\mathrm{diam}(K)}{c\Sigma(K)^{1/k}},$$

with $c$ so that $\delta(K) \geq 1$. We consider separately the cases $\delta(Z_0) \in [m, m+1)$ ($m \in \mathbb{N}$). The restriction $\delta(Z_0) \in [m, m+1)$ allows us to consider in a first step only zero cells lying in some fixed bounded set $C$. In doing so, we then consider separately the cases where the set $C$ is hit by exactly $N$ hyperplanes of the process, for given $N \in \mathbb{N}$. Let

$$\mathcal{K}_{a,\epsilon}(m) := \{K \in \mathcal{K}^d : \vartheta(K) \geq \epsilon,\ \Sigma(K) \in (a, 2a),\ \delta(K) \in [m, m+1)\}.$$

We must estimate the conditional probability

$$p_N := \mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon}(m) \mid \#(X \cap \mathcal{H}_C) = N).$$

By the Poisson property,

$$p_N = \frac{1}{[\Phi(C)\lambda]^N} \int_{\mathcal{H}_C} \cdots \int_{\mathcal{H}_C} \mathbf{1}\{H_1^- \cap \cdots \cap H_N^- \in \mathcal{K}_{a,\epsilon}(m)\}\, \Theta(\mathrm{d}H_1) \cdots \Theta(\mathrm{d}H_N).$$

If the integrand is equal to 1, then

$$P(H_{(N)}) := H_1^- \cap \cdots \cap H_N^- \in \mathcal{K}_{a,\epsilon}(m),$$

hence $\vartheta(P(H_{(N)})) \geq \epsilon$.

Let $B_a$ be an extremal body with $\Sigma(B_a) = a$. By the stability estimate (21),

$$\Phi(P(H_{(N)})) \geq (1 + f(\epsilon))\Phi(B_a).$$

The polytope $P(H_{(N)})$ can have as many as $N$ facets. For an effective estimation, we must restrict its number of vertices. Using an approximation theorem from convex geometry, we can show, for given $\alpha > 0$, the existence of a number $\nu$ independent of $N$ such that the convex hull $Q(H_{(N)})$ of $\nu$ suitably chosen vertices of $P(H_{(N)})$ satisfies

$$\Phi(Q(H_{(N)})) \geq (1 - \alpha)\Phi(P(H_{(N)})).$$

With $g(\epsilon) := f(\epsilon)/(2 + f(\epsilon))$ we obtain

$$\Phi(Q(H_{(N)})) \geq (1 + g(\epsilon))\Phi(B_a).$$

After some work, this leads to the estimate

$$p_N \leq \frac{1}{[\Phi(C)\lambda]^N} \sum_{j=d+1}^{d\nu} \binom{N}{j} [\Phi(C)\lambda - (1 + g(\epsilon))\Phi(B_a)\lambda]^{N-j} [\Phi(C)\lambda]^j.$$

Summation over $N$ finally yields

$$\mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon}(m)) \leq c_1 m^{rd\nu} \exp\left(-(1 + f(\epsilon)/3)\tau a^{r/k}\gamma\right).$$

This estimate can be applied for small numbers $m$. For large $m$, the estimate

$$\mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon}(m)) \leq c_2 \exp\left(-c_3 m^r a^{r/k}\gamma\right)$$

is used, which is obtained in a similar though somewhat easier way.

We have to combine both estimates and extend the considered range of $\Sigma(Z_0)$ from intervals $a(1, 2)$ to intervals $a(1, 1 + h)$. This extension is achieved by a kind of transformation. We end up with the following estimate for the numerator of our conditional probability:

**Lemma 1.** *Let $\epsilon \in (0, 1)$ and $h \in (0, 1/2)$, then*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \in a(1, 1 + h)) \leq c_4 h \exp\left(-(1 + f(\epsilon)/6)\tau a^{r/k}\gamma\right).$$

Since this upper bound for the numerator contains the number $h$ as a factor, it is necessary to estimate the denominator from below by a suitable bound which is also linear in $h$, so that this factor cancels out.

**Lemma 2.** *For each $\beta > 0$, there are constants $h_0 > 0$, $N \in \mathbb{N}$ and $c_5 > 0$ such that, for $a > 0$ and $0 < h < h_0$,*

$$\mathbb{P}(\Sigma(Z_0) \in a(1, 1+h)) \geq c_5 h (a^{r/k}\lambda)^N \exp\left(-(1+\beta)\tau a^{r/k}\lambda\right).$$

The proof is essentially constructive, exhibiting sufficiently many situations in which the event $\Sigma(Z_0) \in a(1, 1+h)$ occurs.

In both lemmas, the number $h$ must be sufficiently small. The final part of the proof extends the estimates from the intervals $a(1, 1+h)$, with small $h$, to general intervals $(a, b)$, by a covering argument.

**Further related results**

For a tessellation induced by a stationary Poisson hyperplane process, one can also define a 'typical cell'. This is a random polytope, heuristically representing a cell randomly picked out from a large region of the tessellation, with equal chances for every cell in the region. For a precise definition, we refer to [42]. Up to translations, the distribution of the zero cell $Z_0$ is the volume-weighted distribution of the typical cell (see [42, Satz 6.1.11]). This relation, however, is not very useful if one one wants to study the asymptotic shape of large typical cells, where 'large' refers to some size functional different from the volume. For some special size functionals, the methods leading to Theorem 2 can be adapted. In [26] it was proved that the shape of the typical cell of a stationary and isotropic Poisson hyperplane tessellation is, with high probability, close to the shape of a ball if the $k$th intrinsic volume ($k \geq 2$) of the typical cell is large. It was also proved that the shape of typical cells of large diameter is close to the shape of a segment.

Associated with every Voronoï tessellation is its dual Delaunay tessellation, by a well-known construction. Starting with a stationary Poisson point process in $\mathbb{R}^d$, we obtain in this way a stationary Poisson–Delaunay tessellation. Its cells are almost surely simplices. Again, the notion of a typical cell is well defined (see [42, Satz 6.2.10] for its distribution). Large typical cells were investigated in [23, 24]. In [23], it was proved that the shape of the typical cell of a stationary Poisson–Delaunay tessellation tends to the shape of a regular simplex, given that the volume of the typical cell tends to infinity. Analogous results for surface area, inradius, and minimal width as size functionals were obtained in [24]. Typical cells of large diameter tend to belong to a special class of simplices, distinct from the regular ones. In the plane, these are the right-angled triangles. The results require the investigation of extremal properties of simplices contained in a fixed ball. Corresponding stability results lead to estimates for probabilities of large deviations from asymptotic shapes.

## References


[1] F. Affentranger - R. Schneider, *Random projections of regular simplices,* Discrete Comput. Geom., **7** (1992), 219–226.

[2] F. Avram, D. Bertsimas, *On central limit theorems in geometrical probability,* Ann. Appl. Probab., **3** (1993), 1033–1046.

[3] I. Bárány, *Random polytopes in smooth convex bodies,* Mathematika, **39** (1982), 81–92.

[4] I. Bárány, *Random polytopes, convex bodies, and approximation,* in A. J. Baddeley - I. Bárány - R. Schneider - W. Weil, *Stochastic Geometry,* C.I.M.E. Summer School, Martina Franca, Italy, 2004 (W. Weil, ed.), Lecture Notes in Mathematics **1892**, pp. 77–118, Springer, Berlin (2007).

[5] I. Bárány - D. G. Larman, *Convex bodies, economic cap coverings, random polytopes,* Mathematika, **35** (1988), 274–291.

[6] I. Bárány - M. Reitzner, *Random polytopes,* (submitted).

[7] I. Bárány - V. Vu, *Central limit theorems for Gaussian polytopes,* Ann. Probab., **35** (2007), 1593–1621.

[8] Y. N. Baryshnikov - R. A. Vitale, *Regular simplices and Gaussian samples,* Discrete Comput. Geom., **11** (1994), 141–147.

[9] K. Böröczky Jr. - M. Henk, *Random projections of regular polytopes,* Arch. Math., **73** (1999), 465–473.

[10] C. Buchta, *An identity relating moments of functionals of convex hulls,* Discrete Comput. Geom., **33** (2005), 125–142.

[11] A. J. Cabo - P. Groeneboom, *Limit theorems for functionals of convex hulls,* Probab. Theory Rel. Fields, **100** (1994), 31–55.

[12] D. L. Donoho, *Neighborly polytopes and sparse solutions of underdetermined linear equations,* Technical Report, Department of Statistics, Stanford University (2004).

[13] D. L. Donoho, *High-dimensional centrally symmetric polytopes with neighborliness proportional to dimension,* Discrete Comput. Geom., **35** (2006), 617–652.

[14] D. L. Donoho - J. Tanner, *Sparse nonnegative solutions of underdetermined linear equations by linear programming,* Proc. Natl. Acad. Sci. USA, **102** (2005), no. 27, 9446–9451.

[15] D. L. Donoho - J. Tanner, *Neighborliness of randomly projected simplices in high dimensions,* Proc. Natl. Acad. Sci. USA, **102** (2005), no. 27, 9452–9457.

[16] D. L. Donoho - J. Tanner, *Counting faces of randomly-projected polytopes when the projection radically lowers dimension,* Technical Report No. 2006-11, Dept. of Statistics, Stanford University; J. Amer. Math. Soc. (to appear)

[17] A. Goldman, *Sur une conjecture de D.G. Kendall concernant la cellule de Crofton du plan et sur sa contrepartie brownienne,* Ann. Probab., **26** (1998), 1727–1750.

[18] P. Groeneboom, *Convex hulls of uniform samples from a convex polygon,* Preprint (2006).

[19] D. Hug - G. O. Munsonius - M. Reitzner, *Asymptotic mean values of Gaussian polytopes,* Beiträge Algebra Geom., **45** (2004), 531–548.

[20] D. Hug - M. Reitzner, *Gaussian polytopes: variances and limit theorems,* Adv. Appl. Prob. (SGSA), **37** (2005), 297–320.

[21] D. Hug - M. Reitzner - R. Schneider, *The limit shape of the zero cell in a stationary Poisson hyperplane tessellation,* Ann. Probab., **32** (2004), 1140–1167.

[22] D. Hug - M. Reitzner - R. Schneider, *Large Poisson–Voronoi cells and Crofton cells,* Adv. Appl. Prob. (SGSA), **36** (2004), 667–690.

[23] D. Hug - R. Schneider, *Large cells in Poisson–Delaunay tessellations,* Discrete Comput. Geom., **31** (2004), 503–514.

[24] D. Hug - R. Schneider, *Large typical cells in Poisson–Delaunay mosaics,* Rev. Roumaine Math. Pures Appl., **50** (2005), 657–670.

[25] D. Hug - R. Schneider, *Asymptotic shapes of large cells in random tessellations,* Geom. Funct. Anal., **17** (2007), 156–191.

[26] D. Hug, D. - R. Schneider, *Typical cells in Poisson hyperplane tessellations,* Discrete Comput. Geom. (to appear).

[27] I. N.Kovalenko, *A proof of a conjecture of David Kendall on the shape of random polygons of large area* (Russian), Kibernet. Sistem. Anal. 1997, 3–10, 187; Engl. transl.: Cybernet. Systems Anal., **33** (1997), 461–467.

[28] I. N. Kovalenko, *An extension of a conjecture of D.G. Kendall concerning shapes of random polygons to Poisson Voronoï cells,* in *Voronoï's Impact on Modern Science* (Engel, P. et al., eds.), Book I. Transl. from the Ukrainian, Kyiv: Institute of Mathematics. Proc. Inst. Math. Natl. Acad. Sci. Ukr., Math. Appl., **212** (1998), 266–274.

[29] I. N. Kovalenko, *A simplified proof of a conjecture of D. G. Kendall concerning shapes of random polygons,* J. Appl. Math. Stochastic Anal., **12** (1999), 301–310.

[30] N. Linial - I. Novik, *How neighborly can a centrally symmetric polytope be?,* Discrete Comput. Geom., **36** (2006), 273–281.

[31] J. Mecke - I. Osburg, *On the shape of large Crofton parallelotopes,* Math. Notae, **41** (2003), 149–157.

[32] R. E. Miles, *A heuristic proof of a long-standing conjecture of D. G. Kendall concerning the shapes of certain large random polygons,* Adv. Appl. Prob. (SGSA), **27** (1995), 397–417.

[33] M. Reitzner, *Random polytopes and the Efron–Stein jackknife inequality,* Ann. Probab., **31** (2003), 2136–2166.

[34] M. Reitzner, *Central limit theorems for random polytopes,* Probab. Theory Relat. Fields, **133** (2005), 483–507.

[35] A. Rényi - R. Sulanke, *Über die konvexe Hülle von n zufällig gewählten Punkten,* Z. Wahrscheinlichkeitsth. verw. Geb., **2** (1963), 75–84.

[36] A. Rényi - R. Sulanke, *Über die konvexe Hülle von n zufällig gewählten Punkten, II,* Z. Wahrscheinlichkeitsth. verw. Geb., **3** (1964), 138–147.

[37] A. Rényi - R. Sulanke, *Zufällige konvexe Polygone in einem Ringgebiet,* Z. Wahrscheinlichkeitsth. verw. Geb., **9** (1968), 146–157.

[38] Y. Rinott, *On normal approximation rates for certain sums of dependent random variables,* J. Comput. Appl. Math., **55** (1994), 135–147.

[39] R. Schneider, *Random approximation of convex sets,* J. Microscopy, **151** (1988), 211–227.

[40] R. Schneider, *Convex Bodies - the Brunn–Minkowski Theory,* Cambridge University Press, Cambridge (1993).

[41] R. Schneider, *Discrete aspects of stochastic geometry,* in *Handbook of Discrete and Computational Geometry* (J. E. Goodman - J. O'Rourke, eds.), 2nd ed., pp. 255–278, CRC Press, Boca Raton (2004).

[42] R. Schneider - W. Weil, *Stochastische Geometrie,* Teubner, Stuttgart (2000).

[43] C. Schütt, *Random polytopes and affine surface area,* Math. Nachr., **170** (1994), 227–249.

[44] C. Schütt - E. Werner, *Polytopes with vertices chosen randomly from the boundary of a convex body,* Israel Seminar 2001–2002 (V. D. Milman, G. Schechtman, eds.), pp. 241–422, Lecture Notes in Math., vol. **1807**, Springer, New York (2003).

[45] D. Stoyan - W. S. Kendall - J. Mecke, *Stochastic Geometry and Its Applications,* 2nd ed., Wiley, Chichester (1995).

[46] J.J. Sylvester, *Question 1491,* Educational Times, London, April (1864).

[47] A. M. Vershik - P. V. Sporyshev, *An asymptotic estimate of the average number of steps of the parametric simplex method,* U.S.S.R. Comput. Math. and Math. Phys., **26** (1986), 104–113.

[48] A. M. Vershik - P. V. Sporyshev, *Asymptotic behavior of the number of faces of random polyhedra and the neighborliness problem,* Selecta Math. Soviet., **11** (1992), 181–201.

[49] V. Vu, *Sharp concentration of random polytopes,* Geom. Funct. Anal., **15** (2005), 1284–1318.

[50] V. Vu, *Central limit theorems for random polytopes in a smooth convex set,* Adv. Math., **207** (2006), 221–243.

[51] W. Weil - J. A. Wieacker, *Stochastic geometry,* in *Handbook of Convex Geometry* (P. M. Gruber - J. M. Wills, eds.), vol. B, pp. 1391–1438, North-Holland, Amsterdam (1993).

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